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On a parameter-controlled method for multiple zeros

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Abstract. For a parameter-controlled Newton-secant method, we investigate the complex dynamics with the stability surfaces and parameter spaces using Möbious conjugacy map applied to the form $((z-A)(z-B))^m$. The basins of attraction of test functions is illustrated according to the color palette. The nonlinear equation related to blood rheology model has been employed to show the basins of attraction.

2020 Mathematics Subject Classifications: 65H05, 65H99

Key Words and Phrases: Conjugacy map, multiple root, nonlinear equation, parameter space, Blood Rheology model

1. Introduction

The root-finding problem[10, 11] in diverse field of science, engineering and artificial intelligence is an important task to handle. Researchers are developing new iterative methods to solve the nonlinear equations. With the advent of high-accuracy computer, the numerical methods are improved efficiently and accurately in [3, 7, 8, 12, 14]. A root α of $f(x)=0$ is called a multiple root with multiplicity m if $f^{(i)}(\alpha)=0,\ i=0,1,2,\cdots,m-1$ and $g^{(m)}(\alpha)\neq 0$.

Let $f: C \to C$ be an analytic function[1, 13] with an integer multiplicity $m \geq 1$. Geum et al. [5] investigated the third order of parameter-controlled Newton-secant scheme given by

$$x_{n+1} = x_n - \frac{m(1-t^m)f(x_n)^2}{f'(x_n)\{f(x_n)-f(y_n)\}}, \tag{1}$$

where $y_n = x_n - m(1-t)\frac{f(x_n)}{f'(x_n)}$.

The iteration method (1) is expressed as a discrete formula

$$x_{n+1} = O_f(x_n), \tag{2}$$

where O_f is the iterative method. We have a complex discrete dynamical system[6].

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$$z_{n+1} = O_f(z_n) = z_n - \lambda \frac{f(z_n)^2}{f'(z_n)\{f(z_n) - f(y_n)\}}, \tag{3}$$

where $y_n = z_n - \mu \frac{f(z_n)}{f'(z_n)}, \mu = m(1-t)$ and $\lambda = m(1-t^m)$.

2. Analysis

Applying Möbius conjugacy map $M(z)$ to $f(z) = ((z-A)(z-B))^m$, we have

$$J(z,t) = M \circ O_f \circ M^{-1}(z) = \frac{z((1+z)r_1 r_2 - r_3 \delta_1)}{(1+z)r_1 r_2 - r_3 \delta_2}, \tag{4}$$

where $M(z) = (z-A)/(z-B),\ A,\ B \in C \cup \{\infty\},\ A \neq B,\ r_1 = (t+z)^m,\ r_2 = (1+tz)^m,\ r_3 = (1+z)^{2m},\ \delta_1 = (t^m + z)$ and $\delta_2 = (1+t^m z)$. Then $O_f(z,t)$ is conjugate [2] to $J(z,t)$.

In case of $m = 1, 2$, we have

$$J(z,t) = \begin{cases} \frac{z^2(t+z)}{1+tz}, & \text{m=1,} \\ \frac{z^{(}1+t^3 z + z(1+z)(2+z) + t^2(2+z+2z^2) + t(-1+z(2+z+z^2)))}{1+t+(3+t+2t^2)z+(1+t)(2+t^2)z^2+(1+t(-1+2t))z^3}, & \text{m=2.} \end{cases} \tag{5}$$

We locate the fixed points of the iterative method $J(z,t)$. Let $J(z,t)-z$ whose roots are the desired fixed point of $J(z,t)$. $z=0$ and $z=1$ are the zeros of $J(z,t)-z$. Since $M(z)$ is a fixed point of $J(z,t)$ for a fixed point z of O_f, the explicit form of $\phi(z,t) = J(z,t) - z$ is given by

$$\phi(z,t) = \frac{(-1+z)z r_3 \delta_3}{(1+z)r_1 r_2 - r_3 \delta_2}, \tag{6}$$

and

$$\phi(z,t) = \begin{cases} \frac{z(z-1)\psi_1(t)}{q_1(t)}, & \text{if} \quad \text{m=1,} \\ \frac{z(z-1)\psi_2(t)}{q_2(t)}, & \text{if} \quad \text{m=2,} \end{cases} \tag{7}$$

where

$$\psi_1(z) = z+1, \quad q_1(z) = 1+tz,$$
$$\psi_2(z) = (1+t)(1+z)^3, \quad q_2(z) = 2(1+z)^4 + t^2 z(1+z)^4 - z(t+z)^2(1+tz)^2.$$

By aid of Mathematica [15], we compute the derivative of $J(z,t)$:

$$J'(z,t) = \frac{T_1 r_3 + T_2 r_1 + (1+z)^2 {r_1}^2 {r_2}^2}{((1+z)r_1 r_2 - r_3 \delta_2)^2}, \tag{8}$$

where $T_1 = (2z + t^m(1 + z^2))r_3 - mz(t + z)^{-1+m}(-1 + z^2)r_2\delta_3$ $T_2 = (-mtz(1 + tz)^{-1+m}(-1 + z^2)r_3\delta_3 + r_2(-((1 + z(4 + z) + t^m(1 + z^2))r_3) + 2m(-1 + z)z(1 + z)^{2m}\delta_3))$, For $m = 1$ and $m = 2$, we have

$$J'(z,t) = \begin{cases} \frac{z(3z+t^2z+2t(1+z^2))}{(1+tz)^2}, & \text{m=1} \\ \frac{z((1+z^6)\delta_0+z((1+z^4)\delta_1+z(\delta_2+z^2\delta_2+z\delta_3)))}{\delta_4{}^2}, & \text{m=2.} \end{cases} \tag{9}$$

$$\begin{aligned}
\delta_0 =& 2(1 + t^2 + 2t^3), \delta_1 = (9 + t(10 + t(16 + t(6 + 7t)))),\\
\delta_2 =& 2(1 + t)(4 + t^2)(3 + t + 2t^2),\\
\delta_3 =& (35 + t(38 + t(47 + t(24 + t(13 + t(2 + t)))))),\\
\delta_4 =& 1 + t + (3 + t + 2t^2)z + (1 + t)(2 + t^2)z^2 + (1 + t(-1 + 2t))z^3.
\end{aligned}$$

The critical point of the parameter-controlled method refers to a point where the derivative of a function is zero, that is $J'(z,t) = 0$. The points $z = \infty$ and $z = 0$ are critical points associated with $(z - A)(z - B)$. The critical points that are not any roots of the polynomial $(z - A)(z - B)$ are called to be free critical points. As the result of following Algorithm 1, the stability surfaces are displayed in Figure 1 and 2.

Algorithm 1

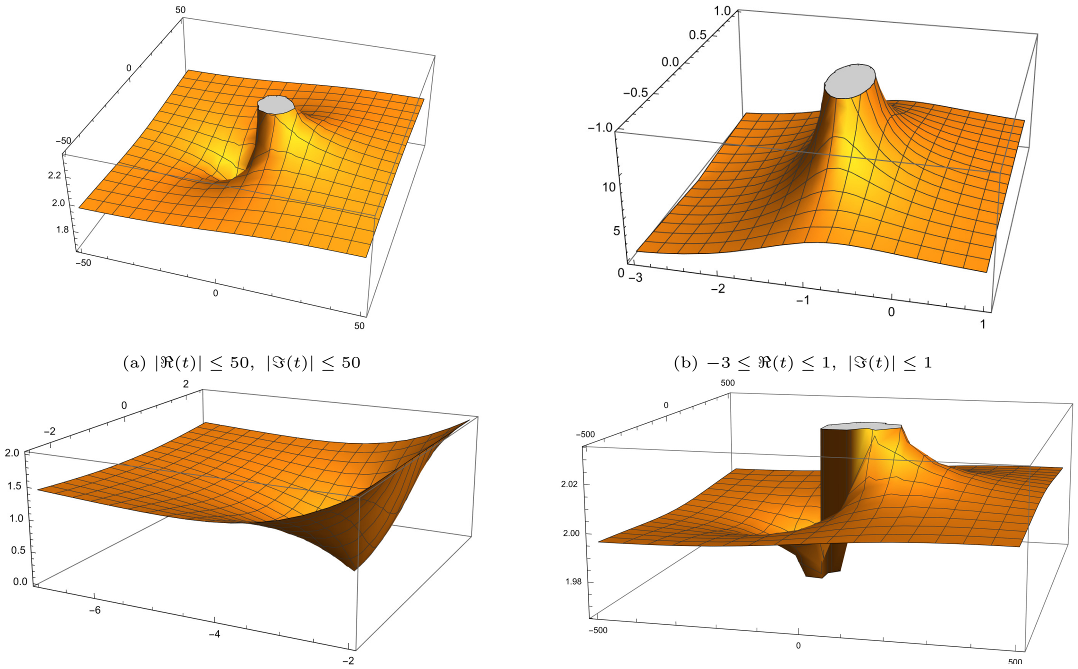
(1) Set $i = 1$
(2) Choose a region $B \in C$ and select a point $v = (Re(v), Im(v))$ in B
(3) For the v, solve $q_i(k) = 0$ using Mathematica [15].
(4) Compute $m_i = |J'(k,t)|$.
(5) Save $(Re(v), Im(v), m_i)$ and choose the next value in B
(6) Repeat steps (2)-(5) until desired result is done.
(7) Set $i = i + 1$ and if $i \leq w$, then repeat steps (2)-(6)
(8) If $i = w$, then stop the process.

3. Experiment

According to Algorithm 2, the numerical parameter spaces for $m = 1$ and $m = 2$ are constructed in Figure 3 and 4. The systematic color palette in Table 1 is utilized to paint a value according to the orbital period of the point z of $J(z,t)$. The tolerance of 10^{-4} after up to 1000 iterations is assigned.

Algorithm 2

(1) Set $i = 1$
(2) Choose a region $B \in C$ and select a point $v = (Re(v), Im(v))$ in B
(3) For the v, find the free critical point.
(4) Compute the orbit of $J(z,t)$ within the maximal iterative number.



(a) $|\Re(t)| \leq 50,\ |\Im(t)| \leq 50$ (b) $-3 \leq \Re(t) \leq 1,\ |\Im(t)| \leq 1$

(c) $-7 \leq \Re(t) \leq -2,\ \ -3 \leq \Im(t) \leq 3$ (d) $|\Re(t)| \leq 500,\ |\Im(t)| \leq 500$

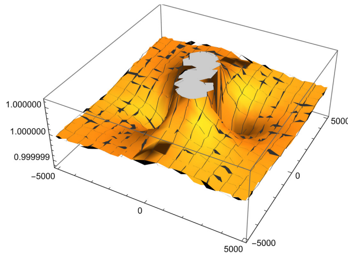
Figure 1: Stability surfaces for $m = 1$.

(5) If the orbit converges to one cycle within the given error, then color the point v according to the color palette in Table 1.
(6) Choose the next value in B
(7) Repeat steps (2)-(6) until desired result is obtained.
(8) Set $i = i + 1$ and if $i \leq w$, then repeat steps (2)-(8)
(9) If $i = w$, then stop the process.

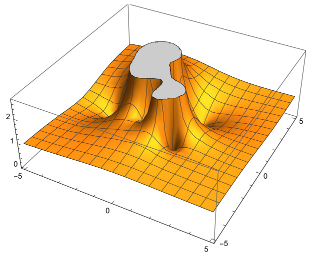
In Figure 5, the basins of attraction associated with $m = 1, 2$ are appeared. As the last example, we choose the equation in blood rheology model[4]

$$z = \left(\frac{x^8}{441} + \frac{8x^5}{63} - \frac{2857144357x^4}{50000000000} + \frac{16x^2}{9} - \frac{906122449x}{250000000} + \frac{3}{10}\right)^4,$$

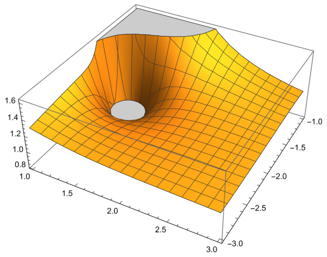
to carry out the experiments. In Figure 6, the basins of attraction for this equation are shown.



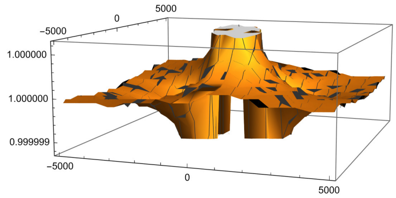
(a) $|\Re(t)| \leq 5000, \ |\Im(t)| \leq 5000$



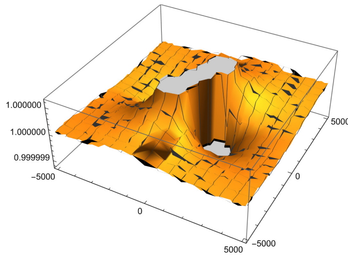
(b) $|\Re(t)| \leq 5, \ |\Im(t)| \leq 5$



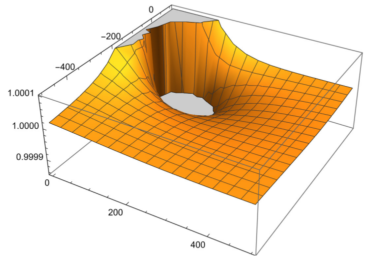
(c) $1 \leq \Re(t) \leq 3, \ -3 \leq \Im(t) \leq -1$



(d) $|\Re(t)| \leq 5000, \ |\Im(t)| \leq 50001$



(e) $|\Re(t)| \leq 5000, \ |\Im(t)| \leq 5000$



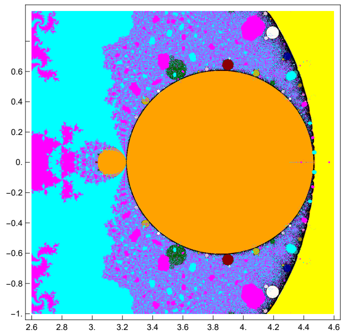
(f) $0 \leq \Re(t) \leq 500, \ -500 \leq \Im(t) \leq 0$

Figure 2: Stability surfaces for $m = 2$.

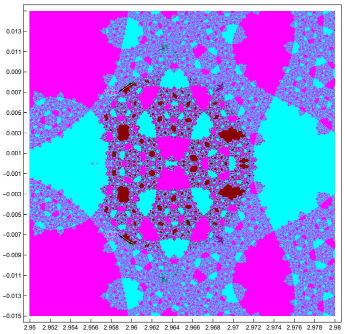
Table 1: Color palette for a n-periodic orbit with $n \in N \cup \{0\}$

n	C_n
$n = 1$	$C_1 = \begin{cases} \text{magenta, for fixed point } \infty \\ \text{cyan, for fixed point } 0 \\ \text{yellow, for fixed point } 1 \\ \text{red, for other strange fixed point ,} \end{cases}$
$2 \leq n \leq 68$	$C_2 =$ orange, $C_3 =$ light green, $C_4 =$ dark red, $C_5 =$ dark blue, $C_6 =$ dark green, $C_7 =$ dark yellow, $C_8 =$ floral white, $C_9 =$ light pink, $C_{10} =$ khaki, $C_{11} =$ dark orange, $C_{12} =$ turquoise, $C_{13} =$ lavender, $C_{14} =$ thistle, $C_{15} =$ plum, $C_{16} =$ orchid, $C_{17} =$ medium orchid, $C_{18} =$ blue violet, $C_{19} =$ dark orchid, $C_{20} =$ purple, $C_{21} =$ power blue, $C_{22} =$ sky blue, $C_{23} =$ deep sky blue, $C_{24} =$ dodger blue, $C_{25} =$ royal blue, $C_{26} =$ medium spring green, $C_{27} =$ spring green, $C_{28} =$ medium sea green, $C_{29} =$ sea green, $C_{30} =$ forest green, $C_{31} =$ olive drab, $C_{32} =$ bisque, $C_{33} =$ moccasin, $C_{34} =$ light salmon, $C_{35} =$ salmon, $C_{36} =$ light coral, $C_{37} =$ Indian red, $C_{38} =$ brown, $C_{39} =$ fire brick, $C_{40} =$ peach puff, $C_{41} =$ wheat, $C_{42} =$ sandy brown, $C_{43} =$ tomato, $C_{44} =$ orange red, $C_{45} =$ chocolate, $C_{46} =$ pink, $C_{47} =$ pale violet red, $C_{48} =$ deep pink, $C_{49} =$ violet red, $C_{50} =$ gainsboro, $C_{51} =$ light gray, $C_{52} =$ dark gray, $C_{53} =$ gray, $C_{54} =$ charteruse, $C_{55} =$ electric indigo, $C_{56} =$ electric lime, $C_{57} =$ lime, $C_{58} =$ silver, $C_{59} =$ teal, $C_{60} =$ pale turquoise, $C_{61} =$ sandy brown, $C_{62} =$ honeydew, $C_{63} =$ misty rose, $C_{64} =$ lemon chiffon, $C_{65} =$ lavender blush, $C_{66} =$ gold, $C_{67} =$ crimson, $C_{68} =$ tan.
$n = 0^*$ or $n > 69$	$C_n =$ black.

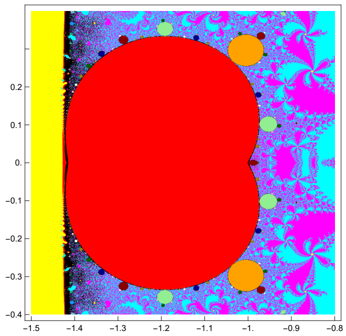
$*$: $n = 0$: the orbit is non-periodic but bounded.



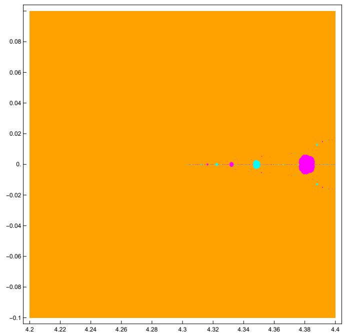
(a) $2.6 \le \Re(t) \le 4.6,\ |\Im(t)| \le 1$



(b) $2.95 \le \Re(t) \le 2.98,\ |\Im(t)| \le 0.015$

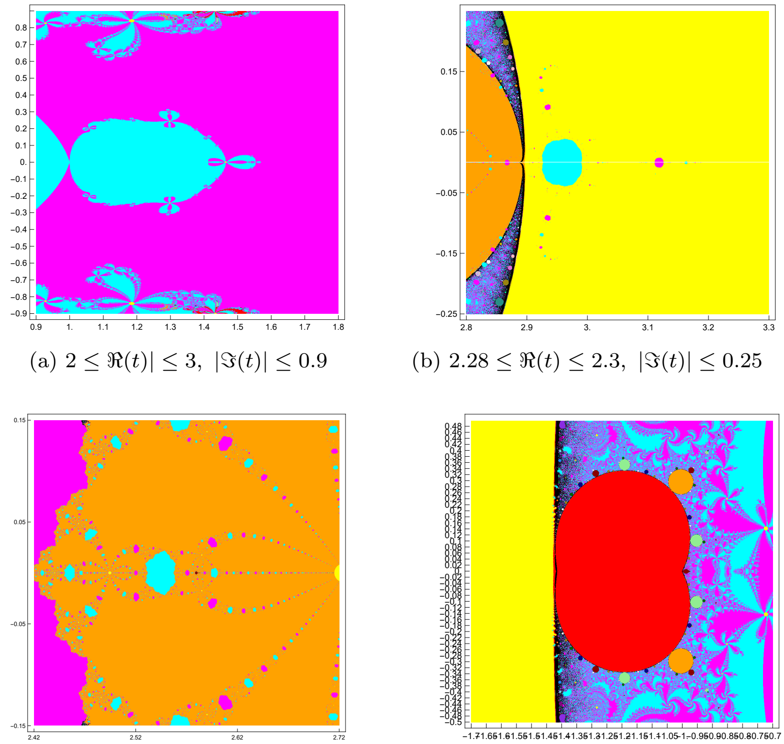


(c) $-1.5 \le \Re(t) \le -0.8,\ |\Im(t)| \le 0.4$



(d) $4.2 \le \Re(t) \le 4.4,\ |\Im(t)| \le 0.1$

Figure 3: Parameter spaces for $m = 1$.



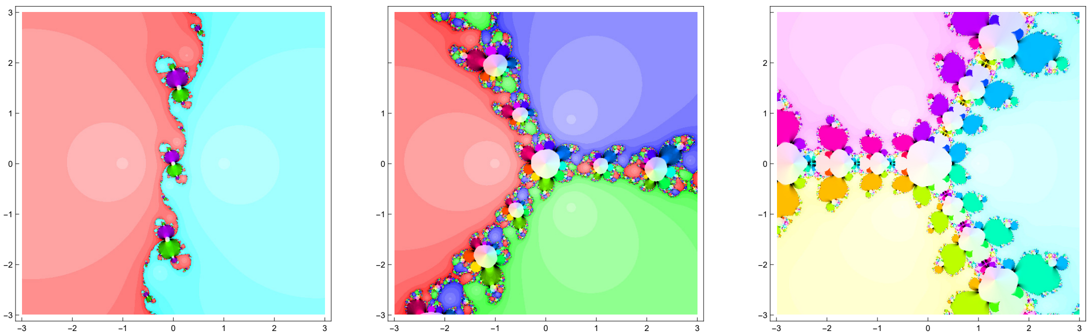
(a) $2 \leq \Re(t)| \leq 3,\ |\Im(t)| \leq 0.9$

(b) $2.28 \leq \Re(t) \leq 2.3,\ |\Im(t)| \leq 0.25$

(c) $-1.5 \leq \Re(t) \leq 0.9,\ |\Im(t)| \leq 0.15$

(d) $0.3 \leq \Re(t) \leq 1.1,\ |\Im(t)| \leq 0.5$

Figure 4: Parameter spaces for $m = 2$.



(a) $p_1(z) = (z^2 - 1)^2$

(b) $p_2(z) = (x^3 + 1)^3(z)$

(c) $p_3(z) = (x^3 - 1)^3$

Figure 5: Basins of attraction of test functions

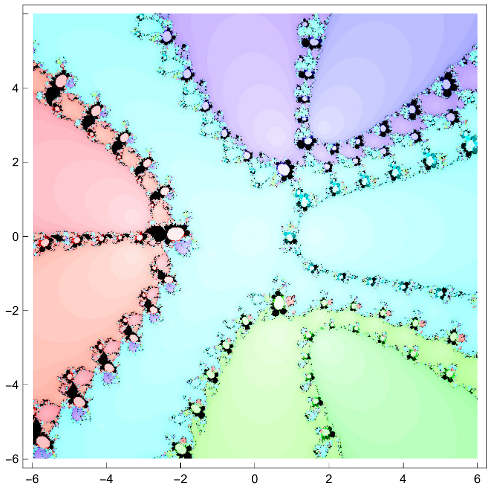


Figure 6: Basins of attraction of Blood Rheology model

4. Conclusion

We have studied the complex dynamics using Möbius conjugacy map applied to $f(z) = ((z-A)(z-B))^m$ with the known multiplicity m. To obtain the useful information on better initial values of any numerical method, we need to experiment the basins of attractions.

As a next work, we focus on developing the higher order methods and illustrating the dynamical behavior to find out the chaotic geometry of their dynamics [6]. For test functions, Euler-Cauchy method is best in the sense that the boundaries of the basins have no lobes in Neta B. [9]. Frow this viewpoint, I will improve the efficient and accurate method for nonlinear equations.

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References

[1] L Ahlfors. *Complex Analysis.* McGraw-Hill Book Inc., 1979.

[2] A Beardon. *Iteration of Rational Functions.* Springer-Verlag, New York, 1991.

[3] C Dong. A family of multipoint iterative functions for finding multiple roots. *Int. J. Comput. Math.*, 21:363–367, 1987.

[4] R.L. Fournier. *Basic transport phenomena in biomedical engineering.* CRC press., 2017.

[5] Y.H Geum and Y.I Kim. Cubic convergence of parameter-controlled newton-secant method for multiple zeros. *J. Comput. Math. Appl.*, 233:931–937, 2009.

[6] A.K. Rathie H.J. Kim and Y.H. Geum. A family of optimal cubic-order multiple-root solvers and their dynamcis. *Euro. J. Pure. Appl. Math. Comput.*, 16(3):1902–1912, 2023.

[7] Y.I Kim and Y.H Geum. A triparametirc family of optimal fourth-order multiple-root finders and their dynamics. *J. Appl. Math.*, 8436759:1–23, 2016.

[8] Y.I Kim and Y.H Geum. Dynamical analysis via möbius conjugacy map on a uni-parametric family of optimal fourth-order multiple-zero solvers with rational weight functions. *Discrete Dyn. Nature Soc.*, 7486125:1–19, 2018.

[9] B Neta. On a fifth-order method for multiple roots of nonlinear equations. *Symmetry*, 15, 2023.

[10] N Osada. An optimal multiple root-finding mehtod of order three. *J. Comput. Appl. Math.*, 51:131–133, 1994.

[11] A. M. Ostrowski. *Solutions of Equations and System of Equations.* Academic Press, New York, 1960.

[12] V Kanwar S Kumar and S Singh. On some modified families of multipoint iterative methods for multiple roots of nonlinear equations. *Appl. Math. Comput.*, 218:7382–7394, 2012.

[13] J. Traub. *Iterative Methods for the Solution of Equations.* Chelsea Publishing Company, 1997.

[14] S Bhatia V Kanwar and M Kansal. New optimal class of higher-order methods for multiple roots, permitting $f'(x_n) = 0$. *Appl. Math. Comput.*, 222(1):564–574, 2013.

[15] S Wolfram. *The Mathematica Book 5th ed.* Wolfram Media, 2003.