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Fekete-Szegö Functional of a Subclass of Bi-Univalent Functions Associated with Gegenbauer Polynomials

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Abstract. In this paper, we introduce and investigate a class of bi-univalent functions, denoted by $\mathcal{F}(n, \alpha, \beta)$, that depends on the Ruscheweyh operator and defined by the use of Gegenbauer Polynomials. For functions in this class, we derive the estimations for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Moreover, we obtain the classical Fekete-Szegö inequality of functions belonging to this class.

2020 Mathematics Subject Classifications: 30C45, 30C50, 33C45, 33C05, 11B39

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1. Introduction

Let $\mathcal{A}$ be the family of all analytic functions f that are defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Any function $f \in \mathcal{A}$ has the following Taylor-Maclarin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{where} \quad z \in \mathbb{D}. \tag{1}$$

Let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. Let the functions f and g be analytic in $\mathbb{D}$, we say the function f is subordinate by the function g in $\mathbb{D}$, denoted by $f(z) \prec g(z)$ for all $z \in \mathbb{D}$, if there exists a Schwartz function w, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent over $\mathbb{D}$ then $f(z) \prec g(z)$ equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D}$. For more information about the Subordination Principle we refer the readers to to the monographs [9], [23] and [24].

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As known univalent functions are injective (one-to-one) functions. Hence, they are invertible and the inverse functions may not be defined on the entire unit disk $\mathbb{D}$. In fact, the Koebe one-quarter Theorem tells us that the image of $\mathbb{D}$ under any function $f \in \mathcal{S}$ contains the disk $D(0, 1/4)$ of center 0 and radius 1/4. Accordingly, every function $f \in \mathcal{S}$ has an inverse $f^{-1} = g$ which is defined as

$$g(f(z)) = z, \quad z \in \mathbb{D}$$

$$f(g(w)) = w, \quad |w| < r(f); \quad r(f) \geq 1/4.$$

Moreover, the inverse function is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{2}$$

For this reason, we define the class Σ as follows. A function $f \in \mathcal{A}$ is said to be bi-univalent if both f and f^{-1} are univalent in $\mathbb{D}$. Therefore, let Σ denote the class of all bi-univalent functions in $\mathcal{A}$ which are given by equation (1). For example, the following functions belong to the class Σ:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \log\sqrt{\frac{1+z}{1-z}}.$$

However, Koebe function, $\dfrac{2z - z^2}{2}$ and $\dfrac{z}{1-z^2}$ do not belong to the class Σ. For more information about univalent and bi-univalent functions we refer the readers to the articles [19], [22], [25], the monograph [10], [12] and the references therein.

In the year 1784, Legendre [18] introduced and studied the orthogonal polynomials. Traditionally, orthogonal polynomials are crucial in approximation theory where are used in polynomial interpolation. Moreover, under specific restrictions, orthogonal polynomials are frequently used in the study of differential equations. In particular in some special cases of Sturm-Liouville differential equation. An example of orthogonal polynomials is a Gegenbauer polynomial. Special cases of Gegenbauer polynomials are Legendre polynomials and the Chebyshev polynomials of the first and second kind. For more information about orthogonal polynomials we refer the readers to the monograph [8]. We define Gegenbauer polynomials in the next section.

The subject of the geometric function theory in complex analysis has been investigated by many researchers in recent years, the typical problem in this field is studying a functional made up of combinations of the initial coefficients of the functions $f \in \mathcal{A}$. For a function in the class $\mathcal{S}$, it is well-known that $|a_n|$ is bounded by n. Moreover, the coefficient bounds give information about the geometric properties of those functions. For instance, the bound for the second coefficients of the class $\mathcal{S}$ gives the growth and distortion bounds for the class. In addition, the Fekete-Szegö functional arises naturally in the investigation of univalency of analytic functions. In the year 1933, Fekete and Szegö [11] found the maximum value of $|a_3 - \lambda a_2^2|$, as a function of the real parameter $0 \leq \lambda \leq 1$ for a univalent function f. Since then, the problem of dealing with the Fekete-Szegö functional for $f \in \mathcal{A}$ with any complex λ is known as the classical Fekete-Szegö problem. There are many researchers investigated the Fekete-Szegö functional and the other coefficient estimates problems, for example see the articles [20], [15], [21], [17], [11], [22], [14] and the references therein.

2. Preliminaries

In this section we present some information that are curial for the main results of this paper. we start by defining our subclass. In the year 1994, Szynal [28] introduced and studied a subclass $\mathfrak{F}(\alpha)$ of the class $\mathcal{A}$ consisting of functions of the form

$$f(z) = \int_{-1}^{1} K(z,x)\, d\sigma(x), \tag{3}$$

where $K(z,x) = \dfrac{z}{(z^2 - 2xz + 1)^{\alpha}}$, $\alpha \geq 0$, $-1 \leq x \leq 1$, and σ is the probability measure on $[-1,1]$. Moreover, the function $K(z,x)$ has the following Taylor-Maclaurin series expansion

$$K(z,x) = z + C_1^{\alpha}(x)z^2 + C_2^{\alpha}(x)z^3 + C_3^{\alpha}(x)z^4 + \cdots,$$

where $C_n^{\alpha}(x)$ denotes the Gegenbauer polynomials of order α and degree n in x. Furthermore, for any real numbers $\alpha, x \in \mathbb{R}$,with $\alpha \geq 0$ and $-1 \leq x \leq 1$, and $z \in \mathbb{D}$ the generating function of Gegenbauer polynomials is given by

$$H_{\alpha}(z,x) = (z^2 - 2xz + 1)^{-\alpha}.$$

Moreover, for any fixed x the function H_{α} is analytic on the unit disk $\mathbb{D}$ and its Taylor-Maclaurin series is given by

$$H_{\alpha}(z,x) = \sum_{n=0}^{\infty} C_n^{\alpha}(x) z^n.$$

In addition, if $f \in \mathfrak{F}(\alpha)$ that is given by (3), the n^{th} coefficient can be written as

$$a_n = \int_{-1}^{1} C_{n-1}^{\alpha}(x)\, d\sigma(x).$$

In addition, Gegenbauer polynomials can be defined in terms of the following recurrence relation:

$$C_n^{\alpha}(x) = \frac{2x(n+\alpha-1)C_{n-1}^{\alpha}(x) - (n+2\alpha-2)C_{n-1}^{\alpha}(x)}{n}, \tag{4}$$

with initial values

$$C_0^{\alpha}(x) = 1, \quad C_1^{\alpha}(x) = 2\alpha x, \quad \text{and} \quad C_2^{\alpha}(x) = 2\alpha(\alpha+1)x^2 - \alpha.$$

It is well-known that the Gegenbauer polynomials and their special cases such as Legendre polynomials $L_n(x)$ and the Chebyshev polynomials of the second kind $T_n(x)$, are orthogonal polynomials, where the values of α are $\alpha = 1/2$ and $\alpha = 1$ respectively, more precisely

$$L_n(x) = C_n^{1/2}(x), \quad \text{and} \quad T_n(x) = C_n^1(x).$$

For more information about the Gegenbauer polynomials and their special cases, we refer the readers to the articles [4], [3], [7], [6], [5], [21], [16], [22], [13], [14], the monograph [10], [12], [27], and the references therein.

In the year 1975, Ruscheweyh [26] introduced the operator $\mathcal{R}$ which defined, using the Hadamard product, as follows

$$\mathcal{R}^\lambda f(z) = f(z) * \frac{z}{(1-z)^{1-\lambda}},$$

where $f \in \mathcal{A}$, $z \in \mathbb{D}$ and real number $\lambda \geq -1$. For $\lambda = n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we get the Rscheweyh derivative $\mathcal{R}^n$ of order n of the function f:

$$\mathcal{R}^n f(z) = z \frac{\left(z^{n-1} f(z)\right)^{(n)}}{n!}.$$

Moreover, the Taylor-Maclaurin series of $\mathcal{R}^n f$ is given by

$$\mathcal{R}^n f(z) = z + \sum_{k=2}^{\infty} \sigma(n,k) a_k z^k,$$

$$\sigma(n,k) = \frac{\Gamma(n+k)}{(k-1)!\Gamma(n+1)}. \tag{5}$$

We say that a function $f \in \Sigma$ in the subclass $\mathcal{F}(n,\alpha,\beta)$ if it satisfies the following subordination conditions, associated with the Gegenbauer Polynomials, for all z, $w \in \mathbb{D}$:

$$(\mathcal{R}^n f(z))' + \beta z (\mathcal{R}^n f(z))'' \prec H_\alpha(z,x) \tag{6}$$

and

$$(\mathcal{R}^n g(w))' + \beta w (\mathcal{R}^n g(w))'' \prec H_\alpha(w,x), \tag{7}$$

where $\alpha > 0$, $\beta > 0$, $n \in \mathbb{N}_0$, $x \in (\frac{1}{2}, 1]$ and $g(w)$ is defined by equation (2).

The following lemma (see[17]) is a well-known fact, so we omit its proof.

Lemma 1. *Let K, $L \in \mathbb{R}$ and p, $q \in \mathbb{C}$. If $|p| < R$ and $|q| < R$,*

$$|(K+L)p + (K-L)q| \leq \begin{cases} 2|K|R, & \textit{if} \ \ |K| \geq |L| \\ 2|L|R, & \textit{if} \ \ |K| \leq |L| \end{cases}$$

Our investigation in this paper is motivated by the work of the researchers presented in the papers [1], [2], and [14]. In this presenting paper, we investigate a subclass of bi-univalent functions Σ in the open unit disk $\mathbb{D}$, which we denote by $\mathcal{F}(n,\alpha,\beta)$ with $\alpha > 0$, $\beta > 0$ and $n \in \mathbb{N}_0$. For functions in this subclass, we obtain the estimates for the initial Taylor-Maclarin coefficients $|a_2|$ and $|a_3|$. Furthermore, we examine the corresponding Fekete-Szegö functional problem for functions in this subclass.

3. Initial Coefficient estimates for the function class $\mathcal{F}(n,\alpha,\beta)$

In this section, we provide bounds for the initial Taylor-Maclaurin coefficients for the functions belong to the class $\mathcal{F}(n,\alpha,\beta)$ which are given by equation (1).

Theorem 1. *Let the function f given by (1) be in the class $\mathcal{F}(n,\alpha,\beta)$. Then*

$$|a_2| \le \frac{2\alpha x\sqrt{x(n!)}}{\sqrt{|(3\alpha(n+2)!(1+2\beta)x^2 - 4(1+\beta)^2(n+1)(n+1)!\{(2+2\alpha)x^2-1\}|}} \tag{8}$$

and

$$|a_3| \le \frac{4\alpha x(n!)}{3(1+2\beta)(n+2)!} + \frac{\alpha^2x^2}{(1+\beta)^2(n+1)^2} \tag{9}$$

Proof. Let f belong to the class $\mathcal{F}(n,\alpha,\beta)$. Then Using (6) and (7) we can find two analytic functions p and q on the unit disk $\mathbb{D}$ such that

$$(\mathcal{R}^n f(z))' + \beta z(\mathcal{R}^n f(z))'' \prec H_\alpha(x, p(z)), \tag{10}$$

and

$$(\mathcal{R}^n g(w))' + \beta w(\mathcal{R}^n g(w))'' \prec H_\alpha(x, q(w)). \tag{11}$$

where the analytic functions p and q are given by

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots \quad \text{where} \quad z \in \mathbb{D},$$

and

$$q(w) = 1 + q_1w + q_2w^2 + q_3w^3 + \cdots \quad \text{where} \quad w \in \mathbb{D}$$

Such that

$$p(0) = q(0) = 0,$$

and for all z, $w \in \mathbb{D}$

$$|p(z)| < 1 \quad \text{and} \quad |q(z)| < 1.$$

Moreover, it is well-known that (see, for details [10]) for all $j \in \mathbb{N}$

$$|p_j| \le 1 \quad \text{and} \quad |q_j| \le 1.$$

Now, upon comparing the coefficients in both sides of (10) and (11) we obtain the following

$$2(1+\beta)\sigma(n,2)a_2 = C_1^{\alpha}(x)p_1, \tag{12}$$

$$3(1+2\beta)\sigma(n,3)a_3 = C_1^{\alpha}(x)p_2 + C_2^{\alpha}(x)p_1^2, \tag{13}$$

$$-2(1+\beta)\sigma(n,2)a_2 = C_1^{\alpha}(x)q_1, \tag{14}$$

and

$$3(1+2\beta)\sigma(n,3)(2a_2^2 - a_3) = C_1^{\alpha}(x)q_2 + C_2^{\alpha}(x)q_1^2 \tag{15}$$

Using equations (12) and (14) we get

$$p_1 = -q_1 \tag{16}$$

Moreover, adding the square of equations (12) and (14) we get

$$8(1+\beta)^2[\sigma(n,2)]^2 a_2^2 = [C_1^{\alpha}(x)]^2(p_1^2 + q_1^2) \tag{17}$$

By adding equations (13) and (15) we get

$$6(1+2\beta)\sigma(n,3)a_2^2 = [C_1^{\alpha}(x)](p_2+q_2) + [C_2^{\alpha}(x)](p_1^2+q_1^2) \tag{18}$$

In view of equation (17), equation (18) can be written as

$$\left(6(1+2\beta)\sigma(n,3)[C_1^{\alpha}(x)]^2 - 8(1+\beta)^2[C_2^{\alpha}(x)][\sigma(n,2)]^2\right)a_2^2 = [C_1^{\alpha}(x)]^3(p_2+q_2) \tag{19}$$

Using equation (5), equation (19) becomes

$$a_2^2 = \frac{4\alpha^2x^3(p_2+q_2)}{3(1+2\beta)(n+1)(n+2)\alpha x^2 - 4(1+\beta)^2(n+1)^2[2(\alpha+1)x^2-1]}.$$

Using the facts $|p_2| \le 1$ and $|q_2| \le 1$, we get the desired estimate of a_2:

$$|a_2| \le \frac{2\alpha x\sqrt{x(n+1)}}{\sqrt{(n+1)^2|(3\alpha(1+2\beta)(n+2)x^2 - 4(1+\beta)^2(n+1)\{(2+2\alpha)x^2-1\}|}}.$$

Next, we look for the bound of $|a_3|$. Subtracting equation (15) from equation (13) and using equation (16), we get

$$a_3 = \frac{C_1^{\alpha}(x)(p_2-q_2)}{6(1+2\beta)\sigma(n,3)} + a_2^2. \tag{20}$$

In view of equation (17), we obtain

$$a_3 = \frac{C_1^\alpha(x)(p_2 - q_2)}{6(1+2\beta)\sigma(n,3)} + \frac{[C_1^\alpha(x)]^2 p_1^2}{4(1+\beta)^2[\sigma(n,2)]^2}.$$

Hence, Using Equation (5) and the facts $|p_2| \le 1$ and $|q_2| \le 1$, we get the desired estimate of a_3:

$$|a_3| \le \frac{4\alpha x}{3(1+2\beta)(n+2)(n+1)} + \frac{\alpha^2 x^2}{(1+\beta)^2(n+1)^2}.$$

This completes the proof of Theorem 1.

Taking $\alpha = 1$, we get the following interesting corollary of Theorem 1. These initial coefficient estimates are related to Chebyshev polynomials of the second kind. The prove is similar to the proof of previous theorem, so we omit the proof's details.

Corollary 1. *Let the function f given by (1) be in the class $\mathcal{F}(n,1,\beta)$. Then*

$$|a_2| \le \frac{2x\sqrt{x(n!)}}{\sqrt{|(3(n+2)!(1+2\beta)x^2 - 4(1+\beta)^2(n+1)(n+1)!(4x^2-1)|}},$$

and

$$|a_3| \le \frac{4x(n!)}{3(1+2\beta)(n+2)!} + \frac{x^2}{(1+\beta)^2(n+1)^2}.$$

On the other hand, taking $\beta = 1$, we get the following corollary.

Corollary 2. *Let the function f given by (1) be in the class $\mathcal{F}(n,\alpha,0)$. Then*

$$|a_2| \le \frac{2\alpha x\sqrt{x(n!)}}{\sqrt{|(9\alpha(n+2)!x^2 - 16(n+1)(n+1)!\{(2+2\alpha)x^2-1\}|}},$$

and

$$|a_3| \le \frac{4\alpha x(n!)}{9(n+2)!} + \frac{\alpha^2 x^2}{4(n+1)^2}.$$

4. Fekete-Szegö problem for the function class $\mathcal{F}(n,\alpha,\beta)$

In this section, we consider the classical Fekete-Szegö problem for our presenting class $\mathcal{F}(n,\alpha,\beta)$.

Theorem 2. *Let the function f given by (1) be in the class $\mathcal{F}(n,\alpha,\beta)$. Then for some $\zeta \in \mathbb{R}$,*

$$|a_3 - \zeta a_2^2| \le \begin{cases} \frac{4\alpha x}{B}, & \text{if } |1-\zeta| \le \frac{\Delta(\alpha,n,\beta)}{4B\alpha^2x^2} \\ \frac{16\alpha^3x^3|1-\zeta|}{\Delta(\alpha,n,\beta)}, & \text{if } |1-\zeta| \ge \frac{\Delta(\alpha,n,\beta)}{4B\alpha^2x^2}, \end{cases} \tag{21}$$

where

$$\Delta(\alpha,n,\beta) = 4\alpha[B - 4(1+\beta)^2(n+1)^2(\alpha+1)]x^2 - 8\alpha(1+\beta)^2(n+1)^2,$$

and

$$B = 3(1+2\beta)(n+2)(n+1)$$

Proof. For some real number ζ, using equation (20) we have

$$a_3 - \zeta a_2^2 = \frac{C_1^\alpha(x)(p_2 - q_2)}{6(1+2\beta)\sigma(n,3)} + (1-\zeta)a_2^2.$$

In view of equation (19), we obtain

$$a_3 - \zeta a_2^2 = \frac{C_1^\alpha(x)(p_2 - q_2)}{6(1+2\beta)\sigma(n,3)} + \frac{(1-\zeta)[C_1^\alpha(x)]^3(p_2+q_2)}{6(1+2\beta)\sigma(n,3)[C_1^\alpha(x)]^2 - 8(1+\beta)^2[C_2^\alpha(x)][\sigma(n,2)]^2}.$$

The last expression can be written as:

$$a_3 - \zeta a_2^2 = C_1^\alpha(x)[(K-L)p_2 + (K+L)q_2],$$

where

$$K = \frac{1}{6(1+a\beta)\sigma(n,3)},$$

and

$$L = \frac{(1-\zeta)[C_1^\alpha(x)]^2}{\triangle(\alpha,n,\beta)}.$$

Using Lemma 1, we get the following

$$|a_3 - \zeta a_2^2| \leq \begin{cases} 2|\frac{C_1^\alpha(x)}{6(1+a\beta)\sigma(n,3)}|, & \text{if } |K| \geq |L| \\ 2|\frac{(1-\zeta)[C_1^\alpha(x)]^3}{\triangle(\alpha,n,\beta)}|, & \text{if } |K| \leq |L| \end{cases}.$$

Using the initial values (4) and equation (5), we get the desired inequality (21). This completes the proof of Theorem 2.

The following corollaries are just consequences of Theorem 2. Taking $\alpha = 1$, we get the Fekete-Szegö inequality that is related to Chebyshev polynomials of the second kind.

Corollary 3. *Let the function f given by (1) be in the class $\mathcal{F}(n,1,\beta)$. Then for some $\zeta \in \mathbb{R}$,*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{4x}{B}, & \textit{if } |1-\zeta| \leq G \\ \frac{16x^3|1-\zeta|}{4B(n+2)(n+1)x^2 - 8(1+\beta)(n+1)^2(4x^2-1)}, & \textit{if } |1-\zeta| \geq G, \end{cases} \tag{22}$$

where

$$G = \frac{4B(n+2)(n+1)x^2 - 8(1+\beta)(n+1)^2(4x^2-1)}{2Bx^2}.$$

Taking $\beta = 1$, we get the following corollary.

Corollary 4. *Let the function f given by (1) be in the class $\mathcal{F}(n,\alpha,0)$. Then for some $\zeta \in \mathbb{R}$,*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{4(n!)\alpha x}{9(n+2)!}, & \textit{if } |1-\zeta| \leq \frac{(n!)H(n,\alpha)}{36(n+2)!\alpha^2x^2} \\ \frac{16\alpha^3x^3|1-\zeta|}{H(n,\alpha)}, & \textit{if } |1-\zeta| \geq \frac{(n!)H(n,\alpha)}{36(n+2)!\alpha^2x^2}, \end{cases} \tag{23}$$

where

$$H(n,\alpha) = 4\alpha x^2(n+1)\left(9(n+2) - 16(n+1)(\alpha+1)\right) - 32\alpha(n+1)^2.$$

5. Conclusion

This research paper has investigated a new subclass of bi-univalent functions, defined in terms of the Ruscheweyh derivative $\mathcal{R}^n$ of order n, by the means of Gegenbauer polynomials. For functions belong to this function class, the author has derived estimates for the Taylor-Maclaurin initial coefficients and Fekete-Szegö functional problem. The work presented in this paper will lead to many different results for subclasses defined by the means of Legendre polynomials $L_n(x) = C_n^{1/2}(x)$ and the Chebyshev polynomials of the second kind $T_n(x) = C_n^1(x)$. Moreover, the presented work in this paper will inspire researchers to extend its concepts to harmonic functions and symmetric q-calculus.

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