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Another Look at Topological BCH-algebras

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Abstract. A BCH-algebra $(H, *, 0)$ furnished with a topology τ on H (also called a BCH-topology on H) is called a topological BCH-algebra (or TBCH-algebra) if the function $* : H \times H \to H$, defined by $*((x, y)) = x * y$ for any $x, y \in H$, is continuous, where the Cartesian product topology on $H \times H$ is furnished by τ. In this paper, we give other structural properties of topological BCH-algebras.

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1. Introduction

In 1983, Hu and Li [5, 6] introduced the notion of a BCH-algebra which is a generalization of BCK and BCI-algebras. In the same paper, the concept of associative BCH-algebra was also introduced. Dar, K. H., and Akram, M. [2] defined the concepts of BCH-ideal, BCH-subalgebra, $*$-commutative, left and right mappings on a BCH-algebra and some properties structures were investigated.

In [8] and [4], the concepts of topological BCK-algebra and topological BCI-algebra were defined and some properties of each newly defined concepts were investigated. In 2017, M. Jansi and V. Thiruveni [7] introduced the concept of topological BCH-algebra (or TBCH-algebra) and investigated some of its algebraic and topological properties. The aim of this paper is to give other structural properties of topological BCH-algebras.

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2. Preliminaries and Known Results

Definition 1. [3] Let (X, τ) be a topological space and let $x \in X$. Any set $U \in \tau$ containing x is called a *neighborhood* (sometimes written as *nbhd* or τ-*nbhd*) of x.

Definition 2. [3] Let (X, τ) be a topological space. Then

(i) (X, τ) is a T_0-space if for any $x, y \in X$ with $x \neq y$, there exists an open set U containing one but not the other;

(ii) (X, τ) is a T_1-space if for any $x, y \in X$ with $x \neq y$, there exist nbhds U and V of x and y, respectively, such that $x \notin V$ and $y \notin U$;

(iii) (X, τ) is a T_2-space (or *Hausdorff* space) if for any $x, y \in X$ with $x \neq y$, there exist disjoint nbhds U and V of x and y, respectively.

Remark 1. [3] $T_2 \Rightarrow T_1 \Rightarrow T_0$ but not conversely.

Theorem 1. [3] *Let (X, τ) be a topological space. X is a T_1-space if and only if for each $x \in X$, $\{x\}$ is a closed set in X.*

Definition 3. [5] A *BCH-algebra* is a nonempty set H endowed with a operation " $*$ " and constant 0 satisfying the following axioms: for all $x, y, z \in H$,

(B1) $x * x = 0$,

(B2) $x * y = 0$ and $y * x = 0$ implies $x = y$.

(B3) $(x * y) * z = (x * z) * y$,

Remark 2. [5, 6] In any BCH-algebra $(X, *, 0)$, the following hold:

(i) $x * 0 = x$;

(ii) $x * 0 = 0$ implies $x = 0$;

(iii) $0 * (x * y) = (0 * x) * (0 * y)$;

(iv) $(x * (x * y)) * y = 0$.

Definition 4. [7] Let $(X, *, 0)$ be a BCH-algebra and U, V be any nonempty subsets of X. We define a subset $U * V$ of X by $U * V = \{x * y : x \in U, y \in V\}$.

Remark 3. Let $(X, *, 0)$ be a BCH-algebra. Then $*(A \times B) = A * B$ for any nonempty subsets A and B of X.

Remark 4. Let $(X, *, 0)$ be a BCH-algebra and $A, B \subseteq X$. If $A \cap B \neq \varnothing$, then $0 \in A * B$.

Definition 5. [2] Let $(X, *, 0)$ be a BCH-algebra. A nonempty subset S of X is a *BCH-subalgebra* if for each $x, y \in S$, $x * y \in S$.

Definition 6. [7] Let $(H, *, 0)$ be a BCH-algebra. A topology τ furnished on H is called a *BCH-topology* on H. In addition, (H, τ) is called a *topological BCH-algebra* (or *TBCH-algebra*) if τ is a BCH-topology on H and the function $* : H \times H \to H$ defined as $*((x, y)) = x * y$ is continuous, where the Cartesian product topology on $H \times H$ is furnished by τ.

Example 1. Let $X = \{0, 1, 2, 3, 4\}$ and define $*$ as follows:

$*$	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Then, $(X, *, 0)$ is a BCH-algebra [1]. Let $\tau = \{X, \varnothing, \{4\}, \{0, 1, 2, 3\}\}$. Then τ is a BCH-topology on X. Moreover,

$$\begin{aligned} *^{-1}(X) &= X \times X \\ *^{-1}(\varnothing) &= \varnothing \\ *^{-1}(\{4\}) &= (\{0,1,2,3\} \times \{4\}) \cup (\{4\} \times \{0,1,2,3\}) \\ *^{-1}(\{0,1,2,3\}) &= (\{0,1,2,3\} \times \{0,1,2,3\}) \cup (\{4\} \times \{4\}). \end{aligned}$$

This implies that $*$ is continuous. Thus, (X, τ) is a TBCH-algebra.

3. Results

Throughout this study, we denote a BCH-algebra $(X, *, 0)$ by X, unless otherwise specified.

Theorem 2. *Let τ be a BCH-topology on X. Then, (X, τ) is a TBCH-algebra if and only if for each $x, y \in X$ and each nbhd W of $x * y$, there exist nbhds U and V of x and y, respectively, such that $U * V \subseteq W$.*

Proof. Let X be a TBCH-algebra. Let $x, y \in X$ and a nbhd W of $x * y$. Since $*$ is continuous, $*^{-1}(W)$ is a nbhd of (x, y) in $X \times X$. By definition of Cartesian product topology, there exist nbhds U and V of x and y, respectively, such that $U \times V \subseteq *^{-1}(W)$. By Remark 3, $U * V = *(U \times V)$. It follows that $U * V \subseteq *(*^{-1}(W)) \subseteq W$.

Conversely, suppose that for each $x, y \in X$ and each nbhd W of $x * y$, there are nbhds U and V of x and y, respectively, such that $U * V \subseteq W$. By definition of Cartesian product topology, $U \times V$ is a nbhd of (x, y) in $X \times X$. By Remark 3, $*(U \times V) = U * V \subseteq W$. Therefore, $*$ is continuous. □

Corollary 1. *Let X be a TBCH-algebra and $A \subseteq X$. If z is an interior point of A, then there exist elements $x, y \in X$ and nbhds N_x, N_y and N_z of x, y and z, respectively, such that $z = x * y$ and $N_x * N_y \subseteq N_z = N_{x*y}$.*

Proof. Suppose z is an interior point of A. Then there exists a nbhd N_z of z such that $N_z \subseteq A$. Since $z \in X$, $z = x * y$ for some $x, y \in X$ (say, $x = z$ and $y = 0$). By Theorem 2, there exist nbhds N_x and N_y of x and y, respectively, such that $N_x * N_y \subseteq N_z = N_{x*y}$. □

The next theorem asserts that the topology associated in a TBCH-algebra having $\{0\}$ as an open set is the discrete topology.

Theorem 3. *Let X be a TBCH-algebra. Then $\{0\}$ is an open set in X if and only if X is a discrete space.*

Proof. Suppose that $\{0\}$ is an open set in X and let $x \in X$. Then, $x * x = 0 \in \{0\}$ by (B1). Since $\{0\}$ is an open set in X, there exist nbhds U and V of x such that $U * V = \{0\}$ by Theorem 2. Let $W = U \cap V$. Then, W is a nbhd of x and $W * W \subseteq U * V$. Hence, $W * W = \{0\}$. Let $y \in W$. Then $x * y = 0 = y * x$. By (B2), $y = x$. Thus, $W = \{x\}$, showing that X is a discrete space.
Conversely, suppose X is the discrete space. Then, $\{0\}$ is an open set in X. □

Corollary 2. *If $\{0\}$ is an open set in a TBCH-algebra X, then every subset of X is both open and closed set in X. In particular, if $|X| \geq 2$, then X is a disconnected space.*

Remark 5. *If a BCH-topological space X is a discrete space, then X is a TBCH-algebra.*

We now show that a BCH-subalgebra of a TBCH-algebra is also a TBCH-algebra.

Theorem 4. *Let X be a TBCH-algebra and H a BCH-subalgebra of X. Then (H, τ_H) is a TBCH-algebra, where τ_H is the relative topology on H.*

Proof. Let $x, y \in H$ and a nbhd W_H of $x * y$ in the subspace H. Note that W_H may be written as the intersection with H of some nbhd W of $x * y$ in X, that is, $W_H = H \cap W$. Since X is a TBCH-algebra, there exist nbhds U and V of x and y, respectively, such that $U * V \subseteq W$ by Theorem 2. Observe that $U_H = H \cap U$ and $V_H = H \cap V$ are nbhds of x and y, respectively, in the subspace H. Furthermore.

$$\begin{aligned} U_H * V_H &= (H \cap U) * (H \cap V) \\ &\subseteq U * V \\ &\subseteq W. \end{aligned}$$

Since H is a BCH-subalgebra, $U_H * V_H \subseteq H * H \subseteq H$ so that $U_H * V_H \subseteq H \cap W = W_H$. By Theorem 2, (H, τ_H) is a TBCH-algebra. □

Theorem 5. *Let $(H_1, *_1, 0)$ and $(H_2, *_2, 0)$ be BCH-algebras such that $H_1 \cap H_2 = \{0\}$ and $H = H_1 \cup H_2$. Then $(H, *, 0)$ is a BCH-algebra, denoted by $H_1 \oplus H_2$, where the operation "$*$" on H is defined for all $x, y \in H$, by*

$$x * y = \begin{cases} x *_1 y & if\ x, y \in H_1 \\ x *_2 y & if\ x, y \in H_2 \\ x & otherwise. \end{cases}$$

Proof. Let $x \in H$. Then

$$x * x = \begin{cases} x *_1 x & if\ x \in H_1 \\ x *_2 x & if\ x \in H_2. \end{cases}$$

Since $(H_1, *_1, 0)$ and $(H_2, *_2, 0)$ are BCH-algebras, $x * x = 0$ by property (B1).

Next, let $x, y \in H$ and suppose that $x * y = 0$ and $y * x = 0$. Consider the following cases:

Case 1: $x, y \in H_1$ (or $x, y \in H_2$).

Then $x * y = x *_1 y = 0$ and $y * x = y *_1 x = 0$. Since $(H_1, *_1, 0)$ is a BCH-algebra, property (B2) yields $x = y$. Similarly, $x = y$ if $x, y \in H_2$.

Case 2: $x \in H_1$ and $y \in H_2$ (or $y \in H_1$ and $x \in H_2$).

Then $0 = x * y = x$ and $0 = y * x = y$. Hence, $x = 0 = y$.

Finally, let $x, y, z \in H$. Consider the following cases:

Case 1: $x, y \in H_1$ (or $x, y \in H_2$)

Then, by the definition of $*$,

$$(x * y) * z = \begin{cases} (x *_1 y) *_1 z & if\ z \in H_1 \\ x *_1 y & if\ z \in H_2. \end{cases}$$

and

$$(x * z) * y = \begin{cases} (x *_1 z) *_1 y & if\ z \in H_1 \\ x *_1 y & if\ z \in H_2. \end{cases}$$

Since $(H_1, *_1, 0)$ is a BCH-algebra, $(x *_1 y) *_1 z = (x *_1 z) *_1 y$ if $z \in H_1$. Hence, $(x * y) * z = (x * z) * y$. Similarly, $(x * y) * z = (x * z) * y$ whenever $x, y \in H_2$.

Case 2: $x \in H_1$ and $y \in H_2$ (or $y \in H_1$ and $x \in H_2$)

Then, by the definition of $*$,

$$(x * y) * z = \begin{cases} x *_1 z & if\ z \in H_1 \\ x & if\ z \in H_2. \end{cases}$$

and

$$(x * z) * y = \begin{cases} x *_1 z & if\ z \in H_1 \\ x & if\ z \in H_2. \end{cases}$$

Therefore, $(x * y) * z = (x * z) * y$. Equality is also obtained if $y \in H_1$ and $x \in H_2$.

Accordingly, $(H, *, 0)$ is a BCH-algebra. □

Lemma 1. *Let $(H, *_1)$ and $(H_2, *_2)$ be BCH-algebras such that $H_1 \cap H_2 = \{0\}$ and let $(H, *)$ be the sum of H_1 and H_2 defined in Theorem 5. Then each of the following holds:*

(*i*) *If U and V are subsets of H_1 (U and V are subsets of H_2), then $U *_1 V = U * V$ (resp. $U *_2 V = U * V$).*

(*ii*) *If $A, B \subseteq H_1$, $C \subseteq H_2$, and $0 \in B$, then $A \subseteq A*B$ and $A*(B \cup C) = A *_1 B = A * B$.*

Proof. (*i*) Suppose U and V are subsets of H_1. Let $x \in U$ and $y \in V$. Since $x*y = x*_1y$, $x*y \in U*V$ if and only if $x*_1y \in U*_1V$. Hence, $U*_1V = U*V$. Similarly, $U*_2V = U*V$ if U and V are subsets of H_2.

(*ii*) Let $x \in A$. Then $x = x * 0 \in A * B$ since $0 \in B$. Hence, $A \subseteq A * B = A *_1 B$.

To establish the equality, first note that $A *_1 B = A * B \subseteq A * (B \cup C)$. Let $a \in A$ and $x \in (B \cup C)$. If $x \in B$, then $a * x = a *_1 x \in A *_1 B$. If $x \in C$, then $a * x = a \in A \subseteq A *_1 B$. Thus, $A * (B \cup C) = A *_1 B = A * B$. □

Theorem 6. *Let $(H_1, *_1, 0)$ and $(H_2, *_2, 0)$ be BCH-algebras such that $H_1 \cap H_2 = \{0\}$ and let $(H, *, 0)$ be the sum of H_1 and H_2 (defined in Theorem 5). Then each of the following holds:*

(*i*) *$(H_1, *_1, 0)$ and $(H_2, *_2, 0)$ are BCH-subalgebras of H.*

(*ii*) *(H, τ^{H_1}) and (H, τ^{H_2}) are TBCH-algebras, where $\tau^{H_1} = \{\varnothing, H_1 \cup H_2, H_1\}$ and $\tau^{H_2} = \{\varnothing, H_1 \cup H_2, H_2\}$.*

(*iii*) *If (H, τ) is a TBCH-algebra and $A, B \in \tau$ for some set $A \subseteq H_1$ and $B \subseteq H_2$ with $0 \in A \cap B$, then τ is the discrete topology on H. In particular, if $H_1, H_2 \in \tau$, then τ is the discrete topology on H.*

(*iv*) *If (H, τ) is a TBCH-algebra and $\tau \subseteq P(H_1) \cup \{H_1 \cup H_2\}$ (or $\tau \subseteq P(H_2) \cup \{H_1 \cup H_2\}$), where $P(H_1)$ and $P(H_2)$ are the power sets of H_1 and H_2, respectively, then $0 \in W$ for every $W \in \tau \setminus \{\varnothing\}$.*

Proof. (*i*) Let $x, y \in H_1$. Then $x * y = x *_1 y \in H_1$ by Theorem 5 and the fact that $(H_1, *_1, 0)$ is a BCH-algebra. Therefore, $(H_1, *_1, 0) = (H_1, *, 0)$ is a BCH-subalgebra of H. Similarly, $(H_2, *_2, 0)$ is a BCH-subalgebra of H.

(*ii*) Clearly, τ^{H_1} are τ^{H_2} are topologies on H. First, consider the space (X, τ^{H_1}). Let $x, y \in H$ and let W be a τ^{H_1}-nbhd of $x * y$. Consider the following cases:

Case 1: $x, y \in H_1$

Then $x * y = x *_1 y \in H_1$. Hence, $W = H_1$ or $W = H_1 \cup H_2$. Then H_1 is a τ^{H_1}-nbhd of both x and y, and by Lemma 1(*i*), $H_1 * H_1 = H_1 *_1 H_1 = H_1 \subset H_1 \cup H_2$.

Case 2: $x, y \in H_2$ or $[x \in H_2$ and $y \in H_1]$

If $x, y \in H_2$, then $x * y = x *_2 y \in H_2$. Hence, $W = H_1 \cup H_2$. The set $V = H_1 \cup H_2$ is a τ^{H_1}-nbhd of both x and y, and $V * V = H_1 \cup H_2$. If $x \in H_2$ and $y \in H_1$, then $x * y = x \in H_2$. Again, $W = H_1 \cup H_2$, $V = H_1 \cup H_1$ is a τ^{H_1}-nbhd of both x and y, and $V * V = H_1 \cup H_2$.

Case 3: $x \in H_1$ and $y \in H_2$

Then $x * y = x \in H_1$. Hence, $W = H_1$ or $W = H_1 \cup H_2$. Let $V_x = H_1$ and $V_y = H_1 \cup H_2$. Then V_x and V_y are τ^{H_1}-nbhds of x and y, respectively, and by Lemma 1(*ii*), $V_x * V_y = H_1 * (H_1 \cup H_2) = H_1 *_1 H_1 = H_1 \subset H_1 \cup H_2$.

Therefore, (H, τ^{H_1}) is a TBCH algebra. Similarly, (H, τ^{H_2}) is a TBCH algebra.

(*iii*) Suppose $A \subseteq H_1$, $B \subseteq H_2$, $0 \in A \cap B$, and $A, B \in \tau$. Since $H_1 \cap H_2 = \{0\}$, it follows that $A \cap B = \{0\}$. Since $A, B \in \tau$, $\{0\} \in \tau$. Thus, by Theorem 3, τ is the discrete topology on H.

(*iv*) Suppose that (H, τ) is a TBCH-algebra and that $\tau \subseteq P(H_1) \cup \{H_1 \cup H_2\}$. Let $W \in \tau \setminus \{\varnothing\}$. Pick any $x \in W$ and $y \in H_2$. Since $x * y = x$, W is a nbhd of $x * y$. By continuity of $*$, there exist nbhds V_x and V_y of x and y, respectively, such that $V_x * V_y \subseteq W$. Now, since $\tau \subseteq P(H_1) \cup \{H_1 \cup H_2\}$, the only nbhd of y is $H_1 \cup H_2$. Hence, $V_y = H_1 \cup H_2$ and by Lemma 1(*ii*), $V_x * V_y = V_x * (H_1 \cup H_2) = V_x *_1 H_1$. Since $x \in H_1$, $x *_1 x = x * x = 0 \in V_x *_1 H_1$. Therefore, $0 \in W$. □

Theorem 7. *Let X be a TBCH-algebra. Then $\{0\}$ is a closed set in X if and only if X is a T_2-space.*

Proof. Suppose $\{0\}$ is a closed set in X. Let $x, y \in X$ with $x \neq y$. Then, $x * y \neq 0$ or $y * x \neq 0$. Without loss of generality, assume that $x * y \neq 0$. Note that $x * y \in X \setminus \{0\}$. By Theorem 2, there exist nbhds U and V of x and y, respectively, such that $U * V \subseteq X \setminus \{0\}$. Suppose $U \cap V \neq \varnothing$. Let $z \in U \cap V$. Then, $z \in U$ and $z \in V$. Hence, by (B1)

$$0 = z * z \in U * V \subseteq X \setminus \{0\}$$

a contradiction. Thus, $U \cap V = \varnothing$ and so X is a T_2-space.

Conversely, assume that X is a T_2-space. Let $x \in X \setminus \{0\}$. Then, there exist nbhds U and V of x and 0, respectively, such that $U \cap V = \varnothing$. Since $0 \notin U$, $x \in U \subseteq X \setminus \{0\}$. This shows that $X \setminus \{0\}$ is open in X. Therefore, $\{0\}$ is a closed set in X. □

The next theorem asserts that T_0, T_1 and T_2 topological spaces are equivalent in a TBCH-algebra.

Theorem 8. *Let X be a TBCH-algebra. Then the following statements are equivalent:*

(i) *X is a T_0-space*

(ii) *X is a T_1-space*

(iii) *X is a T_2-space.*

Proof. (i)⇒(ii): Suppose X is a T_0-space. Let $x, y \in X$ with $x \neq y$. Then $x * y \neq 0$ or $y * x \neq 0$ by (B2). Without loss of generality, assume that $x * y \neq 0$. Since X is a T_0-space, there exists an open set U such that $x * y \in U$ but $0 \notin U$ or $0 \in U$ but $x * y \notin U$. Consider the following cases:
Case 1. $x * y \in U$ (but $0 \notin U$)
By Theorem 2, there exist nbhds G_x and H_y of x and y, respectively, such that $G_x * H_y \subseteq U$. Since $0 \notin U$, $0 \notin G_x * H_y$. By Remark 4, $G_x \cap H_y = \varnothing$. Thus, $y \notin G_x$ and $x \notin H_y$.
Case 2. $0 \in U$ (but $x * y \notin U$).
By (B1), $x * x = 0 \in U$. By Theorem 2, there exist nbhds N_x and M_x of x such that $N_x * M_x \subseteq U$. Since $x * y \notin U$, $x * y \notin N_x * M_x$. It follows that $y \notin M_x$. Similarly, since $y * y = 0 \in U$, there exist nbhds N_y and M_y of y such that $N_y * M_y \subseteq U$. Since $x * y \notin U$, $x * y \notin N_y * M_y$. It follows that $x \notin N_y$. Hence, there exist nbhds M_x and N_y of x and y, respectively, such that $y \notin M_x$ and $x \notin N_y$.

Therefore, X is a T_1-space.
(ii)⇒(iii): Suppose X is a T_1-space. By Theorem 1, $\{0\}$ is a closed set in X. By Theorem 7, X is a T_2-space.
By Remark 1, $T_2 \Rightarrow T_1 \Rightarrow T_0$. Therefore, (i), (ii), and (iii) are equivalent. □

The following corollary follows from Theorems 7 and 8.

Corollary 3. *Let X be a TBCH-algebra. Then the following statements are equivalent:*

(i) *X is a T_0-space*

(ii) *X is a T_1-space*

(iii) *X is a T_2-space*

(iv) *$\{0\}$ is a closed set in X.*

Theorem 9. *Let X be a TBCH-algebra. Then X is a T_2-space if and only if for any $x \in X$ with $x \neq 0$, there exists a nbhd U of x such that $0 \notin U$.*

Proof. Clearly, if X is a T_2-space, then for any $x \in X$ with $x \neq 0$, there exists a nbhd U of x such that $0 \notin U$.

For the converse, suppose that for any $x \in X$ with $x \neq 0$, there exists a nbhd U of x such that $0 \notin U$. Let $a, b \in X$ with $a \neq b$. Then $a * b \neq 0$ or $b * a \neq 0$ by (B2). Without loss of generality, assume that $a * b \neq 0$. Then, by assumption, there exists a nbhd W of $a * b$ such that $0 \notin W$. By Theorem 2, there exist nbhds W_a and W_b of a and b, respectively, such that $W_a * W_b \subseteq W$. Since $0 \notin W$, $0 \notin W_a * W_b$. By Remark 4, $W_a \cap W_b = \varnothing$. Thus, X is a T_2-space. □

Conclusion: Given two BCH-algebras H_1 and H_2 such that $H_1 \cap H_2 = \{0\}$, an operation "$*$" can be defined on $H = H_1 \cup H_2$ so that $(H, *)$ is a BCH-algebra and H_1 and H_2 are BCH-subalgebras. Further, it is shown that T_0, T_1 and T_2 axioms are equivalent in any topological BCH-algebra.

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