EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 2, 2023, 806-818
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

New Bounds For The Eigenvalues Of Matrix Polynomials

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Abstract. We employ several numerical radius inequalities to the square of the Frobenius companion matrices of monic matrix polynomials to provide new bounds for the eigenvalues of these polynomials.

2020 Mathematics Subject Classifications: 26A33, 41A58

Key Words and Phrases: Bounds for the zeroes of polynomials, companion matrix, spectral radius, numerical radius

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1. Introduction

Let $M_{n\times m}(\mathbb{C})$ stands for the algebra of all $n \times m$ complex matrices where, $n, m \in \mathbb{N}$. For $n = m$, we may use the symbol $M_n(\mathbb{C})$. For $A \in M_n(\mathbb{C})$, let $\sigma(A)$, $r(A)$, $w(A)$ and $\|A\|$ denote the spectrum, the spectral radius, the numerical radius and the spectral norm of A, respectively. Recall that $\sigma(A) = \{\lambda \in \mathbb{C} : det(\lambda I - A) = 0\}$, $r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$, $w(A) = \max\limits_{\|x\|=1} |\langle Ax, x\rangle|$ and $\|A\| = \max\{\sqrt{\lambda} : \lambda \in \sigma(A^*A)\}$, where $A^* = [\bar{a}_{ji}]$ for $A = [a_{ij}]$, $a_{ij} \in \mathbb{C}$. The interesting inequality that combines these concepts is

$$|\lambda| \leq r(A) \leq w(A) \leq \|A\|,$$

for any $\lambda \in \sigma(A)$. The polynomial eigenvalue problem (PEP), finding and locating the eigenvalues of matrix polynomials, are very important topics in scientific computation that has attracted the attention of many researchers [2, 3, 5, 8, 9]. The PEP appears in a variety of problems in a wide range of applications. There are numerous examples of physical phenomena where PEPs arise naturally such as structural mechanics, control theory, fluid mechanics.

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DOI: https://doi.org/10.29020/nybg.ejpam.v16i2.4706

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Consider the monic polynomial $P(z) = Iz^m + A_m z^{m-1} + \cdots + A_2 z + A_1$, with degree $m \geq 2$ and matrix coefficients $A_i \in M_n(\mathbb{C})$ for $i = 1, \ldots, m$, where I is the identity matrix in $M_n(\mathbb{C})$.
The Frobenius companion block matrix of $P(z)$ is the $mn \times mn$ matrix given by

$$F(P) = \begin{bmatrix} -A_m & -A_{m-1} & \cdots & -A_2 & -A_1 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}.$$

This matrix builds an important bridge between matrix analysis and the geometry of polynomials.
It is known that λ is an eigenvalue of $P(z)$ iff $\lambda \in \sigma(F(P))$ and so if λ is an eigenvalue of $P(z)$, then

$$|\lambda| \leq r(F(P)) \leq w(F(P)) \leq \|F(P)\|.$$

In order to obtain new upper bounds for the eigenvalues of $P(z)$, we provide new estimates for the numerical radius of $F^2(P)$. The references [3, 5, 7, 8] contain bounds for the eigenvalues of matrix polynomials based on various matrix inequalities. In fact, Higham and Tisseur [5] obtained new bounds for the eigenvalues of matrix polynomials using norm and numerical radius inequalities. Le, Du, and Nguyen [3] established specific (upper and lower) bounds for the eigenvalues of matrix polynomials using the norms of the coefficients matrices of a matrix polynomial. Eigenvalue bounds can be created using l-ifications, or lower order matrix polynomials with the same eigenvalues as a given matrix polynomial, as demonstrated by Melman [8]. Jaradat and Kittaneh [7] , derived new numerical radius inequalities to the Frobenius companion block matrix of $P(z)$ and implemented it in obtaining a new upper bound for eigenvalues of $P(z)$.

2. Main Results

The square of the Frobenius companion block matrix of $P(z)$ can be written as

$$F^2(P) = \begin{bmatrix} B_m & B_{m-1} & \cdots & B_3 & B_2 & B_1 \\ -A_m & -A_{m-1} & \cdots & -A_3 & -A_2 & -A_1 \\ I & 0 & \ldots & 0 & 0 & 0 \\ 0 & I & 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{bmatrix},$$

where $B_j = A_m A_j - A_{j-1}, j = 1, \cdots, m$, with $A_0 = 0$.
To obtain our first new estimate for the numerical radius of $F^2(p)$, we need the following lemmas. The first two Lemmas can be found in [7].

Lemma 1. *Let $A = [A_{ij}] \in M_m(\mathbb{C})$ be the block matrix with $A_{ij} \in M_{ni\times nj}(\mathbb{C})$ and $\sum_{i=1}^{m} n_i = m$, where $1 \leq i, j \leq m$. Then $w(A) \leq w([\alpha_{ij}])$, where*

$$\alpha_{ij} = w\left(\begin{bmatrix} 0 & A_{ij} \\ A_{ji} & 0 \end{bmatrix}\right).$$

In particular, $\alpha_{ii} = w(A_{ii})$ for each $i = 1, 2, \ldots, m$.

Lemma 2. *Let $T_n \in M_n(\mathbb{C})$ be the tridiagonal matrix given by*

$$T_n = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \ldots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \ldots & 0 \\ 0 & \frac{1}{2} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{2} \\ 0 & 0 & \ldots & \frac{1}{2} & 0 \end{bmatrix}.$$

Then

$$w(T_n) = \cos\left(\frac{\pi}{n+1}\right).$$

The following Lemmas can be found in [4, 6] and [1], respectively.

Lemma 3. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the spectral radius of A is*

$$r(A) = \frac{1}{2}(a + d + \sqrt{(a-d)^2 + 4bc}).$$

Lemma 4. *Let $T = [T_{ij}] \in M_n(\mathbb{C})$ with $T_{km} \in M_{km}(\mathbb{C})$. Then*

$$w(T) \leq \frac{1}{2}\sum_{k=1}^{n}\left(w(T_{kk}) + \sqrt{w^2(T_{kk}) + \sum_{\substack{m=1 \\ k\neq m}}^{n} \|T_{km}\|^2}\right).$$

Lemma 5. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the spectral norm of A is*

$$\|A\| = \left[\frac{1}{2}\left(|a|^2 + |b|^2 + |c|^2 + |d|^2 + \sqrt{(|a|^2 + |c|^2 - |b|^2 - |d|^2)^2 + 4|a\bar{b} + c\bar{d}|^2}\right)^{\frac{1}{2}}\right].$$

Now, we introduce our first estimate for the numerical radius of $F^2(P)$.

Theorem 1. *An upper bound of the numerical radius of $F^2(P)$ can be stated as follows:*

$$\begin{aligned} w(F^2(P)) \leq & \frac{1}{4}\left(w(B_m)+w(A_{m-1})+\gamma\right)+\cos^2\left(\frac{\pi}{mn-1}\right)+\frac{1}{2} \\ & +\frac{1}{2}\sqrt{\frac{1}{4}\left(w\left(B_m\right)+w\left(A_{m-1}\right)+\gamma\right)^2+\frac{1}{2}\left(\alpha+\eta+\sqrt{(\alpha-\eta)^2+4\beta^2}\right)} \\ & +\frac{1}{2}\sqrt{\left(2\cos^2\left(\frac{\pi}{mn-1}\right)\ +1\right)^2+\frac{1}{2}\left(\alpha+\eta+\sqrt{(\alpha-\eta)^2+4\beta^2}\right)}, \end{aligned}$$

where

$$\alpha = w^2\left(T\left(I, B_{m-2}\right)\right)+\frac{1}{4}\sum_{i=1}^{m-3}\|B_i\|^2,$$

$$\eta = w^2(T(I, A_{m-3}))+\frac{1}{4}\|A_{m-2}\|^2+\frac{1}{4}\sum_{i=1}^{m-4}\|A_i\|^2,$$

$$\beta = w\left(T\left(B_{m-2}, I\right)\right)\frac{\|A_{m-2}\|}{2}+w\left(T\left(A_{m-3}, I\right)\right)\frac{\|B_{m-3}\|}{2}+\frac{1}{4}\sum_{i=1}^{m-4}\left(\|A_i\|\|B_i\|\right),$$

$$\gamma = \sqrt{\left(w\left(B_m\right)-w\left(A_{m-1}\right)\right)^2+4w^2\left(T\left(B_{m-1}, -A_m\right)\right)}.$$

Proof. For any two matrices, $C, D \in M_n(\mathbb{C})$, let $T(C,D) = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$. By applying Lemma 1 on $F^2(P)$, we have $w(F^2(P)) \leq w(R)$, where R is $mn \times mn$ matrix given by

$$\begin{bmatrix} w(B_m) & w(T(B_{m-1},-A_m)) & w(T(B_{m-2},I)) & w(T(B_{m-3},0)) & w(T(B_{m-4},0)) & \dots & w(T(B_3,0)) & w(T(B_2,0)) & w(T(B_1,0)) \\ w(T(-A_m,B_{m-1})) & w(A_{m-1}) & w(T(0,A_{m-2})) & w(T(A_{m-3},I)) & w(T(A_{m-4},0)) & \dots & w(T(-A_3,0)) & w(T(-A_2,0)) & w(T(-A_1,0)) \\ w(T(I,B_{m-2}) & w(T(0,A_{m-2})) & w(0) & w(T(0,0)) & w(T(0,I)) & \dots & w(T(0,0)) & w(T(0,0)) & w(T(0,0)) \\ w(T(0,B_{m-3})) & w(T(I,A_{m-3})) & w(T(0,0)) & w(0) & w(T(0,0)) & \dots & w(T(0,0)) & w(T(0,0)) & w(T(0,0)) \\ w(T(0,B_{m-4})) & w(T(0,A_{m-4})) & w(T(I,0)) & w(T(0,0)) & w(0) & \dots & w(T(0,0)) & w(T(0,0)) & w(T(0,0)) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w(T(0,B_3)) & w(T(0,-A_3)) & w(T(0,0)) & w(T(0,0)) & w(T(0,0)) & \dots & w(0) & w(T(0,0)) & w(T(0,I)) \\ w(T(0,B_2)) & w(T(0,-A_2)) & w(T(0,0)) & w(T(0,0)) & w(T(0,0)) & \dots & w(T(0,0)) & w(0) & w(T(0,0)) \\ w(T(0,B_1)) & w(T(0,-A_1)) & w(T(0,0)) & w(T(0,0)) & w(T(0,0)) & \dots & w(T(I,0)) & w(T(0,0)) & w(0) \end{bmatrix}$$

$$\begin{bmatrix} w(B_m) & w(T(B_{m-1},-A_m)) & w(T(B_{m-2},I)) & w(T(B_{m-3},0)) & w(T(B_{m-4},0)) & \dots & w(T(B_3,0)) & w(T(B_2,0)) & w(T(B_1,0)) \\ w(T(-A_m,B_{m-1})) & w(A_{m-1}) & w(T(0,A_{m-2})) & w(T(A_{m-3},I)) & w(T(A_{m-4},0)) & \dots & w(T(-A_3,0)) & w(T(-A_2,0)) & w(T(-A_1,0)) \\ w(T(I,B_{m-2}) & w(T(0,A_{m-2})) & w(0) & w(T(0,0)) & w(T(0,I)) & \dots & w(T(0,0)) & w(T(0,0)) & w(T(0,0)) \\ w(T(0,B_{m-3})) & w(T(I,A_{m-3})) & w(T(0,0)) & w(0) & w(T(0,0)) & \dots & w(T(0,0)) & w(T(0,0)) & w(T(0,0)) \\ w(T(0,B_{m-4})) & w(T(0,A_{m-4})) & w(T(I,0)) & w(T(0,0)) & w(0) & \dots & w(T(0,0)) & w(T(0,0)) & w(T(0,0)) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w(T(0,B_3)) & w(T(0,-A_3)) & w(T(0,0)) & w(T(0,0)) & w(T(0,0)) & \dots & w(0) & w(T(0,0)) & w(T(0,I)) \\ w(T(0,B_2)) & w(T(0,-A_2)) & w(T(0,0)) & w(T(0,0)) & w(T(0,0)) & \dots & w(T(0,0)) & w(0) & w(T(0,0)) \\ w(T(0,B_1)) & w(T(0,-A_1)) & w(T(0,0)) & w(T(0,0)) & w(T(0,0)) & \dots & w(T(I,0)) & w(T(0,0)) & w(0) \end{bmatrix}$$

Using the fact that $w\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}\right) = \frac{\|A\|}{2}$, for every matrix $A \in M_n(\mathbb{C})$, then R is equal to

$$\begin{bmatrix} w(B_m) & w(T(B_{m-1},-A_m)) & w(T(B_{m-2},I)) & \frac{\|B_{m-3}\|}{2} & \frac{\|B_{m-4}\|}{2} & \dots & \frac{\|B_3\|}{2} & \frac{\|B_2\|}{2} & \frac{\|B_1\|}{2} \\ w(T(-A_m,B_{m-1})) & w(A_{m-1}) & \frac{\|A_{m-2}\|}{2} & w(T(A_{m-3},I)) & \frac{\|A_{m-4}\|}{2} & \dots & \frac{\|A_3\|}{2} & \frac{\|A_2\|}{2} & \frac{\|A_1\|}{2} \\ w(T(B_{m-2},I) & \frac{\|A_{m-2}\|}{2} & 0 & 0 & \frac{1}{2} & \dots & 0 & 0 & 0 \\ \frac{\|B_{m-3}\|}{2} & w(T(A_{m-3},I)) & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{\|B_{m-4}\|}{2} & \frac{\|A_{m-4}\|}{2} & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{\|B_3\|}{2} & \frac{\|A_3\|}{2} & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 & \frac{1}{2} \\ \frac{\|B_2\|}{2} & \frac{\|A_2\|}{2} & 0 & 0 & \frac{1}{2} & \dots & 0 & 0 & 0 \\ \frac{\|B_1\|}{2} & \frac{\|A_1\|}{2} & 0 & 0 & 0 & \dots & \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

To find $w(R)$ we need to partition the new form of R as

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix},$$

where

$$R_{11} = \begin{bmatrix} w(B_m) & w\left(T\left(B_{m-1},-A_m\right)\right) \\ w\left(T\left(-A_m,B_{m-1}\right)\right) & w(A_{m-1}) \end{bmatrix},$$

$$R_{12} = \begin{bmatrix} w(T(B_{m-2},I)) & \frac{\|B_{m-3}\|}{2} & \frac{\|B_{m-4}\|}{2} & \dots & \frac{\|B_3\|}{2} & \frac{\|B_2\|}{2} & \frac{\|B_1\|}{2} \\ \frac{\|A_{m-2}\|}{2} & w(T(A_{m-3},I)) & \frac{\|A_{m-4}\|}{2} & \dots & \frac{\|A_3\|}{2} & \frac{\|A_2\|}{2} & \frac{\|A_1\|}{2} \end{bmatrix},$$

$$R_{21} = \begin{bmatrix} w(T(B_{m-2},I)) & \frac{\|B_{m-3}\|}{2} & \frac{\|B_{m-4}\|}{2} & \dots & \frac{\|B_3\|}{2} & \frac{\|B_2\|}{2} & \frac{\|B_1\|}{2} \\ \frac{\|A_{m-2}\|}{2} & w(T(A_{m-3},I)) & \frac{\|A_{m-4}\|}{2} & \dots & \frac{\|A_3\|}{2} & \frac{\|A_2\|}{2} & \frac{\|A_1\|}{2} \end{bmatrix}^T,$$

$$R_{22} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{2} & 0 & 0 & \dots & \frac{1}{2} & \dots & 0 \\ 0 & \frac{1}{2} & \ddots & \ddots\dots & \vdots & \vdots\ddots & \vdots \\ 0 & 0 & \frac{1}{2} & \ddots\dots & 0 & 0 & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots\dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & \frac{1}{2} & 0 & 0 \end{bmatrix}_{(mn-2)\times(mn-2)}.$$

To achieve our goal, we apply Lemma 4 on the matrix R. So, we need to estimate $w(R_{11})$,$w(R_{22})$, $\|R_{12}\|$ and $\|R_{21}\|$. A matrix $A \in M_n(\mathbb{C})$ is called Hermitian if $A = A^*$. Since R_{11} is Hermitian, Lemma 3 yields that

$$\begin{aligned} w(R_{11}) &= r(R_{11}) \\ &= \frac{1}{2}\left(w(B_m) + w(A_{m-1}) + \sqrt{\left(w(B_m) - w(A_{m-1})\right)^2 + 4w^2(T(B_{m-1},-A_m)}\right). \end{aligned}$$

The matrix R_{22} can be written as

$$R_{22} = 2\left(A^2 - diag\left(\frac{1}{4},\frac{1}{2},\dots,\frac{1}{2},\frac{1}{4}\right)\right),$$

where

$$A=\begin{bmatrix}0 & \frac{1}{2} & 0 & \dots & 0\\ \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0\\ 0 & \frac{1}{2} & 0 & \dots & 0\\ \vdots & 0 & \vdots & \ddots & \frac{1}{2}\\ 0 & 0 & \dots & \frac{1}{2} & 0\end{bmatrix}.$$

Thus

$$w\left(R_{22}\right)\leq 2\left(w\left(A^2\right)+w(diag\left(\frac{1}{4},\frac{1}{2},\dots,\frac{1}{2},\frac{1}{4}\right)\right).$$

Since A is normal and by using the fact $w\left(A^2\right)=w^2(A)$, we get

$$w\left(R_{22}\right)\leq 2w^2\left(A\right)+1.$$

Lemma 2 yields that

$$w\left(R_{22}\right)\leq 2\cos^2\left(\frac{\pi}{mn-1}\right)+1.$$

Now, to estimate the spectral norm of R_{12}, R_{21} consider

$$R_{21}^{*}R_{21}=\begin{bmatrix}\alpha & \beta\\ \beta & \eta\end{bmatrix},$$

where

$$\alpha=w^2\left(T\left(B_{m-2},I\right)\right)+\frac{1}{4}\sum_{i=1}^{m-3}\|B_i\|^2,$$

$$\beta=w\left(T\left(B_{m-2},I\right)\right)\frac{\|A_{m-2}\|}{2}+w\left(T\left(A_{m-3},I\right)\right)\frac{\|B_{m-3}\|}{2}+\frac{1}{4}\sum_{i=1}^{m-4}(\|A_i\|\|B_i\|),$$

$$\eta=w^2(T(A_{m-3},I))+\frac{1}{4}\|A_{m-2}\|^2+\frac{1}{4}\sum_{i=1}^{m-4}\|A_i\|^2.$$

So,

$$\|R_{21}\|^2=\|R_{21}R_{21}^{*}\|=r\left(R_{21}R_{21}^{*}\right)=\frac{1}{2}\left(\alpha+\eta+\sqrt{(\alpha-\eta)^2+4\beta^2}\right).$$

Since $R_{21}=R_{12}^{T}$, then $\|R_{12}\|=\|R_{21}\|$.
Now, by applying Lemma 4, we have

$$w(F^2(P))\leq w(R)\leq\frac{1}{2}\sum_{L=1}^{2}\left(w(R_{kk})+\sqrt{w^2(R_{kk})+\sum_{\substack{m=1\\ m\neq k}}^{2}\|R_{km}\|^2}\right).$$

This completes the proof.

From Theorem 1, we obtain the first new bound of the eigenvalues of $P(z)$. In fact, if λ is an eigenvalue of $P(z)$, then

$$|\lambda|^2 \leq r(F^2(P)) \leq w(F^2(P)).$$

On the other hand, we derive a new bound for the eigenvalues of matrix polynomials using a similar matrix to $F^2(P)$.
Consider the invertible matrix

$$B = \begin{bmatrix} I & I & I & \dots & I \\ 0 & I & I & \dots & I \\ 0 & 0 & I & \dots & I \\ \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

where

$$B^{-1} = \begin{bmatrix} I & -I & 0 & \dots & 0 \\ 0 & I & -I & \dots & 0 \\ 0 & 0 & I & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & -I \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}.$$

Consider the $mn \times mn$ matrix H, where $H = BF^2(P)B^{-1}$;

$$\text{H}=\begin{bmatrix} B_m - A_m + I & B_{m-1} - B_m + A_m - A_{m-1} & B_{m-2} - B_{m-1} + A_{m-1} - A_{m-2} & \dots & B_2 - B_3 + A_3 - A_2 & B_1 - B_2 + A_2 - A_1 \\ I - A_m & A_m - A_{m-1} & A_{m-1} - A_{m-2} & \dots & A_3 - A_2 - I & A_2 - A_1 \\ I & 0 & 0 & \dots & -I & 0 \\ 0 & I & 0 & \dots & -I & 0 \\ 0 & 0 & \ddots & \dots & -I & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -I & 0 \end{bmatrix}$$

Since H is similar to $F^2(P)$, if λ is an eigenvalue of P(z), we obtain

$$|\lambda|^2 \leq r(H) \leq w(H).$$

In the following, we provide an estimate of the numerical radius of H in order to extract a new upper bound for the eigenvalues of $P(z)$.

Theorem 2. *An upper bound of the numerical radius of H can be stated as follows:*

$$\begin{aligned} w(H) \leq \frac{1}{2}\Bigg(& \xi + 2\cos^2\left(\frac{\pi}{mn-3}\right) + 1 + \frac{2+\sqrt{5}}{4} + \sqrt{\xi^2 + \left(\frac{1}{2}\left(\alpha + \eta + \sqrt{(\alpha-\eta)^2 + 4\beta^2}\right)\right)^2 + \tau^2} \\ & + \sqrt{\left(2\cos^2\left(\frac{\pi}{mn-3}\right) + 1\right)^2 + \left(\frac{1}{2}\left(\alpha + \eta + \sqrt{(\alpha-\eta)^2 + 4\beta^2}\right)\right)^2 + \mu} \\ & + \sqrt{\left(\frac{2+\sqrt{5}}{4}\right)^2 + \tau^2 + \mu}\Bigg), \end{aligned}$$

where

$$\xi = \frac{1}{2}\left(w(B_m - A_{m+I}) + w(A_m - A_{m-1}) + \sqrt{\begin{array}{c}(w(B_m - A_{m+I}) - w(A_m - A_{m-1}))^2 \\ +4w^2(T(B_{m-1} - B_m + A_m - A_{m-1}, I - A_m))\end{array}} \right),$$

$$\alpha = w^2\left(T\left(B_{m-2} - B_{m-1} + A_{m-1} - A_{m-2}, I\right)\right) + \frac{1}{4}\sum_{j=6}^{m} \|B_{j-3} - B_{j-2} + A_{j-2} - A_{j-3}\|^2,$$

$$\eta = \frac{1}{4}\|A_{m-1} - A_{m-2}\|^2 + w^2(T(A_{m-2} - A_{m-3}, I) + \frac{1}{4}\sum_{i=7}^{m} \|A_{j-3} - A_{j-4}\|^2,$$

$$\begin{aligned}\beta =& \frac{1}{2} w\left(T\left(B_{m-2} - B_{m-1} + A_{m-1} - A_{m-2}, I\right)\right)\|A_{m-1} - A_{m-2}\| \\ &+ \frac{1}{2}\|B_{m-3} - B_{m-2} + A_{m-2} - A_{m-3}\| w(T\left(A_{m-2} - A_{m-3}, I\right)) \\ &+ \frac{1}{4}\sum_{j=7}^{m} \|B_{j-4} - B_{j-3} + A_{j-3} - A_{j-4}\|\|A_{j-3} - A_{j-4}\|,\end{aligned}$$

$$\begin{aligned}\tau =& \frac{1}{2}\left(\frac{\|B_2 - B_3 + A_3 - A_2\|^2}{4} + \frac{\|A_3 - A_2 - I\|^2}{4} - \frac{\|B_1 - B_2 + A_2 - A_1\|^2}{4} - \frac{\|A_2 - A_1\|^2}{4} \right) \\ & \sqrt{\begin{array}{c}\left(\frac{\|B_2 - B_3 + A_3 - A_2\|^2}{4} + \frac{\|A_3 - A_2 - I\|^2}{4} - \frac{\|B_1 - B_2 + A_2 - A_1\|^2}{4} - \frac{\|A_2 - A_1\|^2}{4}\right)^2 \\ + \frac{1}{4}\left(\|B_2 - B_3 + A_3 - A_2\|\|B_1 - B_2 + A_2 - A_1\| + \|A_3 - A_2 - I\|\|A_2 - A_1\|\right)^2\end{array}}.\end{aligned}$$

Proof. For any two matrices, $D \in M_n(\mathbb{C})$, let $T(C, D) = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$. Applying Lemma 1 for the matrix H, we have

$$w(H) \leq w(S)$$

where the block matrix S is given by

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix},$$

$$S_{11} = \begin{bmatrix} w(B_m - A_m + I) & w(T(B_{m-1} - B_m + A_m - A_{m-1}, I - A_m)) \\ w(T(I - A_m, B_{m-1} - B_m + A_m - A_{m-1})) & w(A_m - A_{m-1}) \end{bmatrix}_{2\times 2},$$

$$S_{12} = \left(\begin{bmatrix} w(T(I, B_{m-2} - B_{m-1} + A_{m-1} - A_{m-2})) & w(T(0, A_{m-1} - A_{m-2})) \\ w(T(0, B_{m-3} - B_{m-2} + A_{m-2} - A_{m-3})) & w(T(I, A_{m-2} - A_{m-3})) \\ \vdots & w(T(0, A_{m-3} - A_{m-4})) \\ \vdots & \vdots \\ w(T(0, B_3 - B_4 + A_4 - A_3)) & w(T(0, A_4 - A_3)) \end{bmatrix}^t \right)_{2\times(mn-4)},$$

$$S_{13}=\begin{bmatrix} w(T(B_2-B_3+A_3-A_2,0)) & w(T(B_1-B_2+A_2-A_1,0)) \\ w(T(A_3-A_2-I,0)) & w(T(A_2-A_1,0)) \end{bmatrix}_{2\times 2},$$

$$S_{21}=\begin{bmatrix} w(T(I,B_{m-2}-B_{m-1}+A_{m-1}-A_{m-2})) & w(T(0,A_{m-1}-A_{m-2})) \\ w(T(0,B_{m-3}-B_{m-2}+A_{m-2}-A_{m-3})) & w(T(I,A_{m-2}-A_{m-3})) \\ \vdots & w(T(0,A_{m-3}-A_{m-4})) \\ \vdots & \vdots \\ w(T(0,B_3-B_4+A_4-A_3)) & w(T(0,A_4-A_3)) \end{bmatrix}_{(mn-4)\times 2},$$

$$S_{22}=\begin{bmatrix} w(0) & w(T(0.0)) & w(T(0.I)) & w(T(0.0)) & \dots & w(T(0.0)) & w(T(0.0)) \\ w(T(0.0)) & w(0) & w(T(0.0)) & w(T(0.I)) & \ddots & w(T(0.0)) & w(T(0.0)) \\ w(T(I.0)) & w(T(0.0)) & w(0) & w(T(0.0)) & \ddots & w(T(0.0)) & w(T(0.0)) \\ w(T(0.0)) & w(T(I.0)) & w(T(0.0)) & w(0) & \ddots & w(T(0.I)) & w(T(0.0)) \\ w(T(0.0)) & w(T(0.0)) & w(T(I.0)) & w(T(0.0)) & \ddots & w(T(0.0)) & w(T(0.I)) \\ \vdots & \vdots & \vdots & \ddots & \ddots & w(0) & w(T(0.0)) \\ w(T(0.0)) & w(T(0.0)) & \cdots & w(T(0.0)) & w(T(I.0)) & w(T(0.0)) & w(0) \end{bmatrix}_{(mn-4)\times(mn-4)},$$

$$S_{23}=\begin{bmatrix} w(T(-I,0)) & w(T(0,0)) \\ w(T(-I,0)) & w(T(0,0)) \\ \vdots & \vdots \\ w(T(-I,0)) & w(T(0,0)) \end{bmatrix}_{(mn-4)\times 2},$$

$$S_{31}=\begin{bmatrix} w\left(T\left(0,B_2-B_3+A_3-A_2\right)\right) & w(T\left(0,A_3-A_2-I\right)) \\ w\left(T\left(0,B_1-B_2+A_2-A_1\right)\right) & w(T\left(0,A_2-A_1\right)) \end{bmatrix}_{2\times 2},$$

$$S_{32}=\begin{bmatrix} w(T(0,-I)) & w(T(0,-I)) & \cdots & w(T(I,-I)) & w(T(0,-I)) \\ w(T(0,0)) & w(T(0,0)) & \cdots & w(T(0,0)) & w(T(I,0)) \end{bmatrix}_{2\times(mn-4)},$$

and

$$S_{33}=\begin{bmatrix} w\left(T\left(-I,0\right)\right) & w\left(T\left(0,-I\right)\right) \\ w\left(T\left(-I,0\right)\right) & w\left(0\right) \end{bmatrix}_{2\times 2}.$$

We achieve our goal by applying Lemma 4 on the matrix S. So, we need to estimate $w(S_{11})$, $w(S_{22})$, $w(S_{33})$, $\|S_{12}\|$, $\|S_{13}\|$, $\|S_{21}\|\|S_{31}\|$, $\|S_{23}\|$ and $\|S_{32}\|$. Since S_{11} is Hermitian, applying Lemma 3 to get $w(S_{11})=r(S_{11})=\xi$, where

$$\begin{aligned}\xi=&\frac{1}{2}\left(w(B_m-A_m+I)+w(A_m-A_{m-1})\right)\\ &+\frac{1}{2}\sqrt{\left(w\left(B_m-A_m+I\right)-w\left(A_m-A_{m-1}\right)\right)^2+4w^2\left(T\left(B_{m-1}-B_m+A_m-A_{m-1},I-A_m\right)\right)}.\end{aligned}$$

Using the fact that $w\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}\right) = \frac{\|A\|}{2}$ for every matrix $A \in M_n(\mathbb{C})$, the matrix S_{22} can be written as

$$S_{22} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \ddots & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \ddots & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \ddots & 0 & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

and

$$S_{33} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}.$$

So,

$$S_{22} = 2\left(A^2 - diag\left(\frac{1}{4}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{4}\right)\right)$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \dots & 0 \\ \vdots & 0 & \vdots & \ddots & \frac{1}{2} \\ 0 & 0 & \dots & \frac{1}{2} & 0 \end{bmatrix}.$$

Now,

$$w(S_{22}) \leq 2\left(w(A^2) + w(diag\ \left(\frac{1}{4}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{4}\right)\right).$$

Given that A is normal, thus $w(A^2) = w^2(A)$. Then applying Lemma 2, to get

$$w\left(S_{22}\right) \leq 2\cos^2\left(\frac{\pi}{mn-3}\right) + 1.$$

Also, we have

$$w(S_{33}) = r\left(S_{33}\right) = \frac{2+\sqrt{5}}{4}.$$

Since S_{12} can be written as

$$\mathrm{S}_{12} = \begin{bmatrix} w\left(T\left(B_{m-2} - B_{m-1} + A_{m-1} - A_{m-2}, I\right)\right) & \frac{\|B_{m-3}-B_{m-2}+A_{m-2}-A_{m-3}\|}{2} & \dots & \dots & \frac{\|B_3-B_4+A_4-A_3\|}{2} \\ \frac{\|A_{m-1}-A_{m-2}\|}{2} & w(T\left(A_{m-2} - A_{m-3}, I\right)) & \frac{\|A_{m-3}-A_{m-4}\|}{2} & \dots & \frac{\|A_4-A_3\|}{2} \end{bmatrix}.$$

and $S^*{}_{12} = S^t{}_{12}$, we have $S_{12}S_{12}^* = \begin{bmatrix} \alpha & \beta \\ \beta & \eta \end{bmatrix}$, where

$$\alpha = w^2\left(T\left(B_{m-2} - B_{m-1} + A_{m-1} - A_{m-2}, I\right)\right) + \frac{1}{4}\sum_{j=6}^{m}\|B_{j-3} - B_{j-2} + A_{j-2} - A_{j-3}\|^2,$$

$$\eta = \frac{1}{4}\|A_{m-1} - A_{m-2}\|^2 + w^2(T(A_{m-2} - A_{m-3}, I)) + \frac{1}{4}\sum_{j=7}^{m}\|A_{j-3} - A_{j-4}\|^2,$$

and

$$\begin{aligned}\beta =& \frac{1}{2}w\left(T\left(B_{m-2} - B_{m-1} + A_{m-1} - A_{m-2}, I\right)\right)\|A_{m-1} - A_{m-2}\| \\ &+ \frac{1}{2}\|B_{m-3} - B_{m-2} + A_{m-2} - A_{m-3}\| w(T(A_{m-2} - A_{m-3}, I)) \\ &+ \frac{1}{4}\sum_{j=7}^{m}\|B_{j-4} - B_{j-3} + A_{j-3} - A_{j-4}\|\|A_{j-3} - A_{j-4}\|.\end{aligned}$$

Applying Lemma 3 , to get

$$\|S_{12}\|^2 = \|S_{12}S^*{}_{12}\| = r\left(S_{12}S^*{}_{12}\right) = \frac{1}{2}\left(\alpha + \eta + \sqrt{(a-\eta)^2 + 4\beta^2}\right).$$

Since $S_{12} = S_{21}^t$, we have

$$\|s_{21}\|^2 = \frac{1}{2}\left(\alpha + \eta + \sqrt{(a-\eta)^2 + 4\beta^2}\right).$$

Now, we need to find the numerical radius of S_{13}, S_{31}.

$$S_{13} = \begin{bmatrix} \frac{\|B_2 - B_3 + A_3 - A_2\|}{2} & \frac{\|B_1 - B_2 + A_2 - A_1\|}{2} \\ \frac{\|A_3 - A_2 - I\|}{2} & \frac{\|A_2 - A_1\|}{2} \end{bmatrix},$$

$$S_{13} = \begin{bmatrix} \frac{\|B_2 - B_3 + A_3 - A_2\|}{2} & \frac{\|A_3 - A_2 - I\|}{2} \\ \frac{\|B_1 - B_2 + A_2 - A_1\|}{2} & \frac{\|A_2 - A_1\|}{2} \end{bmatrix},$$

By Lemma 1, we have $\tau = \|S_{31}\| = \|S_{13}\|$, where

$$\tau = \frac{1}{2}\left(\frac{\|B_2 - B_3 + A_3 - A_2\|^2}{4} + \frac{\|A_3 - A_2 - I\|^2}{4} - \frac{\|B_1 - B_2 + A_2 - A_1\|^2}{4} - \frac{\|A_2 - A_1\|^2}{4}\right)$$
$$\sqrt{\begin{array}{c}\left(\frac{\|B_2 - B_3 + A_3 - A_2\|^2}{4} + \frac{\|A_3 - A_2 - I\|^2}{4} - \frac{\|B_1 - B_2 + A_2 - A_1\|^2}{4} - \frac{\|A_2 - A_1\|^2}{4}\right)^2 \\ + \frac{1}{4}\left(\|B_2 - B_3 + A_3 - A_2\|\|B_1 - B_2 + A_2 - A_1\| + \|A_3 - A_2 - I\|\|A_2 - A_1\|\right)^2\end{array}}$$

To find the numerical radius of S_{32}, we have

$$\|S_{32}\|^2 = \|S_{32}S^*{}_{32}\| = \left\|\begin{matrix} \frac{mn}{4} - 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{matrix}\right\|.$$

So,

$$\mu = \|S_{23}\|^2 = \|S_{32}S^*{}_{32}\| = r(S_{32}S^*{}_{32}) = \frac{1}{2}\left(\frac{mn-3}{4} + \sqrt{\left(\frac{m-5}{4}\right)^2 + \frac{1}{4}}\right).$$

Finally, applying Lemma 4 to get

$$w\left(H\right) \leq w\left(S\right) \leq \frac{1}{2}\Bigg(w\left(S_{11}\right) + w\left(S_{22}\right) + w\left(S_{33}\right) + \sqrt{w^2(S_{11}) + \}|S12\|^2 + \|S_{13}\|^2}$$

$$+ \sqrt{w^2(S_{22}) + \}\|S21\|^2 + \|S_{23}\|^2} + + \sqrt{w^2(S_{33}) + \}\|S31\|^2 + \|S_{32}\|^2}$$

Thus,

$$w(H) \leq \frac{1}{2}\left(\xi + 2\cos^2\left(\frac{\pi}{mn-3}\right) + 1 + \sqrt{\xi^2 + \left(\frac{1}{2}\left(\alpha + \eta + \sqrt{(\alpha-\eta)^2 + 4\beta^2}\right)\right)^2 + \tau^2}\right.$$

$$+ \frac{2+\sqrt{5}}{4} + \sqrt{\left(2\cos^2\left(\frac{\pi}{mn-3}\right) + 1\right)^2 + \left(\frac{1}{2}\left(\alpha + \eta + \sqrt{(\alpha-\eta)^2 + 4\beta^2}\right)\right)^2 + \mu}$$

$$+ \sqrt{\left(\frac{2+\sqrt{5}}{4}\right)^2 + \tau^2 + \mu}.$$

Conclusion

We have established new effective bounds for the eigenvalues of matrix polynomials by employing the similarity of matrices and matrix inequalities including the numerical radius, spectral radius and matrix norms. It is worth noting that our results can be used in many applications in geometry and matrix analysis.

Acknowledgements

The authors express their gratitude to the dear referees, who wish to remain anonymous, and the editor for their helpful suggestions, which improved the final version of this paper.

Conflicts of Interest

The authors declare no conflict of interest.

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