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Spectral Properties of Power Graph of Dihedral Groups

Mamika Ujianita Romdhini[1,*], Athirah Nawawi[2], Faisal Al-Sharqi[3,4], Ashraf Al-Quran[5]

[1] *Department of Mathematics, Faculty of Mathematics and Natural Science, Universitas Mataram, Mataram 83125, Indonesia*
[2] *Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*
[3] *Department of Mathematics, Faculty of Education for Pure Sciences, University Of Anbar, Ramadi, Anbar, Iraq*
[4] *College of Engineering, National University of Science and Technology, Dhi Qar, Iraq*
[5] *Basic Sciences Department, Preparatory Year Deanship, King Faisal University, Al-Ahsa, Saudi Arabia*

Abstract. This paper focuses on the power graph of dihedral groups of order $2n$, D_{2n}, where $n \geq 3$. We show the characteristic polynomial of the power graph corresponding to the adjacency, Laplacian, signless Laplacian, and normalized form of these matrices.

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1. Introduction

Spectral graph theory describes graphs based on specific matrices, such as adjacency, Laplacian, or signless Laplacian matrices. The spectrum of these matrices can characterize a graph. These various matrices, in general, give insight into the graph based on their spectrum. This paper examines a power graph, one of the finite groups that can be represented by graphs.

A power graph of the group G is denoted by Γ_G and defined as a graph whose vertex set is all the elements of G and two distinct vertices v_p and v_q are adjacent if and only if $v_p^x = v_q$ or $v_q^y = v_p$ for positive integers x and y [5]. The vertex set for Γ_G is the non-abelian dihedral group of order $2n$, where $n \geq 3$, denoted by $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$ [3]. Let $G_1 = \{e\}$, $G_2 = \{a^i : 1 \leq i \leq n-1\}$, and $G_3 = \{a^i b : 1 \leq i \leq n\}$. Note that

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Email addresses: mamika@unram.ac.id (M. U. Romdhini), athirah@upm.edu.my (A. Nawawi), faisal.ghazi@uoanbar.edu.iq (F. Al-Sharqi), aalquran@kfu.edu.sa (A. Al-Quran)

$D_{2n} = G_1 \cup G_2 \cup G_3$. Throughout this paper, the power graph for the dihedral group is denoted by $\Gamma_{D_{2n}}$. It is clear that $\Gamma_{D_{2n}}$ is a connected graph [2].

Furthermore, the discussion on the degree formula of the power graph of some finite group can be found in [14]. Later, Takshak, et al. [15] showed the new finding that if Γ_G is a power graph of a finite group G, then it is a divisor graph. Kumar et al. [7] have presented a complete and excellent survey of the power graph for some finite groups. Meanwhile, the degree of $\Gamma_{D_{2n}}$ has been presented by [2] as in the following theorem:

Theorem 1. *[2] If $\Gamma_{D_{2n}}$ is the power graph of D_{2n}, then*

(i) the degree of a in $\Gamma_{D_{2n}}$ is $d_e = 2n - 1$,

(ii) the degree of a^i in $\Gamma_{D_{2n}}$ is $d_{a^i} = n - 1$,

(iii) the degree of a^ib in $\Gamma_{D_{2n}}$ is $d_{a^ib} = 1$,

The above theorem gives the information that vertex e is adjacent to all other vertices in $\Gamma_{D_{2n}}$. Every vertex in G_2 is adjacent to e and all other members in G_2. Meanwhile, all vertices in G_3 are only adjacent to e [2].

Several authors have discussed the graphs that are defined on dihedral groups. They worked on the spectral problem of the commuting and non-commuting graphs, which can be seen in [9–13], Accordingly, Romdhini et al. [8] investigated signless Laplacian spectral of interval-valued fuzzy graphs. Moreover, an analysis of the relationship between graphs and unitary commutative rings' prime spectrum is presented by [1]. Motivated by this, this research aims to formulate the characteristic polynomial of the power graph of the dihedral group associated with the adjacency, Laplacian, signless Laplacian, and normalized form of these matrices. The definition of the various matrices can be seen in the following definitions.

Definition 1. *([4]) The adjacency matrix of order $n \times n$ associated with $\Gamma_{D_{2n}}$ is given by $A(\Gamma_{D_{2n}}) = [a_{ij}]$ whose (i, j)-th entry*

$$a_{ij} = \begin{cases} 1, & \textit{if } v_i \neq v_j \textit{ and they are adjacent} \\ 0, & \textit{otherwise} \end{cases}$$

Definition 2. *([4]) The diagonal degree matrix of order $n \times n$ associated with $\Gamma_{D_{2n}}$ is given by $D(\Gamma_{D_{2n}}) = [d_{ij}]$ whose (i, j)-th entry*

$$d_{ij} = \begin{cases} d_{v_i}, & \textit{if } v_i = v_j \\ 0, & \textit{otherwise} \end{cases}$$

where d_{v_i} is the degree of vertex v_i, a number of vertices adjacent to v_i in $\Gamma_{D_{2n}}$.

Definition 3. *([4]) The Laplacian matrix of order $n \times n$ associated with $\Gamma_{D_{2n}}$ is given by $L(\Gamma_{D_{2n}}) = D(\Gamma_{D_{2n}}) - A(\Gamma_{D_{2n}})$.*

Definition 4. *([4]) The signless Laplacian matrix of order $n \times n$ associated with $\Gamma_{D_{2n}}$ is given by $SL(\Gamma_{D_{2n}}) = D(\Gamma_{D_{2n}}) + A(\Gamma_{D_{2n}})$.*

Definition 5. *([4]) The normalized adjacency matrix of order $n \times n$ associated with $\Gamma_{D_{2n}}$ is given by*

$$NA(\Gamma_{D_{2n}}) = \sqrt{D(\Gamma_{D_{2n}})}^{-1} A(\Gamma_{D_{2n}}) \sqrt{D(\Gamma_{D_{2n}})}^{-1}.$$

Definition 6. *([4]) The normalized Laplacian (NL) matrix of order $n \times n$ associated with $\Gamma_{D_{2n}}$ is given by*

$$NL(\Gamma_{D_{2n}}) = \sqrt{D(\Gamma_{D_{2n}})}^{-1} L(\Gamma_{D_{2n}}) \sqrt{D(\Gamma_{D_{2n}})}^{-1} = I_n - NA(\Gamma_{D_{2n}}).$$

Definition 7. *([4]) The normalized signless Laplacian (NSL) matrix of order $n \times n$ associated with $\Gamma_{D_{2n}}$ is given by*

$$NSL(\Gamma_{D_{2n}}) = \sqrt{D(\Gamma_{D_{2n}})}^{-1} SL(\Gamma_{D_{2n}} \sqrt{D(\Gamma_{D_{2n}})}^{-1} = I_n + NA(\Gamma_{D_{2n}}).$$

The characteristic polynomial of $A(\Gamma_{D_{2n}})$ is defined by $P_{A(\Gamma_{D_{2n}})}(\lambda) = det\left(\lambda I_{2n} - A(\Gamma_{D_{2n}})\right)$, where I_{2n} is an $2n \times 2n$ identity matrix. Likewise, the notation for other matrices can also be applied in the same way.

To formulate the characteristic polynomial of $\Gamma_{D_{2n}}$, row and column operations need to be performed. Assume that R_i and C_i are the $i-$th row and column of the matrix, respectively. In this case, R_i' and C_i' will be the new $i-$th row and column of the matrix, respectively, as obtained from R_i and C_i. The following theorem is a result of [6] as our guideline to simplify the characteristic polynomial of $\Gamma_{D_{2n}}$.

Theorem 2. *[6] If a square matrix M can be partitioned into four blocks $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A is a nonsingular, then*

$$|M| = \begin{vmatrix} A & B \\ O & D - CA^{-1}B \end{vmatrix} = |A| \left|D - CA^{-1}B\right|.$$

2. Main Results

This section presents the main results on the characteristic polynomial of $\Gamma_{D_{2n}}$. We begin with the adjacency matrix as the matrix representation of $\Gamma_{D_{2n}}$.

Theorem 3. *Let $\Gamma_{D_{2n}}$ be the power graph for D_{2n}, then the characteristic polynomial of $A(\Gamma_{D_{2n}})$ is*

$$P_{A(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}(\lambda+1)^{n-2}\left(\lambda^3 - (n-2)\lambda^2 + (1-2n)\lambda + n(n-2)\right).$$

Proof. From Theorem 1 we know that vertex e is adjacent to all other vertices in $\Gamma_{D_{2n}}$ and vertices in G_3 are only adjacent to e. Meanwhile, every vertex in G_2 is adjacent to e and all other vertices in G_2. Following definition 1, we can construct $A(\Gamma_{D_{2n}})$ of the size

$2n \times 2n$:

$$A(\Gamma_{D_{2n}}) = \begin{array}{c} \\ e \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{ccccccccc} e & a & a^2 & \dots & a^{n-1} & b & ab & \dots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \end{array}. \tag{1}$$

Matrix $A(\Gamma_{D_{2n}})$ can be partitioned into nine block matrices as follows:

$$A(\Gamma_{D_{2n}}) = \begin{bmatrix} 0 & J_{1\times(n-1)} & J_{1\times n} \\ J_{(n-1)\times 1} & (J-I)_{n-1} & 0_{(n-1)\times n} \\ J_{n\times 1} & 0_{n\times(n-1)} & 0_n \end{bmatrix}.$$

The characteristic polynomial of $A(\Gamma_{D_{2n}})$ is

$$P_{A(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda & -J_{1\times(n-1)} & -J_{1\times n} \\ -J_{(n-1)\times 1} & (\lambda+1)I_{n-1} - J_{n-1} & 0_{(n-1)\times n} \\ -J_{n\times 1} & 0_{n\times(n-1)} & \lambda I_n \end{vmatrix}. \tag{2}$$

We apply the following steps to simplify the determinant in Equation 2:

(i) $R_{n+1+i} \longrightarrow R_{n+1+i} - R_{n+1}$, for $i = 1, 2, \dots, n-1$.

(ii) $C_{n+1} \longrightarrow C_{n+1} + C_{n+2} + \dots + C_{2n}$.

(iii) $C_1 \longrightarrow C_1 + \frac{1}{\lambda} C_{n+1}$.

(iv) $R_{2+i} \longrightarrow R_{2+i} - R_2$, for $i = 1, 2, \dots, n-2$.

(v) $C_2 \longrightarrow C_2 + C_3 + \dots + C_n$.

Then we get

$$P_{A(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \frac{\lambda^2+n}{\lambda} & 1-n & -J_{1\times(n-2)} & -n & -J_{1\times n} \\ -1 & \lambda-(n-2) & -J_{1\times(n-2)} & 0 & 0_{1\times(n-1)} \\ 0_{(n-2)\times 1} & 0_{(n-2)\times 1} & (\lambda+1)I_{n-2} & 0_{(n-2)\times 1} & 0_{(n-2)\times(n-1)} \\ 0 & 0 & 0_{1\times(n-2)} & \lambda & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & \lambda I_{n-1} \end{vmatrix}.$$

Consequently, by Theorem 2, we can obtain the characteristic polynomial of $A(\Gamma_{D_{2n}})$ as follows:

$$P_{A(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}(\lambda+1)^{n-2}\left(\lambda^3 - (n-2)\lambda^2 + (1-2n)\lambda + n(n-2)\right).$$

The previous theorem is devoted to the adjacency matrix. Now we are moving to the Laplacian matrix as the representation of $\Gamma_{D_{2n}}$.

Theorem 4. *Let $\Gamma_{D_{2n}}$ be the power graph for D_{2n}, then the characteristic polynomial of $L(\Gamma_{D_{2n}})$ is*

$$P_{L(\Gamma_{D_{2n}})}(\lambda) = \lambda(\lambda-2n)(\lambda-n)^{n-2}(\lambda-1)^n.$$

Proof. The Laplacian matrix of $\Gamma_{D_{2n}}$ construction depends on the degree and adjacency matrices of $\Gamma_{D_{2n}}$. Now we need to construct a $2n \times 2n$ degree matrix of $\Gamma_{D_{2n}}$ as follows:

$$D(\Gamma_{D_{2n}}) = \begin{array}{c} \\ e \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{ccccccccc} e & a & a^2 & \dots & a^{n-1} & b & ab & \dots & a^{n-1}b \end{array} \\ \begin{pmatrix} 2n-1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & n-1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & n-1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n-1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \end{array}. \tag{3}$$

Based on Definition 3, the Laplacian matrix of $\Gamma_{D_{2n}}$ is

$$L(\Gamma_{D_{2n}}) = D(\Gamma_{D_{2n}}) - A(\Gamma_{D_{2n}})$$

$$= \begin{array}{c} \\ e \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{ccccccccc} e & a & a^2 & \dots & a^{n-1} & b & ab & \dots & a^{n-1}b \end{array} \\ \begin{pmatrix} 2n-1 & -1 & -1 & \dots & -1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 & 0 & 0 & \dots & 0 \\ -1 & -1 & n-1 & \dots & -1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & n-1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \end{array}.$$

Moreover, $L(\Gamma_{D_{2n}})$ can be partitioned into six block matrices as given below:

$$L(\Gamma_{D_{2n}}) = \begin{pmatrix} 2n-1 & -J_{1\times(n-1)} & -J_{1\times n} \\ -J_{(n-1)\times 1} & nI_{n-1} - J_{n-1} & 0_{(n-1)\times n} \\ -J_{n\times 1} & 0_{n\times(n-1)} & I_n \end{pmatrix}.$$

The characteristic polynomial of $L(\Gamma_{D_{2n}})$ can be obtained from the following determinant:

$$P_{L(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda-(2n-1) & J_{1\times(n-1)} & J_{1\times n} \\ J_{(n-1)\times 1} & (\lambda-(n-2))I_{n-1} - J_{n-1} & 0_{(n-1)\times n} \\ J_{n\times 1} & 0_{n\times(n-1)} & (\lambda-1)I_n \end{vmatrix}. \tag{4}$$

We apply the following steps to equation 4:

(i) $R_{n+1+i} \longrightarrow R_{n+1+i} - R_{n+1}$, for $i = 1, 2, \ldots, n-1$.

(ii) $C_{n+1} \longrightarrow C_{n+1} + C_{n+2} + \ldots + C_{2n}$.

(iii) $C_1 \longrightarrow C_1 - \frac{1}{\lambda-1} C_{n+1}$.

(iv) $R_{2+i} \longrightarrow R_{2+i} - R_2$, for $i = 1, 2, \ldots, n-2$.

(v) $C_2 \longrightarrow C_2 + C_{2+1} + \ldots + C_n$,

then we get

$$P_{L(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \frac{\lambda^2-2n\lambda+n-1}{\lambda-1} & n-1 & J_{1\times(n-2)} & n & J_{1\times(n-1)} \\ 1 & \lambda-1 & J_{1\times(n-2)} & 0 & 0_{1\times(n-1)} \\ 0_{(n-2)\times 1} & 0_{(n-2)\times 1} & (\lambda-n)I_{n-2} & 0_{(n-2)\times 1} & 0_{(n-2)\times(n-1)} \\ 0 & 0 & 0_{1\times(n-2)} & \lambda-1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & (\lambda-1)I_{n-1} \end{vmatrix}.$$

Following Theorem 2, we then can obtain

$$P_{L(\Gamma_{D_{2n}})}(\lambda) = \lambda(\lambda-2n)(\lambda-n)^{n-2}(\lambda-1)^n.$$

The next theorem presents the characteristic polynomial of $\Gamma_{D_{2n}}$ associated with the signless Laplacian matrix.

Theorem 5. *Let $\Gamma_{D_{2n}}$ be the power graph for D_{2n}, then the characteristic polynomial of $SL(\Gamma_{D_{2n}})$ is*

$$P_{SL(\Gamma_{D_{2n}})}(\lambda) = (\lambda-1)^{n-1}(\lambda-n+2)^{n-2}\left(\lambda^3 + (3-4n)\lambda^2 + 2n(2n-3)\lambda - 2(n-1)(n-2)\right).$$

Proof. By Equations 1 and 4, and Definition 4, the signless Laplacian matrix of $\Gamma_{D_{2n}}$ is

$$SL(\Gamma_{D_{2n}}) = D(\Gamma_{D_{2n}}) + A(\Gamma_{D_{2n}})$$

$$
= \begin{array}{c} \\ e \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} e & a & a^2 & \dots & a^{n-1} & b & ab & \dots & a^{n-1}b \end{array} \\
\begin{pmatrix}
2n-1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\
1 & n-1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\
1 & 1 & n-1 & \dots & 1 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \dots & n-1 & 0 & 0 & \dots & 0 \\
1 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\
1 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1
\end{pmatrix}
\end{array}.
$$

Moreover, $SL(\Gamma_{D_{2n}})$ can be partitioned into nine block matrices as given below:

$$
SL(\Gamma_{D_{2n}}) = \begin{pmatrix} 2n-1 & J_{1\times(n-1)} & J_{1\times n} \\ J_{(n-1)\times 1} & (n-2)I_{n-1} + J_{n-1} & 0_{(n-1)\times n} \\ J_{n\times 1} & 0_{n\times(n-1)} & I_n \end{pmatrix}.
$$

The characteristic polynomial of $SL(\Gamma_{D_{2n}})$ can be obtained from the following determinant:

$$
P_{SL(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda-(2n-1) & -J_{1\times(n-1)} & -J_{1\times n} \\ -J_{(n-1)\times 1} & (\lambda-(n-2))I_{n-1} - J_{n-1} & 0_{(n-1)\times n} \\ -J_{n\times 1} & 0_{n\times(n-1)} & (\lambda-1)I_n \end{vmatrix}. \tag{5}
$$

We apply the following steps into Equation 5:

(i) $R_{n+1+i} \longrightarrow R_{n+1+i} - R_{n+1}$, for $i = 1, 2, \dots, n-1$.

(ii) $C_{n+1} \longrightarrow C_{n+1} + C_{n+2} + \ldots + C_{2n}$.

(iii) $C_1 \longrightarrow C_1 + \left(\frac{1}{\lambda-1}\right) C_{n+1}$.

(iv) $R_{2+i} \longrightarrow R_{2+i} - R_2$, for $i = 1, 2, \dots, n-2$.

(v) $C_2 \longrightarrow C_2 + C_{2+1} + \ldots + C_n$,

then we get

$$
P_{SL(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \frac{\lambda^2-2n\lambda+n-1}{\lambda-1} & 1-n & -J_{1\times(n-2)} & -n & -J_{1\times(n-1)} \\ -1 & \lambda-2n+3 & -J_{1\times(n-2)} & 0 & 0_{1\times(n-1)} \\ 0_{(n-2)\times 1} & 0_{(n-2)\times 1} & (\lambda-n+2)I_{n-2} & 0_{(n-2)\times 1} & 0_{(n-2)\times(n-1)} \\ 0 & 0 & 0_{1\times(n-2)} & \lambda-1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & (\lambda-1)I_{n-1} \end{vmatrix}.
$$

From Theorem 2, we derive the characteristic polynomial of $SL(\Gamma_{D_{2n}})$ as follows:

$$P_{SL(\Gamma_{D_{2n}})}(\lambda) = (\lambda-1)^{n-1}(\lambda-n+2)^{n-2}\left(\lambda^3+(3-4n)\lambda^2+2n(2n-3)\lambda-2(n-1)(n-2)\right).$$

The normalized form of the adjacency, Laplacian, and signless Laplacian matrices of $\Gamma_{D_{2n}}$ are presented in the following three theorems.

Theorem 6. *Let $\Gamma_{D_{2n}}$ be the power graph for D_{2n}, then the characteristic polynomial of $NA(\Gamma_{D_{2n}})$ is*

$$P_{NA(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}\left(\lambda+\frac{1}{n-1}\right)^{n-2}\left(\lambda^3-\frac{(n-2)}{n-1}\lambda^2-\frac{n+1}{2n-1}\lambda+\frac{n(n-2)}{(n-1)(2n-1)}\right).$$

Proof. By Definition 5, we need to construct $(\sqrt{D})^{-1}(\Gamma_{D_{2n}})$. Using Equation 3, we can construct $(\sqrt{D})^{-1}(\Gamma_{D_{2n}})$ as follows:

$$(\sqrt{D})^{-1}(\Gamma_{D_{2n}}) = \begin{array}{c} \\ e \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{ccccccccc} e & a & a^2 & \dots & a^{n-1} & b & ab & \dots & a^{n-1}b \end{array} \\ \begin{pmatrix} \frac{1}{\sqrt{2n-1}} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{n-1}} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{\sqrt{n-1}} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{\sqrt{n-1}} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \end{array}. \tag{6}$$

Based on Definition 5, $NA(\Gamma_{D_{2n}})$ is a $2n \times 2n$ matrix as given below:

$$\begin{array}{c} \\ e \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{ccccccccc} e & a & a^2 & \dots & a^{n-1} & b & ab & \dots & a^{n-1}b \end{array} \\ \begin{pmatrix} 0 & \frac{1}{\sqrt{(2n-1)(n-1)}} & \frac{1}{\sqrt{(2n-1)(n-1)}} & \dots & \frac{1}{\sqrt{(2n-1)(n-1)}} & \frac{1}{\sqrt{2n-1}} & \frac{1}{\sqrt{2n-1}} & \dots & \frac{1}{\sqrt{2n-1}} \\ \frac{1}{\sqrt{(2n-1)(n-1)}} & 0 & \frac{1}{n-1} & \dots & \frac{1}{n-1} & 0 & 0 & \dots & 0 \\ \frac{1}{\sqrt{(2n-1)(n-1)}} & \frac{1}{n-1} & 0 & \dots & \frac{1}{n-1} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{(2n-1)(n-1)}} & \frac{1}{n-1} & \frac{1}{n-1} & \dots & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{\sqrt{2n-1}} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{\sqrt{2n-1}} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2n-1}} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \end{array}. \tag{7}$$

In other words, $NA(\Gamma_{D_{2n}})$ can be partitioned into nine block matrices as follows:

$$NA(\Gamma_{D_{2n}}) = \begin{bmatrix} 0 & \frac{1}{\sqrt{(2n-1)(n-1)}}J_{1\times(n-1)} & \frac{1}{\sqrt{2n-1}}J_{1\times n} \\ \frac{1}{\sqrt{(2n-1)(n-1)}}J_{(n-1)\times 1} & \frac{1}{n-1}(J-I)_{n-1} & 0_{(n-1)\times n} \\ \frac{1}{\sqrt{2n-1}}J_{n\times 1} & 0_{n\times(n-1)} & 0_n \end{bmatrix}.$$

The characteristic polynomial of $NA(\Gamma_{D_{2n}})$ is

$$P_{NA(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda & -\frac{1}{\sqrt{(2n-1)(n-1)}}J_{1\times(n-1)} & -\frac{1}{\sqrt{2n-1}}J_{1\times n} \\ -\frac{1}{\sqrt{(2n-1)(n-1)}}J_{(n-1)\times 1} & \left(\lambda+\frac{1}{n-1}\right)I_{n-1} - \frac{1}{n-1}J_{n-1} & 0_{(n-1)\times n} \\ -\frac{1}{\sqrt{2n-1}}J_{n\times 1} & 0_{n\times(n-1)} & \lambda I_n \end{vmatrix}. \tag{8}$$

We apply the following steps into Equation 8:

(i) $R_{n+1+i} \longrightarrow R_{n+1+i} - R_{n+1}$, for $i = 1, 2, \ldots, n-1$.

(ii) $C_{n+1} \longrightarrow C_{n+1} + C_{n+2} + \ldots + C_{2n}$.

(iii) $C_1 \longrightarrow C_1 + \frac{1}{\lambda\sqrt{2n-1}}C_{n+1}$.

(iv) $R_{2+i} \longrightarrow R_{2+i} - R_2$, for $i = 1, 2, \ldots, n-2$.

(v) $C_2 \longrightarrow C_2 + C_{2+1} + \ldots + C_n$,

then we get

$$P_{NA(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \frac{(2n-1)\lambda^2-n}{(2n-1)\lambda} & -\frac{n-1}{\sqrt{(2n-1)(n-1)}} & -\frac{1}{\sqrt{(2n-1)(n-1)}}J_{1\times(n-2)} & -\frac{n}{\sqrt{2n-1}} & -\frac{1}{\sqrt{2n-1}}J_{1\times n} \\ -\frac{1}{\sqrt{(2n-1)(n-1)}} & \lambda - \frac{(n-2)}{n-1} & -\frac{1}{n-1}J_{1\times(n-2)} & 0 & 0_{1\times(n-1)} \\ 0_{(n-2)\times 1} & 0_{(n-2)\times 1} & (\lambda+\frac{1}{n-1})I_{n-2} & 0_{(n-2)\times 1} & 0_{(n-2)\times(n-1)} \\ 0 & 0 & 0_{1\times(n-2)} & \lambda & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & \lambda I_{n-1} \end{vmatrix}. \tag{9}$$

By Theorem 2, we can obtain $P_{NA(\Gamma_{D_{2n}})}(\lambda)$ as follows:

$$P_{NA(\Gamma_{D_{2n}})}(\lambda) = \lambda^{n-1}\left(\lambda+\frac{1}{n-1}\right)^{n-2}\left(\lambda^3 - \frac{(n-2)}{n-1}\lambda^2 - \frac{n+1}{2n-1}\lambda + \frac{n(n-2)}{(n-1)(2n-1)}\right).$$

Theorem 7. *Let $\Gamma_{D_{2n}}$ be the power graph for D_{2n}, then the characteristic polynomial of $NL(\Gamma_{D_{2n}})$ is*

$$P_{NL(\Gamma_{D_{2n}})}(\lambda) = \lambda(\lambda-1)^{n-1}\left(\lambda-1-\frac{1}{n-1}\right)^{n-2}\left(\lambda^2 + \frac{(1-2n)}{n-1}\lambda + \frac{n(n+1)}{(2n-1)(n-1)}\right).$$

Proof. By Definition 6, and Equations 6 and 7, we can construct $NL(\Gamma_{D_{2n}})$ of the size $2n \times 2n$ as follows:

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} e & a & a^2 & \dots & a^{n-1} & b & ab & \dots & a^{n-1}b \end{array} \\
\begin{array}{c} e \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} &
\begin{pmatrix}
1 & -\frac{1}{\sqrt{(2n-1)(n-1)}} & -\frac{1}{\sqrt{(2n-1)(n-1)}} & \dots & -\frac{1}{\sqrt{(2n-1)(n-1)}} & -\frac{1}{\sqrt{2n-1}} & -\frac{1}{\sqrt{2n-1}} & \dots & -\frac{1}{\sqrt{2n-1}} \\
-\frac{1}{\sqrt{(2n-1)(n-1)}} & 1 & -\frac{1}{n-1} & \dots & -\frac{1}{n-1} & 0 & 0 & \dots & 0 \\
-\frac{1}{\sqrt{(2n-1)(n-1)}} & -\frac{1}{n-1} & 1 & \dots & -\frac{1}{n-1} & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{1}{\sqrt{(2n-1)(n-1)}} & -\frac{1}{n-1} & -\frac{1}{n-1} & \dots & 1 & 0 & 0 & \dots & 0 \\
-\frac{1}{\sqrt{2n-1}} & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\
-\frac{1}{\sqrt{2n-1}} & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{1}{\sqrt{2n-1}} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1
\end{pmatrix}
\end{array}.$$

$NL(\Gamma_{D_{2n}})$ can be partitioned into nine block matrices as follows:

$$NL(\Gamma_{D_{2n}}) = \begin{bmatrix} 1 & -\frac{1}{\sqrt{(2n-1)(n-1)}} J_{1\times(n-1)} & -\frac{1}{\sqrt{2n-1}} J_{1\times n} \\ -\frac{1}{\sqrt{(2n-1)(n-1)}} J_{(n-1)\times 1} & \left(1+\frac{1}{n-1}\right) I_{n-1} - \frac{1}{n-1} J_{n-1} & 0_{(n-1)\times n} \\ -\frac{1}{\sqrt{2n-1}} J_{n\times 1} & 0_{n\times(n-1)} & I_n \end{bmatrix}.$$

The characteristic polynomial of $NL(\Gamma_{D_{2n}})$ is

$$P_{NL(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda - 1 & \frac{1}{\sqrt{(2n-1)(n-1)}} J_{1\times(n-1)} & \frac{1}{\sqrt{2n-1}} J_{1\times n} \\ \frac{1}{\sqrt{(2n-1)(n-1)}} J_{(n-1)\times 1} & \left(\lambda - 1 - \frac{1}{n-1}\right) I_{n-1} + \frac{1}{n-1} J_{n-1} & 0_{(n-1)\times n} \\ \frac{1}{\sqrt{2n-1}} J_{n\times 1} & 0_{n\times(n-1)} & (\lambda-1) I_n \end{vmatrix}. \tag{10}$$

By applying row and column operations into Equation 10:

(i) $R_{n+1+i} \longrightarrow R_{n+1+i} - R_{n+1}$, for $i = 1, 2, \dots, n-1$.

(ii) $C_{n+1} \longrightarrow C_{n+1} + C_{n+2} + \dots + C_{2n}$.

(iii) $C_1 \longrightarrow C_1 - \frac{1}{(\lambda-1)\sqrt{2n-1}} C_{n+1}$.

(iv) $R_{2+i} \longrightarrow R_{2+i} - R_2$, for $i = 1, 2, \dots, n-2$.

(v) $C_2 \longrightarrow C_2 + C_{2+1} + \dots + C_n$,

then we get

$$P_{NL(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \frac{-n}{(\lambda-1)(2n-1)} + \lambda - 1 & \frac{n-1}{\sqrt{(2n-1)(n-1)}} & \frac{1}{\sqrt{(2n-1)(n-1)}} J_{1\times(n-2)} & \frac{n}{\sqrt{2n-1}} & \frac{1}{\sqrt{2n-1}} J_{1\times n} \\ \frac{1}{\sqrt{(2n-1)(n-1)}} & \lambda - 1 + \frac{(n-2)}{n-1} & \frac{1}{n-1} J_{1\times(n-2)} & 0 & 0_{1\times(n-1)} \\ 0_{(n-2)\times 1} & 0_{(n-2)\times 1} & (\lambda - 1 - \frac{1}{n-1}) I_{n-2} & 0_{(n-2)\times 1} & 0_{(n-2)\times(n-1)} \\ 0 & 0 & 0_{1\times(n-2)} & \lambda - 1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & (\lambda-1) I_{n-1} \end{vmatrix}. \tag{11}$$

Consequently, based on Theorem 2, we can obtain $P_{NL(\Gamma_{D_{2n}})}(\lambda)$ as follows:

$$P_{NL(\Gamma_{D_{2n}})}(\lambda) = \lambda(\lambda-1)^{n-1}\left(\lambda - 1 - \frac{1}{n-1}\right)^{n-2}\left(\lambda^2 + \frac{(1-2n)}{n-1}\lambda + \frac{n(n+1)}{(2n-1)(n-1)}\right).$$

Theorem 8. *Let $\Gamma_{D_{2n}}$ be the power graph for D_{2n}, then the characteristic polynomial of $NSL(\Gamma_{D_{2n}})$ is*

$$P_{NSL(\Gamma_{D_{2n}})}(\lambda) = (\lambda-1)^{n-1}\left(\lambda - 1 + \frac{1}{n-1}\right)^{n-2}\left(\lambda^3 + \frac{(5-4n)}{n-1}\lambda^2 + \frac{(9n^2-19n+8)}{(2n-1)(n-1)}\lambda - \frac{2(n-2)}{2n-1}\right).$$

Proof. By Definition 7, and Equations 6 and 7, we can construct $NSL(\Gamma_{D_{2n}})$ of the size $2n \times 2n$ as given below:

$$\begin{array}{c} \\ e \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} e & a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \end{array} \\
\begin{pmatrix}
1 & \frac{1}{\sqrt{(2n-1)(n-1)}} & \frac{1}{\sqrt{(2n-1)(n-1)}} & \cdots & \frac{1}{\sqrt{(2n-1)(n-1)}} & \frac{1}{\sqrt{2n-1}} & \frac{1}{\sqrt{2n-1}} & \cdots & \frac{1}{\sqrt{2n-1}} \\
\frac{1}{\sqrt{(2n-1)(n-1)}} & 1 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 & 0 & \cdots & 0 \\
\frac{1}{\sqrt{(2n-1)(n-1)}} & \frac{1}{n-1} & 1 & \cdots & \frac{1}{n-1} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{\sqrt{(2n-1)(n-1)}} & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & 1 & 0 & 0 & \cdots & 0 \\
\frac{1}{\sqrt{2n-1}} & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\frac{1}{\sqrt{2n-1}} & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{\sqrt{2n-1}} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\end{array}.$$

In other words, $NSL(\Gamma_{D_{2n}})$ can be partitioned into nine block matrices as follows:

$$NSL(\Gamma_{D_{2n}}) = \begin{bmatrix} 1 & \frac{1}{\sqrt{(2n-1)(n-1)}}J_{1\times(n-1)} & \frac{1}{\sqrt{2n-1}}J_{1\times n} \\ \frac{1}{\sqrt{(2n-1)(n-1)}}J_{(n-1)\times 1} & \left(1 - \frac{1}{n-1}\right)I_{n-1} + \frac{1}{n-1}J_{n-1} & 0_{(n-1)\times n} \\ \frac{1}{\sqrt{2n-1}}J_{n\times 1} & 0_{n\times(n-1)} & I_n \end{bmatrix}.$$

The characteristic polynomial of $NSL(\Gamma_{D_{2n}})$ is

$$P_{NSL(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda - 1 & -\frac{1}{\sqrt{(2n-1)(n-1)}}J_{1\times(n-1)} & -\frac{1}{\sqrt{2n-1}}J_{1\times n} \\ -\frac{1}{\sqrt{(2n-1)(n-1)}}J_{(n-1)\times 1} & \left(\lambda - 1 + \frac{1}{n-1}\right)I_{n-1} - \frac{1}{n-1}J_{n-1} & 0_{(n-1)\times n} \\ -\frac{1}{\sqrt{2n-1}}J_{n\times 1} & 0_{n\times(n-1)} & (\lambda-1)I_n \end{vmatrix}. \tag{12}$$

We apply the row and column operations to Equation 12:

(i) $R_{n+1+i} \longrightarrow R_{n+1+i} - R_{n+1}$, for $i = 1, 2, \ldots, n-1$.

(ii) $C_{n+1} \longrightarrow C_{n+1} + C_{n+2} + \ldots + C_{2n}$.

(iii) $C_1 \longrightarrow C_1 + \frac{1}{(\lambda-1)\sqrt{2n-1}} C_{n+1}$.

(iv) $R_{2+i} \longrightarrow R_{2+i} - R_2$, for $i = 1, 2, \ldots, n-2$.

(v) $C_2 \longrightarrow C_2 + C_{2+1} + \ldots + C_n$,

consequently we have

$$P_{NSL(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \frac{-n}{(\lambda-1)(2n-1)} + \lambda - 1 & -\frac{n-1}{\sqrt{(2n-1)(n-1)}} & -\frac{1}{\sqrt{(2n-1)(n-1)}} J_{1\times(n-2)} & -\frac{n}{\sqrt{2n-1}} & -\frac{1}{\sqrt{2n-1}} J_{1\times n} \\ -\frac{1}{\sqrt{(2n-1)(n-1)}} & \lambda - 1 - \frac{(n-2)}{n-1} & -\frac{1}{n-1} J_{1\times(n-2)} & 0 & 0_{1\times(n-1)} \\ 0_{(n-2)\times 1} & 0_{(n-2)\times 1} & (\lambda - 1 + \frac{1}{n-1}) I_{n-2} & 0_{(n-2)\times 1} & 0_{(n-2)\times(n-1)} \\ 0 & 0 & 0_{1\times(n-2)} & \lambda - 1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & (\lambda-1) I_{n-1} \end{vmatrix}.$$

According to Theorem 2, we can obtain $P_{NSL(\Gamma_{D_{2n}})}(\lambda)$ as follows:

$$P_{NSL(\Gamma_{D_{2n}})}(\lambda) = (\lambda-1)^{n-1} \left(\lambda - 1 + \frac{1}{n-1}\right)^{n-2} \left(\lambda^3 + \frac{(5-4n)}{n-1}\lambda^2 + \frac{(9n^2 - 19n + 8)}{(2n-1)(n-1)}\lambda - \frac{2(n-2)}{2n-1}\right).$$

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