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On the Solutions for Grötzsch Annulus Extremal Problem

Xie Zhichun* and Huang Xinzhong

School of Mathematical Science, Huaqiao University, Quanzhou, Fujian, 362021, P. R. C.

Abstract. Using the properties of planar quasiconformal mappings, we obtain the solutions for the Grötzsch extremal problem on the annulus. We also point out the shortcoming used in [3] for solving this extremal problem. Moreover, by the hyperbolic area distortion and the property of domain module, one criterion for the solution of the Grötzsch annulus extremal problem is given under some conditions.

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1. Introduction

Let $\Omega, \Omega' \subset C$ be planar domains, a sense-preserving homeomorphism $f : \Omega \to \Omega'$ is said to be $K-$quasiconformal mapping in Ω, if it satisfies: (1) f is ACL in Ω; (2)

*Corresponding author.

Email addresses: xiezhichun@hqu.edu.cn (X. Zhichun), huangxz@hqu.edu.cn (H. Xinzhong)

$f_{\bar{z}}(z) = \mu(z)f_z(z)\ a.e., z \in \Omega$, where $ess \sup_{z\in\Omega} |\mu(z)| = k < 1, K = \frac{1+k}{1-k}$. Suppose that $f(z)$ is a quasiconformal mapping in Ω, let $D_f(z) = \frac{|f_z|+|f_{\bar{z}}|}{|f_z|-|f_{\bar{z}}|}$. The Grötzsch extremal problem is to find a quasiconformal mapping $f_0(z)$ among all quasiconformal mappings $f(z)$ in Ω, such that

$$ess \sup_{z\in\Omega} D_{f_0}(z) = \inf_f \sup_z D_f(z). \tag{1}$$

It is known by [1] that the solution of the Grötzsch extremal problem from square to rectangle is an affine mapping, and the Grötzsch extremal problem from annulus $\{z|r \leq |z| \leq 1\}$ onto $\{z|R \leq |z| \leq 1\}$ was investigated in [2] and [3]. The following result was proved in [2].

Theorem 1. *If $f(z)$ be a quasiconformal homeomorphism in the unit disk onto itself such that*

$$f(0) = 0, \qquad \lim_{z\to 0} \frac{|f(z)|}{|z|^{\frac{1}{K}}} = 1.$$

Then the solution for the Grötzsch annulus extremal problem is $f(z) = e^{i\theta} z|z|^{\frac{1}{K}-1}$, where θ is a real number.

Obviously, the normalized condition $f(0) = 0$ is unnecessary for solving the Grötzsch annulus extremal problem. By the method of extremal length, the following result was proved in [3].

Lemma 1. *If $f(z)$ is a $K-$quasiconformal homeomorphism from $\{z|r \leq |z| \leq 1\}$ to $\{z|r^{\frac{1}{K}} \leq |z| \leq 1\}$, then*

$$f(z) = \lambda z|z|^{\frac{1}{K}-1},$$

where λ is a constant and $|\lambda| = 1$.

We point out that Lemma 1 is not true. For example, let $g(z) = r^{\frac{1}{K}} e^{i(\theta + k sin\theta)}$, where $z = re^{i\theta}$, $k = \frac{K-1}{K+1}, K \geq 1$, then $g(z)$ is a $\frac{2K^2}{K+1}-$quasiconformal homeomorphism from $\{z|r \leq |z| \leq 1\}$ onto $\{z|r^{\frac{1}{K}} \leq |z| \leq 1\}$. Therefore, Lemma 1 is meaningful only under considering the Grötzsch extremal problem on the annulus. On the other hand, [3] tried to prove the solution of the Grö tzsch extremal problem on the annulus by using the method of extremal length, the property of the module of ring domain and used the following equality

$$\frac{1}{K\mathrm{mod}(R_R)} = \frac{1}{\mathrm{mod}(R_{R^{\frac{1}{K}}})}.$$

However, above equality is also not true, since we find that $\frac{1}{K\mathrm{mod}(R_R)} = \frac{1}{K\frac{1}{2\pi}\log\frac{1}{R}}$ and $\frac{1}{\mathrm{mod}(R_{R^{\frac{1}{K}}})} = \frac{1}{\frac{1}{K}\frac{1}{2\pi}\log\frac{1}{R}}$.

In this paper, the solution for the Grötzsch extremal problem on the annulus is solved by using analytic method and the extremal condition. On the other hand, [3] also tried to find the solution of the Grötzsch extremal problem for area distortion problem. We also point out that the proved result is not correct. At last, we obtain one criterion for the solution of the Grötzsch annulus extremal problem considering hyperbolic area distortion.

2. Main Results and their Proofs

The notations in this paper are adopted as in [3]. Let $A(r_1, r_2) = \{z|r_1 \le |z| \le r_2\}, A(r_1, r_2; \theta_1, \theta_2) = \{z|r_1 \le |z| \le r_2, \theta_1 \le \arg z \le \theta_2\}$. We will prove the following result.

Theorem 2. *Let Q be K-quasiconformal mappings from $\{z|r_1 \le |z| \le 1\}$ to $\{z|r_1^{\frac{1}{K}} \le |z| \le 1\}$, and if $f \in Q$ satisfies*

$$ess\sup_z D_f(z) = \inf_{g\in Q}\sup_z D_g(z),$$

then

$$f(z) = r^{\frac{1}{K}} e^{i(\theta+\alpha)},\ z = re^{i\theta},\ \textit{where } \alpha \textit{ is a constant.}$$

Proof. For any $R_1, r_1 < R_1 < 1$, let $A = \{z | r_1 \leq |z| \leq R_1\}, B = \{z | R_1 \leq |z| \leq 1\}$, by the property of the module of ring domain [4], we have

$$\begin{aligned} \frac{1}{K}\frac{1}{2\pi}\log\frac{1}{r_1} &= \frac{1}{K}(\text{mod}(A) + \text{mod}(B)) \\ &\leq \text{mod}(f(A)) + \text{mod}(f(B)) \leq \text{mod}(A(r_1^{\frac{1}{K}}, 1)) = \frac{1}{K}\frac{1}{2\pi}\log\frac{1}{r_1}, \end{aligned}$$

thus,

$$\text{mod}(f(A)) + \text{mod}(f(B)) = \text{mod}(A(r_1^{\frac{1}{K}}, 1)) = \frac{1}{K}\frac{1}{2\pi}\log\frac{1}{r_1}.$$

By Lemma 1.3 in [3], we see that $f(A)$ and $f(B)$ are concentric annulus, we have $|f(R_1 e^{i\theta})| = R', 0 \leq \theta < 2\pi$. By the quasi-invariant property of module, we have

$$\begin{cases} \text{mod}(f(A)) \geq \frac{1}{K}\text{mod}(A), \\ \text{mod}(f(B)) \geq \frac{1}{K}\text{mod}(B), \end{cases}$$

that is

$$\begin{cases} R' \geq R_1^{\frac{1}{K}}, \\ R' \leq R_1^{\frac{1}{K}}. \end{cases}$$

Thus, $R' = R_1^{\frac{1}{K}}$. Therefore, we obtain that $f(z) = r^{\frac{1}{K}} e^{i\varphi(r,\theta)}, z = re^{i\theta}$. Next, we will prove that $\varphi(r,\theta)$ depends only on θ. By calculation, we have

$$\begin{aligned} f_{\bar{z}} &= \frac{1}{2} r^{\frac{1}{K}-1} e^{i(\varphi+\theta)} (\frac{1}{K} + ir\varphi_r - \varphi_\theta), \\ f_z &= \frac{1}{2} r^{\frac{1}{K}-1} e^{i(\varphi-\theta)} (\frac{1}{K} + ir\varphi_r + \varphi_\theta). \end{aligned}$$

Obviously, $w = \lambda z |z|^{\frac{1}{K}-1}$ is a $K - q.c.$ from $\{z | r_1 \leq |z| \leq 1\}$ onto $\{z | r_1^{\frac{1}{K}} \leq |z| \leq 1\}$. If $f(z)$ is an extremal $K - q.c.$, which satisfies the conditions of Theorem 2, we have

$$|\frac{f_{\bar{z}}}{f_z}| = |\frac{\frac{1}{K} + ir\varphi_r - \varphi_\theta}{\frac{1}{K} + ir\varphi_r + \varphi_\theta}| \leq \frac{K-1}{K+1},$$

thus,

$$K^2\varphi_\theta^2 - (K^2+1)\varphi_\theta + 1 + K^2 r^2 \varphi_r^2 \le 0, \tag{2}$$

If $\varphi_r \ne 0$, by the inequality (2), we have

$$K^2\varphi_\theta^2 - (K^2+1)\varphi_\theta + 1 \le -K^2 r^2 \varphi_r^2 < 0,$$

that is

$$(K^2\varphi_\theta - 1)(\varphi_\theta - 1) < 0,$$

so we obtain that

$$\frac{1}{K^2} < \varphi_\theta < 1.$$

For any $r \in (0,1)$, we have

$$2\pi = \varphi(r,2\pi) - \varphi(r,0) = \int_0^{2\pi} \varphi_\theta(r,\theta)\,d\theta < \int_0^{2\pi} d\theta = 2\pi.$$

This is impossible. So we have proved that $\varphi_r = 0$. Therefore, we may assume that

$$f(z) = r^{\frac{1}{K}} e^{i\psi(\theta)}.$$

At last, we will prove that $\psi(\theta) = \theta + \alpha$, where α is a constant. For any $\theta_1, \theta_2, 0 \le \theta_1 < \theta_2 \le 2\pi$, let

$$A_1 = A(r_1, 1; \theta_1, \theta_2), \quad A_2 = A(r_1, 1; \theta_2, \theta_1 + 2\pi),$$

then

$$f(A_1) = A(r_1^{\frac{1}{K}}, 1; \psi(\theta_1), \psi(\theta_2)),$$
$$f(A_2) = A(r_1^{\frac{1}{K}}, 1; \psi(\theta_2), \psi(\theta_1) + 2\pi).$$

By the quasi-invariant property of module, we have

$$\begin{cases} \frac{1}{K}\mathrm{mod}(A_1) \le \mathrm{mod}(f(A_1)), \\ \frac{1}{K}\mathrm{mod}(A_2) \le \mathrm{mod}(f(A_2)), \end{cases}$$

that is

$$\begin{cases} \psi(\theta_2) - \psi(\theta_1) \le \theta_2 - \theta_1, \\ \psi(\theta_2) - \psi(\theta_1) \ge \theta_2 - \theta_1, \end{cases}$$

therefore,

$$\psi(\theta_2) - \psi(\theta_1) = \theta_2 - \theta_1.$$

For any $\theta, 0 \le \theta \le 2\pi$, since

$$\psi'(\theta) = \lim_{h\to 0} \frac{\psi(\theta+h) - \psi(\theta)}{h} = \lim_{h\to 0} \frac{(\theta+h) - \theta}{h} = 1,$$

then

$$\psi(\theta) = \theta + \alpha, \alpha \text{ is a constant,}$$

thus, we obtain that

$$f(z) = r^{\frac{1}{K}} e^{i(\theta+\alpha)}.$$

The theorem is proved.

By Theorem 2, the following result can be proved.

Corollary 1. *Let Q be $K-$quasiconformal mappings from $\{z | r_1 \le |z| \le 1\}$ to $\{z || r_1^K \le |z| \le 1\}$, and if $f \in Q$ satisfies*

$$ess \sup_z D_f(z) = \inf_{g\in Q} \sup_z D_g(z),$$

then

$$f(z) = r^K e^{i(\theta+\alpha)}, \quad z = re^{i\theta},$$

where α is a constant.

On the other hand, [3] also considered the Grötzsch extremal problem on annulus under some restriction of area distortion, and proved the following result.

Theorem 3. *Let* $f : \Delta = \{z||z| < 1\} \to \Delta$ *be a* $K-$*quasiconformal mapping and* $f(\Delta) = \Delta, f(0) = 0$. *For any* $\Delta_r = \{z||z| < r\} \subset \Delta$, *if*

$$\frac{Area(f(\Delta_r))}{Area(\Delta_r)^{\frac{1}{K}}} \geq \pi^{1-\frac{1}{K}},$$

then, $f(z) = \lambda z|z|^{\frac{1}{K}-1}$ *for any* $z \in R_r = \{z|r < |z| < 1\}$, *where* λ *is a constant with* $|\lambda| = 1$.

We will point out that Theorem 3 is not correct. For example, if we again take $g(z) = r^{\frac{1}{K}} e^{i(\theta + k\sin\theta)}, z = re^{i\theta}$, it is easy to see that $\frac{\text{Area}(g(\Delta_r))}{\text{Area}(\Delta_r)^{\frac{1}{K}}} = \pi^{1-\frac{1}{K}}$, but $g(z)$ is not the required form, thus Theorem 3 is fault.

Next, by considering the relationship between the hyperbolic area of ring domain and its module, we can obtain one criterion for extremal mapping under the hyperbolic area distortion condition.

Suppose that $E \subset \Delta$ is a measurable subset, let

$$|E|_{hyp} = \iint_E \frac{1}{(1-|z|^2)^2} |dz|^2,$$

be the hyperbolic area of E, and Area(E) be its Euclidean area. We need the following result made in [5].

Lemma 2. *Let* R *be a ring domain bounded by two mutually disjoint closed curves* B_0 *and* B_1. *For any arbitrary concentric ring* R^* *bounded by concentric circles* B_0^* *and* B_1^*, *if* $Area(R) = Area(R^*)$, *then* $mod(R) \leq mod(R^*)$.

Lemma 3. *If* $f(x)$ *is a positive continuous function in* $[0, 2\pi]$, *then*

$$\int_0^{2\pi} \frac{1}{f(x)} dx \geq \frac{4\pi^2}{\int_0^{2\pi} f(x) dx},$$

with equality if and only if $f(x) \equiv c > 0$.

Proof. If $f(x) \in C[0,2\pi]$ and $f(x) > 0$, we have

$$[\int_0^{2\pi} \sqrt{f(x)} \cdot \frac{1}{\sqrt{f(x)}} dx]^2 \leq \int_0^{2\pi} f(x) dx \cdot \int_0^{2\pi} \frac{1}{f(x)} dx,$$

thus

$$4\pi^2 \leq \int\limits_0^{2\pi} f(x)dx \cdot \int\limits_0^{2\pi} \frac{1}{f(x)}dx,$$

that is,

$$\int_0^{2\pi} \frac{1}{f(x)} dx \geq \frac{4\pi^2}{\int_0^{2\pi} f(x) dx}.$$

Obviously, with equality if and only if $f(x) \equiv c > 0$. The proof of Lemma 3 is finished.

Next we will prove the following

Theorem 4. *Suppose that $S_1 = A(r_1, r_2), 0 < r_1 < r_2 < 1$, and $S_2 \subset \Delta$ is a ring domain bounded by two mutually disjoint closed curves C_1 and C_2, where $C_1 = \{z||z| = r_1\}$ and $C_2 = \{z|z = f(\theta)e^{i\theta}\}$, where $f(\theta)$ is a continuous function in $[0,2\pi]$ and $f(0) = f(2\pi)$, are the inner and outer boundaries of S_2. If*

$$|S_1|_{hyp} \geq |S_2|_{hyp},$$

then

$$mod(S_1) \geq mod(S_2).$$

With equality if and only if $S_2 = S_1 = A(r_1, r_2)$.

Proof. Since

$$\begin{aligned} |S_1|_{hyp} &= \iint_{S_1} \frac{1}{(1-|z|^2)^2} |dz|^2 = \int_0^{2\pi} \int_{r_1}^{r_2} \frac{r}{(1-r^2)^2} dr d\theta \\ &= \pi(\frac{1}{1-r_2^2} - \frac{1}{1-r_1^2}), \end{aligned}$$

$$\begin{aligned} |S_2|_{hyp} &= \iint_{S_2} \frac{1}{(1-|z|^2)^2}|dz|^2 = \int_0^{2\pi}\int_{r_1}^{f(\theta)} \frac{r}{(1-r^2)^2}\,dr d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\left(\frac{1}{1-f^2(\theta)} - \frac{1}{1-r_1^2}\right)d\theta, \end{aligned}$$

by the condition $|S_1|_{hyp} \geq |S_2|_{hyp}$, we have

$$\frac{\pi}{1-r_2^2} \geq \frac{1}{2}\int_0^{2\pi}\frac{1}{1-f^2(\theta)}\,d\theta.$$

By Lemma 3, we obtain

$$\pi\frac{1}{1-r_2^2} \geq \frac{1}{2}\int_0^{2\pi}\frac{1}{1-f^2(\theta)}\,d\theta \geq \frac{1}{2}\frac{4\pi^2}{\int_0^{2\pi} 1-f^2(\theta)\,d\theta},$$

thus,

$$\frac{1}{2\pi}\frac{1}{1-r_2^2} \geq \frac{1}{\int_0^{2\pi} 1-f^2(\theta)\,d\theta},$$

by calculation, we get

$$\mathrm{Area}(S_1 \cup \Delta_{r_1}) \geq \frac{1}{2}\int_0^{2\pi} f(\theta)^2\,d\theta = \mathrm{Area}(S_2 \cup \Delta_{r_1}),$$

thus

$$\mathrm{Area}(S_1) \geq \mathrm{Area}(S_2),$$

owing to Lemma 2, we obtain

$$\mathrm{mod}(S_1) \geq \mathrm{mod}(S_2).$$

Next, we will discuss the equality.

If $\mathrm{mod}(S_1) = \mathrm{mod}(S_2)$, we conclude that $\mathrm{Area}(S_1) = \mathrm{Area}(S_2)$. Otherwise, if $\mathrm{Area}(S_1) > \mathrm{Area}(S_2)$, then there exists an $A(r_1, r') \subset S_1 = A(r_1, r_2), r' < r_2$, such that $\mathrm{Area}(A(r_1, r')) = \mathrm{Area}(S_2)$. By Lemma 2, we get

$$\mathrm{mod}(S_1) > \mathrm{mod}(A(r_1, r')) \geq \mathrm{mod}(S_2),$$

this is a contradiction with $mod(S_1) = mod(S_2)$. Therefore, if $\mathrm{mod}(S_1) = \mathrm{mod}(S_2)$, by Lemma 3 and the proof above, we have

$$\begin{cases} |S_1|_{hyp} = |S_2|_{hyp}, \\ f(\theta) \equiv c = r_2. \end{cases} \quad \Leftrightarrow S_1 = S_2.$$

The proof is complete.

We have pointed out that the area distortion used in [3, Theorem 2.1] could not characterize extremal quasiconformal mapping. We will prove the following

Theorem 5. *Let $w = f(z)$ be a quasiconformal mapping from $S = \{z|R_1 \le |z| \le R_2, 0 < R_1 < R_2 < 1\}$ onto S', where $S' \subset D = \{w||w| < 1\}$ is a ring domain bounded by two mutually disjoint closed curves $\Gamma_1 = \{w||w| = R_1^{\frac{1}{K}}\}$ and $\Gamma_2 = \{w|w = f(R_2 e^{i\theta}), 0 \le \theta \le 2\pi\}$. If $f(z)$ satisfies*

$$\frac{|f(S)|_{hyp}}{|S|_{hyp}} = \frac{(R_2^{\frac{2}{K}} - R_1^{\frac{2}{K}})(1 - R_2^2)(1 - R_1^2)}{(R_2^2 - R_1^2)(1 - R_2^{\frac{2}{K}})(1 - R_1^{\frac{2}{K}})},$$

and $K[f] = K$, then $f(z)$ is an extremal quasiconformal mapping from S onto S', and

$$f(z) = r^{\frac{1}{K}} e^{i(\theta+\alpha)}, \quad z = re^{i\theta},$$

where α is a constant.

Proof. Let $S'' = A(R_1^{\frac{1}{K}}, R_2^{\frac{1}{K}})$, by calculation, we have

$$|S|_{hyp} = \pi \frac{R_2^2 - R_1^2}{(1 - R_2^2)(1 - R_1^2)}, \quad |S''|_{hyp} = \pi \frac{R_2^{\frac{2}{K}} - R_1^{\frac{2}{K}}}{(1 - R_2^{\frac{2}{K}})(1 - R_1^{\frac{2}{K}})},$$

by the hypothesis that

$$\frac{|f(S)|_{hyp}}{|S|_{hyp}} = \frac{(R_2^{\frac{2}{K}} - R_1^{\frac{2}{K}})(1 - R_2^2)(1 - R_1^2)}{(R_2^2 - R_1^2)(1 - R_2^{\frac{2}{K}})(1 - R_1^{\frac{2}{K}})},$$

we have

$$\begin{aligned}
|f(S)|_{hyp} &= |S|_{hyp}\frac{(R_2^{\frac{2}{K}}-R_1^{\frac{2}{K}})(1-R_2^2)(1-R_1^2)}{(R_2^2-R_1^2)(1-R_2^{\frac{2}{K}})(1-R_1^{\frac{2}{K}})}\\
&= \pi\frac{R_2^2-R_1^2}{(1-R_2^2)(1-R_1^2)}\cdot\frac{(R_2^{\frac{2}{K}}-R_1^{\frac{2}{K}})(1-R_2^2)(1-R_1^2)}{(R_2^2-R_1^2)(1-R_2^{\frac{2}{K}})(1-R_1^{\frac{2}{K}})}\\
&= \pi\frac{R_2^{\frac{2}{K}}-R_1^{\frac{2}{K}}}{(1-R_2^{\frac{2}{K}})(1-R_1^{\frac{2}{K}})}\\
&= |S''|_{hyp}.
\end{aligned}$$

Using Theorem 4, we get

$$\mathrm{mod}(S'')\geq \mathrm{mod}(f(S)),$$

and by the quasiconformallity of $f(z)$, we have

$$\mathrm{mod}(f(S))\geq \frac{1}{K}\mathrm{mod}(S),$$

thus, $\mathrm{mod}(f(S))=\frac{1}{K}\mathrm{mod}(S)=\mathrm{mod}(S'')$, again by Theorem 4, we obtain

$$f(S)=S''=A(R_1^{\frac{1}{K}},R_2^{\frac{1}{K}}).$$

Therefore, as it is proved in Theorem 2, we have

$$f(z)=r^{\frac{1}{K}}e^{i(\theta+\alpha)}, z=re^{i\theta},$$

where α is a constant.

Using Theorem 5, we have the following

Corollary 2. *Let $w=f(z)$ be a quasiconformal mapping from S onto $S'=\{w|R_1^K\leq |w|\leq R_2^K, 0<R_1<R_2<1\}$, where $S\subset D=\{z||z|<1\}$ is a ring domain bounded by two mutually disjoint closed curves $\Gamma_1=\{z||z|=R_1\}$ and $\Gamma_2=\{z||f(z)|=R_2^K\}$. If $f(z)$ satisfies*

$$\frac{|S|_{hyp}}{|f(S)|_{hyp}}=\frac{(R_2^2-R_1^2)(1-R_2^{2K})(1-R_1^{2K})}{(R_2^{2K}-R_1^{2K})(1-R_2^2)(1-R_1^2)},$$

and $K[f]=K$, then $f(z)$ is an extremal quasiconformal mapping from S onto S', and

$$f(z)=r^K e^{i(\theta+\alpha)},\ z=re^{i\theta},$$

where α is a constant.

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