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# Twice Differentiable Ostrowski Type Tensorial Norm Inequalities for Continuous Functions of Selfadjoint Operators in Hilbert Spaces

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**Abstract.** In this paper several tensorial norm inequalities for continuous functions of selfadjoint operators in Hilbert spaces have been obtained. Multiple inequalities are obtained with variations due to the convexity properties of the mapping $f$

$$\left\|(1\otimes B-A\otimes 1)^{-1}[\exp(1\otimes B)-\exp(A\otimes 1)]-\exp\left(\frac{A\otimes 1+1\otimes B}{2}\right)\right\|$$

$$\leqslant \|1\otimes B-A\otimes 1\|^2\,\frac{\|f''\|_{I,+\infty}}{24}.$$

**2020 Mathematics Subject Classifications**: 26D05, 26D07, 26D20

**Key Words and Phrases**: Tensorial product, Selfadjoint operators, Convex functions

## 1. Introduction and Preliminaries

The notion of a tensor has its origin in the 19th century, where it was formulated by Gibbs, though he didn't formally use the word tensor but a dyadic. In modern language, it can be seen as the origin of the tensor definition and its introduction to the mathematics. Interplay of inequalities in mathematics is vast, and as such it has applications in tensors as well. Mathematics and other scientific fields are highly influenced by inequalities. Many types of inequalities exist, but those involving Jensen, Ostrowski, Hermite–Hadamard, and Minkowski hold particular significance among them. More about inequalities and its history can be found in these books [21, 23]. Multiple papers have been published concerning the generalizations of the said inequalities, see the following and references therein for more information [1–5, 7–9, 25–29].

Since our paper is about tensorial Ostrowski type inequalities, we give the brief introduction to the topic. In 1938, A. Ostrowski [22] , proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)dt$ and the value $f(x)$, $x\in[a,b]$.

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**Theorem 1.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$ such that $f' : (a, b) \to \mathbb{R}$ is bounded on $(a, b)$ and $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right] \|f'\|_\infty (b-a),$$

*for all $x \in [a, b]$ and the constant $\frac{1}{4}$ is the best possible.*

If we take $x = \frac{a+b}{2}$ we get the midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{1}{4} \|f'\|_\infty (b-a),$$

with $\frac{1}{4}$ as best possible constant.
In order to derive similar inequalities of the tensorial type, we need the following introduction and preliminaries.
Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of $A_i$ is contained in $I_i$ for $i = 1, ..., k$. We say that such a k-tuple is in the domain of $f$. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of $A_i$ for $i = 1, ..., k$ by following [6] , we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_k) dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k)$$

as bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.
If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction extends the definition of Kornyi [20] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever $f$ can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

Since we will be using tensorial products, we will define in the following what tensors and tensorial products are in short, for more consult the following book [17].
Let U, V and W be vector spaces over the same field $F$. A mapping $\Phi : U \times V \to W$ is called a bilinear mapping if it is linear in each variable separately. Namely, for all $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$ and $a, b \in F$,
$\Phi(au_1 + bu_2, v) = a\Phi(u_1, v) + b\Phi(u_2, v)$,
$\Phi(u, av_1 + bv_2) = a\Phi(u, v_1) + b\Phi(u, v_2)$. If $W = F$, a bilinear mapping $\Phi : U \times V \to F$ is

called a bilinear function.
Let $\otimes : U \times V \to W$ be a bilinear mapping. The pair $(W, \otimes)$ is called a tensor product space of $U$ and $V$ if it satisfies the following conditions:
1. Generating property $< Im\otimes >= W$;
2. Maximal span property $dim < Im\otimes >= dimU \cdot dimV$.
The member $w \in W$ is called a tensor, but not all tensors in W are products of two vectors of the form $u \otimes v$.
The notation $< Im\otimes >$ denotes the span.
**Example**
Let $u = (x_1, .., x_m) \in \mathbb{R}^m$ and $v = (y_1, ..., y_n) \in \mathbb{R}^n$. We can view $u$ and $v$ as column vectors. Namely,

$$u = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, v = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

are $m \times 1$ and $n \times 1$ matrices respectively.
We define $\otimes : \mathbb{R}^m \times \mathbb{R}^n \to M_{m,n}$,

$$u \otimes v = uv^t = \begin{bmatrix} x_1y_1 \cdots x_1y_n \\ \vdots \\ x_my_1 \cdots x_my_n \end{bmatrix},$$

an $m \times n$ matrix with entries $A_{ij} = x_iy_j$. $(M_{m,n}, \otimes)$ is a tensor product space of $\mathbb{R}^m$ and $\mathbb{R}^n$. Tensors do not need to be matrices. This is just one model given. For more consult the following book [17].

Recall the following property of the tensorial product

$$(AC) \otimes (BD) = (A \otimes B)(C \otimes D)$$

that holds for any $A, B, C, D \in B(H)$.
From the property we can deduce easily the following consequences

$$A^n \otimes B^n = (A \otimes B)^n, n \geqslant 0,$$

$$(A \otimes 1)(1 \otimes B) = (1 \otimes B)(A \otimes 1) = A \otimes B,$$

which can be extended, for two natural numbers $m, n$ we have

$$(A \otimes 1)^n(1 \otimes B)^m = (1 \otimes B)^n(A \otimes 1)^m = A^n \otimes B^m.$$

The current research concerning tensorial inequalities can be seen in the following papers, [10–14, 16]. The following Lemma which we require can be found in a paper of Silvestru [15].

**Lemma 1.** *Assume $A$ and $B$ are selfadjoint operators with $Sp(A) \subset I, Sp(B) \subset J$ and having the spectral resolutions . Let $f; h$ be continuous on $I, g, k$ continuous on $J$ and $\phi$ and $\psi$ continuous on an interval $K$ that contains the sum of the intervals $f(I) + g(J); h(I) + k(J)$,then*

$$\phi(f(A) \otimes 1 + 1 \otimes g(B))\psi(h(A) \otimes 1 + 1 \otimes k(B))$$
$$= \int_I \int_J \phi(f(t) + g(s))\psi(h(t) + k(s))dE_t \otimes dF_s.$$

In the paper written by Ozdemir et al. [19] , the authors used the following Lemma. We will utilize it to produce results in the tensorial setting.

**Lemma 2.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0$ where $a, b \in I$ with $a < b$. If $f'' \in L[a, b]$ then the following equality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{16}\bigg[\int_0^1 l^2 f''\left(l\frac{a+b}{2} + (1-l)a\right) dl$$
$$+ \int_0^1 (l-1)^2 f''\left(lb + (1-l)\frac{a+b}{2}\right) dl\bigg].$$

In the following Theorem, we give a fundamental result which we will use in our paper to produce inequalities.

## 2. Main results

**Theorem 2.** *Assume that $f$ is continuously differentiable on $I$, $A$ and $B$ are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\int_0^1 f((1-\lambda)A \otimes 1 + \lambda 1 \otimes B)d\lambda - f\left(\frac{A \otimes 1 + 1 \otimes B}{2}\right)$$
$$= \frac{(1 \otimes B - A \otimes 1)^2}{16}\bigg[\int_0^1 l^2 f''\left(\left(1 - \frac{l}{2}\right) A \otimes 1 + \frac{l}{2} 1 \otimes B\right) dl$$
$$+ \int_0^1 (l-1)^2 f''\left(\left(\frac{1-l}{2}\right) A \otimes 1 + \left(\frac{1+l}{2}\right) 1 \otimes B\right) dl\bigg]$$

*Proof.* We start with Lemma 2. Introducing the substitution $x = \lambda b + (1 - \lambda)a$ on the left hand side integral. Then we assume that $A$ and $B$ have the spectral resolutions

$$A = \int_I t dE_t, B = \int_I s dF_s.$$

*If we take the integral $\int_I \int_I dE_t \otimes dF_s$, then we get*

$$\int_I \int_I \int_0^1 \left(f((1-\lambda)t + \lambda s)d\lambda - f\left(\frac{s+t}{2}\right)\right)dE_t \otimes dF_s$$

$$= \int_I \int_I \left( \frac{(s-t)^2}{16} \left[ \int_0^1 l^2 f'' \left( \left(1 - \frac{l}{2}\right) t + \frac{l}{2} s \right) dl \right.\right.$$

$$\left.\left. + \int_0^1 (l-1)^2 f'' \left( \left( \frac{1-l}{2} \right) a + \left( \frac{1+l}{2} \right) s \right) dl \right] \right) dE_t \otimes dF_s.$$

*By utilizing Fubini's Theorem for the left and right hand side with Lemma 1 for appropriate choices of the functions involved, we have successively*

$$\int_I \int_I \int_0^1 f((1-\lambda)t + \lambda s) d\lambda dE_t \otimes dF_s$$

$$= \int_0^1 \int_I \int_I f((1-\lambda)t + \lambda s) dE_t \otimes dF_s d\lambda$$

$$= \int_0^1 f((1-\lambda) A \otimes 1 + \lambda 1 \otimes B) d\lambda,$$

$$\int_I \int_I \left( \frac{(s-t)^2}{16} \int_0^1 l^2 f'' \left( \left(1 - \frac{l}{2}\right) t + \frac{l}{2} s \right) dl dE_t \otimes dF_s\right.$$

$$= \int_0^1 \int_I \int_I \frac{(s-t)^2}{16} l^2 f'' \left( \left(1 - \frac{l}{2}\right) t + \frac{l}{2} s \right) dE_t \otimes dF_s dl$$

$$= \int_0^1 \frac{(1 \otimes B - A \otimes 1)^2}{16} l^2 f'' \left( \left(1 - \frac{l}{2}\right) A \otimes 1 + \frac{l}{2} 1 \otimes B \right) dl,$$

$$\int_I \int_I \left( \frac{(s-t)^2}{16} \int_0^1 l^2 f'' \left( \left( \frac{1-l}{2} \right) t + \left( \frac{1+l}{2} \right) s \right) dl dE_t \otimes dF_s,\right.$$

$$= \int_0^1 \int_I \int_I \frac{(s-t)^2}{16} l^2 f'' \left( \left( \frac{1-l}{2} \right) t + \left( \frac{1+l}{2} \right) s \right) dE_t \otimes dF_s dl$$

$$= \int_0^1 \frac{(1 \otimes B - A \otimes 1)^2}{16} l^2 f'' \left( \left( \frac{1-l}{2} \right) A \otimes 1 + \left( \frac{1+l}{2} \right) 1 \otimes B \right) dl.$$

**Theorem 3.** *Assume that $f$ is continuously differentiable on $I$ with $\|f''\|_{I,+\infty} := \sup_{t \in I} |f''(t)| < +\infty$ and $A, B$ are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\left\| \int_0^1 f((1-\lambda) A \otimes 1 + \lambda 1 \otimes B) d\lambda - f\left( \frac{A \otimes 1 + 1 \otimes B}{2} \right) \right\|$$

$$\leqslant \|1 \otimes B - A \otimes 1\|^2 \frac{\|f''\|_{I,+\infty}}{24}.$$

*Proof.* *If we take the operator norm, we get*

$$\left\| \int_0^1 f((1-\lambda) A \otimes 1 + \lambda 1 \otimes B) d\lambda - f\left( \frac{A \otimes 1 + 1 \otimes B}{2} \right) \right\|$$

$$\leqslant \frac{\|1\otimes B - A\otimes 1\|^2}{16}\Bigg\| \int_0^1 l^2 f''\left(\left(1-\frac{l}{2}\right)A\otimes 1 + \frac{l}{2}1\otimes B\right)dl$$
$$+\int_0^1 (l-1)^2 f''\left(\left(\frac{1-l}{2}\right)A\otimes 1 + \left(\frac{1+l}{2}\right)1\otimes B\right)dl\Bigg\|.$$

*Using the triangle inequality and the properties of the integral and the norm, we get*

$$\frac{\|1\otimes B - A\otimes 1\|^2}{16}\Bigg\| \int_0^1 l^2 f''\left(\left(1-\frac{l}{2}\right)A\otimes 1 + \frac{l}{2}1\otimes B\right)dl$$
$$+\int_0^1 (l-1)^2 f''\left(\left(\frac{1-l}{2}\right)A\otimes 1 + \left(\frac{1+l}{2}\right)1\otimes B\right)dl\Bigg\|$$
$$\leqslant \frac{\|1\otimes B - A\otimes 1\|^2}{16}\left( \int_0^1 l^2 \left\| f''\left(\left(1-\frac{l}{2}\right)A\otimes 1 + \frac{l}{2}1\otimes B\right)\right\| dl\right.$$
$$\left.+\int_0^1 (l-1)^2 \left\| f''\left(\left(\frac{1-l}{2}\right)A\otimes 1 + \left(\frac{1+l}{2}\right)1\otimes B\right)\right\| dl\right).$$

*Observe that by Lemma 1,*

$$\left| f''\left(\left(1-\frac{l}{2}\right)\otimes 1 + \frac{l}{2}1\otimes B\right)\right| = \int_I\int_I \left| f''\left(\left(1-\frac{l}{2}\right)t + \frac{l}{2}s\right)\right| dE_t\otimes dF_s.$$

*Since*

$$\left| f''\left(\left(1-\frac{l}{2}\right)t + \frac{l}{2}s\right) \leqslant \|f''\|_{I,+\infty}\right.$$

*for all* $l\in[0,1]$ *and* $t,s\in I$. *If we take the integral* $\int_I\int_I$ *over* $dE_t\otimes dF_s$, *then we get*

$$\left| f''\left(\left(1-\frac{l}{2}\right)\otimes 1 + \frac{l}{2}1\otimes B\right)\right| = \int_I\int_I \left| f''\left(\left(1-\frac{l}{2}\right)t + \frac{l}{2}s\right)\right| dE_t\otimes dF_s$$
$$\leqslant \|f'\|_{I,+\infty}\int_I\int_I dE_t\otimes dF_s = \|f'\|_{I,+\infty}.$$

*This implies that*

$$\left| f''\left(\left(1-\frac{l}{2}\right)\otimes 1 + \frac{l}{2}1\otimes B\right)\right| \leqslant \|f''\|_{I,+\infty}$$

*for* $l\in[0,1]$, *similarly we have*

$$\left\| f''\left(\left(\frac{1-l}{2}\right)A\otimes 1 + \left(\frac{1+l}{2}\right)1\otimes B\right)\right\| \leqslant \|f''\|_{I,+\infty}.$$

*Which combined gives us the following*

$$\frac{\|1\otimes B - A\otimes 1\|^2}{16}\left( \int_0^1 l^2 \left\| f''\left(\left(1-\frac{l}{2}\right)A\otimes 1 + \frac{l}{2}1\otimes B\right)\right\| dl\right.$$

$$+\int_0^1 (l-1)^2 \left\| f''\left(\left(\frac{1-l}{2}\right) A\otimes 1 + \left(\frac{1+l}{2}\right) 1\otimes B\right)\right\| dl\right)$$
$$\leqslant \frac{\|1\otimes B - A\otimes 1\|^2}{16}\left(\int_0^1 l^2 \|f''\|_{I,+\infty}\, dl + \int_0^1 (l-1)^2 \|f''\|_{I,+\infty}\, dl\right).$$

*Solving the resulting integrals and simplifying, we obtain the desired result.*

**Theorem 4.** *Assume that $f$ is continuously differentiable on $I$ and $f''$ is convex and $A, B$ are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\left\| \int_0^1 f((1-\lambda)A\otimes 1 + \lambda 1\otimes B)d\lambda - f\left(\frac{A\otimes 1 + 1\otimes B}{2}\right)\right\|$$
$$\leqslant \frac{\|1\otimes B - A\otimes 1\|^2}{48}\left(\|f''(A)\| + \|f''(B)\|\right).$$

*Proof. Since $|f''|$ is convex on $I$, then we get*

$$\left| f''\left(\left(1-\frac{l}{2}\right)t + \frac{l}{2}s\right)\right| \leqslant \left(1-\frac{l}{2}\right)|f''(t)| + \frac{l}{2}|f''(s)|$$

*for all $l\in[0,1]$ and $t, s\in I$.*
*If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get*

$$\left| f''\left(\left(1-\frac{l}{2}\right)A\otimes 1 + \frac{l}{2} 1\otimes B\right)\right| = \int_I\int_I \left| f''\left(\left(1-\frac{l}{2}\right)t + \frac{l}{2}s\right)\right| dE_t\otimes dF_s$$
$$\leqslant \int_I\int_I \left[\left(1-\frac{l}{2}\right)|f''(t)| + \frac{l}{2}|f''(s)|\right] dE_t\otimes dF_s$$
$$= \left(1-\frac{l}{2}\right)|f''(A)|\otimes 1 + \frac{l}{2} 1\otimes |f''(B)|$$

*for all $l\in[0,1]$.*
*If we take the norm in the inequality, we get the following*

$$\left\| f''\left(\left(1-\frac{l}{2}\right)A\otimes 1 + \frac{l}{2}1\otimes B\right)\right\| \leqslant \left\|\left(1-\frac{l}{2}\right)|f''(A)|\otimes 1 + \frac{l}{2}1\otimes|f''(B)|\right\|$$
$$\leqslant \left(1-\frac{l}{2}\right)\||f''(A)|\otimes 1\| + \frac{l}{2}\|1\otimes|f''(B)|\|$$
$$= \left(1-\frac{l}{2}\right)\|f''(A)\| + \frac{l}{2}\|f''(B)\|.$$

*Similarly, we get*

$$\left\| f''\left(\left(\frac{1-l}{2}\right)A\otimes 1 + \left(\frac{1+l}{2}\right)1\otimes B\right)\right\| \leqslant \frac{1-l}{2}\|f''(A)\| + \frac{1+l}{2}\|f''(B)\|.$$

*Which when applied to the inequality obtained in the previous Theorem, we obtain the following*

$$\frac{\|1\otimes B-A\otimes 1\|^2}{16}\bigg(\int_0^1 l^2\left\|f''\left(\left(1-\frac{l}{2}\right)A\otimes 1+\frac{l}{2}1\otimes B\right)\right\|dl$$

$$+\int_0^1 (l-1)^2\left\|f''\left(\left(\frac{1-l}{2}\right)A\otimes 1+\left(\frac{1+l}{2}\right)1\otimes B\right)\right\|dl\bigg)$$

$$\leqslant \frac{\|1\otimes B-A\otimes 1\|^2}{16}\bigg(\int_0^1 l^2\left(\left(1-\frac{l}{2}\right)\|f''(A)\|+\frac{l}{2}\|f''(B)\|\right)dl$$

$$+\int_0^1 (l-1)^2\left(\frac{1-l}{2}\|f''(A)\|+\frac{1+l}{2}\|f''(B)\|\right)dl\bigg).$$

*Which when simplified after integrating the terms, we obtain the original inequality.*

We recall that the function $f : I \to \mathbb{R}$ is quasi-convex, if $f((1-\lambda)t+\lambda s) \leqslant \max(f(t), f(s)) = \frac{1}{2}(f(t) + f(s) + |f(s) - f(t)|)$ for all $t, s \in I$ and $\lambda \in [0, 1]$.

**Theorem 5.** *Assume that $f$ is continuously differentiable on $I$ with $|f''|$ is quasi-convex on $I$, $A$ and $B$ are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\left\|\int_0^1 f((1-\lambda)A\otimes 1+\lambda 1\otimes B)d\lambda - f\left(\frac{A\otimes 1+1\otimes B}{2}\right)\right\|$$

$$\leqslant \frac{\|1\otimes B-A\otimes 1\|^2}{48}(\||f''(A)|\otimes 1+1\otimes|f''(B)|\|+\||f''(A)|\otimes 1-1\otimes|f''(B)|\|).$$

*Proof. Since $|f''|$ is quasi-convex on $I$, then we get*

$$\left|f''\left(\left(1-\frac{l}{2}\right)t+\frac{l}{2}s\right)\right|\leqslant\frac{1}{2}(|f''(t)|+|f''(s)|+||f''(t)|-|f''(s)||)$$

*for all $l \in [0, 1]$ and $t, s \in I$. If we take the integral $\int_I\int_I$ over $dE_t \otimes dF_s$, then we get*

$$\left|f''\left(\left(1-\frac{l}{2}\right)A\otimes 1+\frac{l}{2}1\otimes B\right)\right|$$

$$\leqslant \int_I\int_I |f''\left(\left(1-\frac{l}{2}\right)t+\frac{l}{2}s\right)|dE_t\otimes dF_s$$

$$\leqslant \frac{1}{2}\int_I\int_I (|f''(t)|+|f''(s)|+||f''(t)|-|f''(s)||)dE_t\otimes dF_s$$

$$=\frac{1}{2}(|f''(A)|\otimes 1+1\otimes|f''(B)|+||f''(A)|\otimes 1-1\otimes|f''(B)||)$$

*for all* $l \in [0,1]$.
*If we take the norm, then we get*

$$\left\|f''\left(\left(1-\frac{l}{2}\right)A\otimes 1+\frac{l}{2}1\otimes B\right)\right\|$$

$$\leqslant \left\|\frac{1}{2}(|f''(A)|\otimes 1+1\otimes|f''(B)|+||f''(A)|\otimes 1-1\otimes|f''(B)||)\right\|$$

$$\leqslant \frac{1}{2}\left(\||f''(A)|\otimes 1+1\otimes|f''(B)|\|+\||f''(A)|\otimes 1-1\otimes|f''(B)|\|\right)$$

*for all* $l \in [0,1]$. *In a similar way, we obtain*

$$\left\|f''\left(\frac{1-l}{2}A\otimes 1+\frac{1+l}{2}1\otimes B\right)\right\|$$

$$\leqslant \left\|\frac{1}{2}(|f''(A)|\otimes 1+1\otimes|f''(B)|+||f''(A)|\otimes 1-1\otimes|f''(B)||)\right\|$$

$$\leqslant \frac{1}{2}\left(\||f''(A)|\otimes 1+1\otimes|f''(B)|\|+\||f''(A)|\otimes 1-1\otimes|f''(B)|\|\right)$$

*for all* $l \in [0,1]$.
*Using these inequalities in the inequality obtained during Theorem 4, we obtain the following*

$$\left(\int_0^1 l^2\left\|f''\left(\left(1-\frac{l}{2}\right)A\otimes 1+\frac{l}{2}1\otimes B\right)\right\|dl\right.$$
$$\left.+\int_0^1 (l-1)^2\left\|f''\left(\left(\frac{1-l}{2}\right)A\otimes 1+\left(\frac{1+l}{2}\right)1\otimes B\right)\right\|dl\right)$$

$$\leqslant \int_0^1 l^2\left(\frac{1}{2}\left(\||f''(A)|\otimes 1+1\otimes|f''(B)|\|+\||f''(A)|\otimes 1-1\otimes|f''(B)|\|\right)\right)dl$$
$$+\int_0^1 (l-1)^2\left(\frac{1}{2}\left(\||f''(A)|\otimes 1+1\otimes|f''(B)|\|+\||f''(A)|\otimes 1-1\otimes|f''(B)|\|\right)\right)dl.$$

*Which when simplified, we obtain the desired inequality.*

## 3. Some comments

It is known that if $U$ and $V$ are commuting, that is $UV = VU$, then the exponential function satisfies the property

$$\exp(U)\exp(V) = \exp(V)\exp(U) = \exp(U+V).$$

Also, if $U$ is invertible and $a, b \in \mathbb{R}$ and $a < b$ then

$$\int_a^b \exp(tU)dt = U^{-1}[\exp(bU) - \exp(aU)].$$

Moreover, if $U$ and $V$ are commuting and $V - U$ is invertible, then

$$\int_0^1 \exp((1-k)U + kV)dk = \int_0^1 \exp(k(V-U))\exp(U)dk$$
$$= (\exp(k(V-U))dk)\exp(U)$$
$$= (V-U)^{-1}[\exp(V-U) - I]\exp(U) = (V-U)^{-1}[\exp(V) - \exp(U)].$$

Since the operators $U = A \otimes 1$ and $V = 1 \otimes B$ are commutative and if $1 \otimes B - A \otimes 1$ is invertible, then

$$\int_0^1 \exp((1-k)A \otimes 1 + k1 \otimes B)dk$$
$$= (1 \otimes B - A \otimes 1)^{-1}[\exp(1 \otimes B) - \exp(A \otimes 1)].$$

**Corollary 1.** *If $A, B$ are selfadjoint operators with $Sp(A), Sp(B) \subset [m, M]$ and $1 \otimes B - A \otimes 1$ is invertible, then by Theorem 3, we get*

$$\left\| (1 \otimes B - A \otimes 1)^{-1}[\exp(1 \otimes B) - \exp(A \otimes 1)] - \exp\left(\frac{A \otimes 1 + 1 \otimes B}{2}\right) \right\|$$
$$\leqslant \|1 \otimes B - A \otimes 1\|^2 \frac{\exp(M)}{24}.$$

**Corollary 2.** *Since for $f(t) = \exp(t), t \in \mathbb{R}$, $|f''|$ is convex, then by Theorem 4*

$$\left\| (1 \otimes B - A \otimes 1)^{-1}[\exp(1 \otimes B) - \exp(A \otimes 1)] - \exp\left(\frac{A \otimes 1 + 1 \otimes B}{2}\right) \right\|$$
$$\leqslant \frac{\|1 \otimes B - A \otimes 1\|^2}{48} (\|\exp(A)\| + \|\exp(B)\|)$$

## 4. Conclusion

Tensors have become important in various fields, for example in physics because they provide a concise mathematical framework for formulating and solving physical problems in fields such as mechanics, electromagnetism, quantum mechanics, and many others. As such inequalities are crucial in numerical aspects. Reflected in this work is the tensorial Ozdemir's Lemma, which as a consequence enabled us to obtain Ostrowski type inequalities in Hilbert space. New Ostrowski type inequalities are given, examples of specific convex functions and their inequalities using our results are given in the section some examples. Plans for future research can be reflected in the fact that the obtained inequalities in this work can be sharpened or generalized by using other methods. An interesting perspective can be seen in incorporating other techniques for Hilbert space inequalities with the techniques shown in this paper. One direction is the technique of the Mond-Pecaric inequality, on which we will work on.

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