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Certified Vertex Cover of a Graph

Javier A. Hassan[1,*], Maria Andrea O. Bonsocan[2], Regimar A. Rasid[1], Amil-Shab S. Sappari[1]

[1] *Mathematics and Sciences Department, College of Arts and Sciences, MSU Tawi-Tawi College of Technology and Oceanography, Bongao, Tawi-Tawi, Philippines*
[2] *Department of Mathematics, Ateneo de Davao University, E. Jacinto Street, 8016 Davao City, Philippines*

Abstract. Let G be a graph. Then $Q \subseteq V(G)$ is called a certified vertex cover of G if Q is a vertex cover of G and every $x \in Q$, x has either zero or at least two neighbors in $V(G) \setminus Q$. The certified vertex cover number of G, denoted by $\beta_{cer}(G)$, is the minimum cardinality of a certified vertex cover of G. In this paper, we investigate this newly defined concept on some special graphs and on the join of two graphs. We characterize certified vertex cover in these graphs and subsequently derive the simplified formulas for calculating the certified vertex cover number. Moreover, we present some bounds and properties of certified vertex cover.

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1. Introduction

Vertex cover of a graph is an important concept in graph theory. A vertex cover of a graph is a subset of vertices that covers all the edges in the graph. In other words, every edge in the graph is incident to at least one vertex in the vertex cover. The concept of vertex covering set has practical applications in various fields, such as network design, optimization, and resource allocation. The size of the smallest vertex cover is called the vertex cover number of the graph. Finding the minimum vertex cover or the vertex cover number of a graph is an important problem in graph theory. In practical scenarios, finding a minimum vertex cover helps in minimizing costs or maximizing efficiency. Researchers had studied vertex cover parameter and its variants on different types of graphs (see [1, 12, 13]).

*Corresponding author.
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Email addresses: `javierhassan@msutawi-tawi.edu.ph` (J. A. Hassan)
`mariaandrea.bonsocan@g.msuiit.edu.ph` (M. A. Bonsocan)
`regimarrasid@msutawi-tawi.edu.ph` (R. A. Rasid)
`amilshabsappari@msutawi-tawi.edu.ph` (A. S. Sappari)

Recently, V. Bilar et al. [1] introduced and investigated another variant of a vertex cover called vertex cover hop domination. Let G be a graph. Then a subset C of a vertex-set of G is called a vertex cover hop dominating if C is both a vertex cover and a hop dominating of G. The vertex cover hop domination number of G, denoted by $\gamma_{vch}(G)$, is the minimum cardinality among all vertex cover hop dominating sets in G. They have determined its relations with other parameters in graph theory such as vertex cover, hop domination parameter. Moreover, they have characterized the vertex cover hop dominating sets in some special graphs, join, and corona of two graphs, and obtained the exact values or bounds of the parameter of these graphs. Some studies related to vertex cover hop domination can be found in [2–11].

In this study, we initiate the study of certified vertex cover of a graph. We do believe, this parameter and its results would lead to another interesting studies and applications in the future. Further, this study would serve as reference to future researchers who will study on concept or problem related to vertex cover of a graph.

2. Terminology and Notation

Let $G = (V(G), E(G))$ be a simple and undirected graph. Then $S \subseteq V(G)$ is called a vertex cover of G if every edge in G is incident to some vertex in S. The minimum cardinality of a vertex cover of G, denoted by $\beta(G)$, is called the vertex cover number of G.

A set $Q \subseteq V(G)$ is called a certified set of G if every vertex $x \in Q$, x has either zero or at least two neighbors in $V(G) \setminus Q$.

Let G and H be any two graphs. The *join* of G and H, denoted by $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

3. Results

We begin this section by introducing the concept of certified vertex cover of a graph.

Definition 1. Let G be a graph. Then $Q \subseteq V(G)$ is called a certified vertex cover of G if Q is a vertex cover of G and every $x \in Q$, x has either zero or at least two neighbors in $V(G) \setminus Q$. The certified vertex cover number of G, denoted by $\beta_{cer}(G)$, is the minimum cardinality of a certified vertex cover of G.

Example 1. Consider the graph G below.

Let $Q = \{u_4, u_5, u_6, u_7\}$. Then vertex u_4 has three neighbors outside Q and vertices u_5, u_6 and u_7 have zero neighbor outside Q. It follows that Q is a certified vertex cover of G. Observe that every edge of G is incident to some vertex in Q. Thus, Q is a vertex cover of G, showing that Q is a certified vertex cover of G. Moreover, it can be verified that $\beta_{cer}(G) = 4$.

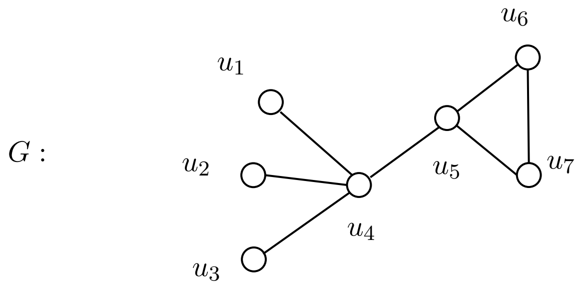


Figure 1: Graph G with $\beta_{cer}(G) = 4$

Proposition 1. *Let G be a graph. Then*

(i) $\beta(G) \leq \beta_{cer}(G)$;

(ii) $1 \leq \beta_{cer}(G) \leq |V(G)|$;

(iii) a certified set of G may not be a vertex cover of G;

(iv) a vertex cover of G may not be a certified set of G; and

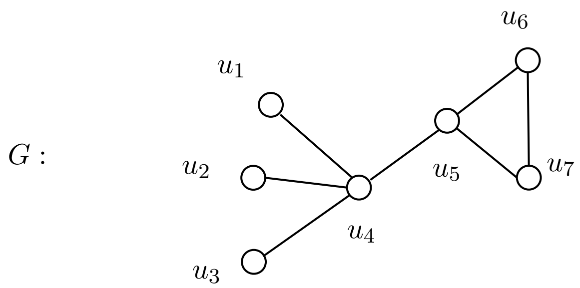
(v) if $Q \subseteq V(G)$ is both certified and minimum vertex cover of G, then Q is a minimum certified vertex cover of G and $\beta_{cer}(G) = |Q|$.

Proof. (i) Let G be a graph and let Q be a minimum certified vertex cover of G. Then Q is vertex cover of G. Since $\beta(G)$ is the smallest cardinality of a vertex cover of G, it follows that $\beta(G) \leq |Q| = \beta_{cer}(G)$.

(ii) Since $\beta(G) \geq 1$ for any graph G, $\beta_{cer}(G) \geq 1$ by (i). Clearly, $\beta_{cer}(G) \leq |V(G)|$. Therefore,

$$1 \leq \beta_{cer}(G) \leq |V(G)|.$$

(iii) Consider again the graph G below:



Let $Q = \{u_4, u_5\}$. Then u_4 and u_5 have three and two neighbors in $V(G) \setminus Q$, respectively. It follows that Q is a certified set of G. However, Q is not a vertex cover of G since edge u_6u_7 is not incident to either u_4 or u_5. Hence, the assertion follows.

(iv) Consider again the graph G in (iii) and let $S = \{u_4, u_5, u_6\}$. Then every edge of G is incident to some elements of S, showing that S is a vertex cover of G. Now, observe that vertex u_5 and u_6 have only one neighbor u_7 outside S. It follows that S is not a certified set of G. Hence, (iv) holds.

(v) Let Q be both certified and minimum vertex cover of G. Suppose that Q is not a minimum certified vertex cover of G. Then there exists $T \subset V(G)$ such that T is a certified vertex cover and $|Q| > |T|$. Note that T is a vertex cover of G, a contradiction to the fact Q is a minimum vertex cover of G. Therefore, Q is a minimum certified vertex cove of G, and so $\beta_{cer}(G) = |Q|$. □

Lemma 1. *Let n be a positive integer and let Q be a certified vertex cover of P_n, where $V(P_n) = \{v_1, v_2, ..., v_n\}$. Then $Q = V(P_n)$ if and only if one of the following holds:*

(i) $v_i, v_j \in Q$ *where* $d_{P_n}(v_i, v_j) = 1$ *for some* $i, j \in \{1, 2, \dots, n\}$.

(ii) $v_1 \in Q$.

(iii) $v_n \in Q$.

Proof. Suppose that Q is a certified vertex cover of P_n. If $Q = V(P_n) = \{v_1, v_2, \dots, v_n\}$, then (i),(ii) and (iii) follow.

Conversely, suppose that (i) holds. That is, $v_i, v_j \in Q$ such that $d_{P_n}(v_i, v_j) = 1$ for some $i, j \in \{1, 2, \dots, n\}$. Assume that $i < j$. If $j = n$, then v_i has only one neighbor v_{i-1} in $V(P_n)\backslash Q$. Since Q is a certified set, v_{i-1} must be in Q. If v_{i-1} is in Q, then applying the same argument, v_{i-2} must also be in Q. Continuing this process, all other vertices in $V(P_n)\backslash Q$ must be in Q. Hence, $Q = V(P_n)$.

Suppose that $j \neq n$. Since $d_{P_n}(v_i, v_j) = 1$, both v_i and v_j have only one neighbor v_{i-1} and v_{j+1} in $V(P_n)\backslash Q$, respectively. Applying the same argument, all other vertices in $V(P_n)\backslash Q$ must also be in Q. Thus, $Q = V(P_n)$. Similarly, the same result follows when $i > j$.

Now, suppose that (ii) holds, that is, $v_1 \in Q$. Observe that v_1 has only one neighbor in $V(P_n) \setminus Q$, which is v_2. Since Q is a certified set in P_n, v_2 must be in Q. However, v_2 has only one neighbor in $V(P_n) \setminus Q$, which is v_3. Since Q is a certified set in P_n, v_3 must also be in Q. Continuing this process, all other vertices in $V(P_n)\backslash Q$ must be included as elements of Q. Therefore, $Q = V(P_n)$. Similarly, the assertion follows when (iii) holds. □

The following Lemma can be proved similarly.

Lemma 2. *Let n be a positive integer and let Q be a certified vertex cover of C_n, where $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Then $Q = V(C_n)$ if and only if $v_i, v_j \in Q$ such that*

$$d_{C_n}(v_i, v_j) = 1.$$

Theorem 1. *Let n be a positive integer. Then* $\beta_{cer}(P_n) = \begin{cases} n & , \quad n = 1 \text{ or even} \\ \lfloor \frac{n}{2} \rfloor & , \text{ otherwise.} \end{cases}$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Clearly, $\beta_{cer}(P_1) = 1$, $\beta_{cer}(P_2) = 2$ and $\beta_{cer}(P_4) = 4$. Suppose that $n \geq 6$ and even. Let Q be a certified vertex cover of P_n. If $v_1, v_n \in Q$ or $v_i, v_j \in Q$, where $d_{P_n}(v_i, v_j) = 1$, then $Q = V(P_n)$ by Lemma 1, and the proof is complete.

Now, suppose that $v_1, v_n \notin Q$ and $d_{P_n}(v_s, v_t) \geq 2$ for every $s, t \in \{2, \ldots, n–1\}$. Since Q is a vertex cover of P_n, it follows that either $\{v_2, v_4, \ldots, v_{n-2}, v_{n-1}\} \subseteq Q$ or $\{v_2, v_4, \ldots, v_{n-2}, v_n\} \subseteq Q$. By Lemma 1, either of this case, we have $Q = V(P_n)$. Therefore, $\beta_{cer}(P_n) = n$ for all $n \geq 6$ and even.

Next, suppose that $n \geq 5$ and odd. Let $S = \{v_2, v_4, \ldots, v_{n-1}\}$. Then S is a minimum vertex cover of P_n. Clearly, S is a certified set of P_n. Therefore, $\beta_{cer}(P_n) = |S| = \lfloor \frac{n}{2} \rfloor$ since n is odd. □

Theorem 2. *Let n be a positive integer. Then* $\beta_{cer}(C_n) = \begin{cases} n & , \text{ if } n \text{ is odd} \\ \frac{n}{2} & , \text{ if } n \text{ is even.} \end{cases}$

Proof. Let n be a positive integer and V$(C_n) = \{v_1, v_2, \ldots, v_n\}$. Let $n = 3$. Then $\beta(C_3) = 2$, and so $\beta_{cer}(C_3) \geq 2$ by Proposition 1. Suppose that $\beta_{cer}(C_3) = 2$. Then there exists $v_i \in V(C_3) \backslash N$ for some $i \in \{1, 2, 3\}$, where N is a certified vertex cover of C_3. Assume that $i = 1$. Then both $v_2, v_3 \in N$ have only one neighbor v_1 outside N, a contradiction. Similarly, when $i = 2$ or $i = 3$. Therefore, $\beta_{cer}(C_3) = 3$.

Now, suppose that $n \geq 5$ and odd. Let Q be a certified vertex cover of C_n and let $v_i, v_j \in Q$ for some $i, j \in \{1, 2, \ldots, n\}$. If $d_{C_n}(v_i, v_j) = 1$, then by Lemma 2, $Q = V(C_n)$ and we are done. Now, since Q is a vertex cover, $d_{C_n}(v_i, v_j) < 3$. Thus, $d_{C_n}(v_i, v_j) = 2$. WLOG, assume that $v_1 \in Q$. Then $v_1, v_3, \ldots, v_{n-2} \in Q$. Since Q is vertex cover, either v_{n-1} or v_n must be in Q. If $v_{n-1} \in Q$, then $v_1, v_3, \ldots, v_{n-2}, v_{n-1} \in Q$. Since $d_{C_n}(v_{n-1}, v_{n-2}) = 1$, it follows that $Q = V(C_n)$ by Lemma 2. Similarly, when $v_n \in Q$, then $Q = V(C_n)$. Therefore, $\beta_{cer}(C_n) = n$ for all $n \geq 3$ and odd.

Next, suppose that $n \geq 4$ and even. Let $Q_1 = \{v_1, v_3, \ldots, v_{n-1}\}$. Then Q_1 is a minimum vertex cover of C_n. Clearly, Q_1 is a certified set of C_n. Since n is even, it follows that $\beta_{cer}(C_n) = |Q_1| = \frac{n}{2}$ for all $n \geq 4$ and even. □

Theorem 3. *Let m be a positive integer. Then N is a certified vertex cover of K_m if and only if $N = V(K_m)$.*

Proof. Let N be a certified vertex cover of K_m. Then N is a vertex cover of K_m by definition. Thus, $\beta_{cer}(K_m) \geq m-1$ since $\beta(G) \leq \beta_{cer}(G)$ for any graph G. If $\beta_{cer}(G) = m$, then $N = V(K_m)$, and we are done. Suppose that $\beta_{cer}(K_m) = m-1$. Then there exists a unique $x \in V(K_m)$ such that $x \notin N$. However, each vertex in N has only one neighbor x outside N, which is a contradiction to the fact that N is a certified set of K_m. Therefore, $N = V(K_m)$ for all $m \geq 1$.

The converse is clear. □

Corollary 1. *Let m be a positive integer. Then $\beta_{cer}(K_m) = m$.*

Theorem 4. *Let G and H be two graphs with no trivial components. Then $Q \subseteq V(G+H)$ is a certified vertex cover of $G+H$ if and only if $Q = Q_G \cup Q_H$ and satisfies one of the following conditions:*

(i) $Q_G = V(G)$ and Q_H is a certified vertex cover of H.

(ii) $Q_H = V(H)$ and Q_G is a certified vertex cover of G.

Proof. Suppose that Q is a certified vertex cover of $G+H$. If $Q = V(G+H)$, then we are done. Assume that $Q \neq V(G+H)$. Since Q is a vertex cover of $G+H$, either $Q_G = V(G)$ and $Q_H \neq V(H)$ or $Q_G \neq V(G)$ and $Q_H = V(H)$. Suppose that $Q_G = V(G)$ and $Q_H \neq V(H)$. Since Q is a certified vertex cover of $G+H$, Q_H must be a certified vertex cover of H. Thus, (i) holds. Similarly, when $\mathrm{Q}_G \neq V(G)$ and $Q_H = V(H)$, then (ii) holds.

Conversely, suppose that (i) holds. Let $x \in Q$. Then either $x \in Q_G = V(G)$ or $x \in Q_H \subseteq V(H)$. Assume that $x \in Q_G = V(G)$. Since H has no trivial components and $Q_H \subseteq V(H)$ is a certified set in H, it follows that x has either zero or at least two neighbors in $V(H) \setminus Q_H$. Since x is arbitrary, Q is a certified set in $G+H$. Since Q_H is a vertex cover of H, $Q = V(G) \cup Q_H$ is a vertex cover of $G+H$. Therefore, Q is a certified vertex cover of $G+H$. Similarly, the assertion follows when $x \in Q_H \subseteq V(H)$. The assertion also follows, when (ii) holds. □

Corollary 2. *Let G and H be graphs with no trivial components. Then*

$$\beta_{cer}(G+H) = min\{|V(G)| + \beta_{cer}(H), |V(H)| + \beta_{cer}(G)\}.$$

Theorem 5. *Let G be a trivial graph and H be a graph with no trivial components. Then $Q \subseteq V(G+H)$ is a certified vertex cover of $G+H$ if and only if $Q = Q_G \cup Q_H$, where $Q_G = V(G)$ and Q_H is a certified vertex cover of H.*

Proof. Suppose that Q is a certified vertex cover of $G+H$. If $Q = V(G+H)$, then we are done. Assume that $Q \neq V(G+H)$. Since Q is a vertex cover of $G+H$, either $Q_G = V(G)$ and $Q_H \neq V(H)$ or $Q_G \neq V(G)$ and $Q_H = V(H)$. Since G is trivial and Q is a certified set in $G+H$, $Q_G \neq V(G)$ and $Q_H = V(H)$ is not possible. Hence, $Q_G = V(G)$ and $Q \neq V(H)$. Since Q is a certified vertex cover of $G+H$, Q_H must be a certified vertex cover of H.

Conversely, suppose that $Q = Q_G \cup Q_H$, where $Q_G = V(G)$ and Q_H is a certified vertex cover of H. Let $y \in Q$. Then either $y \in Q_G = V(G)$ or $y \in Q_H \subseteq V(H)$. Assume that $y \in Q_G = V(G)$. Since H has no trivial components and $Q_H \subseteq V(H)$ is a certified set in H, it follows that y has either zero or at least two neighbors in $V(H) \setminus Q_H$. Since y is arbitrary, Q is a certified set in $G+H$. Since Q_H is a vertex cover of H, it follows that $Q = V(G) \cup Q_H$ is a certified vertex cover of $G+H$. Similarly, the assertion follows when $y \in Q_H \subseteq V(H)$. □

Corollary 3. *Let G be a trivial graph and H be a graph with no trivial components. Then*

$$\beta_{cer}(G+H) = \beta_{cer}(H) + 1$$

In particular, each of the following holds:

(i) $\beta_{cer}(F_n) = \begin{cases} n+1 & , \text{if } n \geq 2 \text{ and even} \\ \lfloor \frac{n}{2} \rfloor + 1 & , \text{if } n \geq 3 \text{ and odd.} \end{cases}$

(ii) $\beta_{cer}(W_n) = \begin{cases} n+1 & , \text{ if } n \text{ is odd} \\ \frac{n}{2} + 1 & , \text{ if } n \text{ is even.} \end{cases}$

4. Conclusion

The concept of certified vertex cover has been introduced and initially investigated in this study. Defining the concept introduces a new idea in graph theory. The minimum cardinality of a certified vertex cover of some graphs has been determined in this study. Additionally, characterizations of vertex covering sets of certain graphs have aided in determining the exact values of parameters of some graphs. Exploring graphs that haven't been addressed in this study could prove to be interesting, offering a fresh perspective on the concept. Investigating these unexplored graphs might reveal new insights and provide a deeper understanding of the concept. Moreover, interested researchers may study the complexity and algorithms of solving the certified vertex cover number of a graph.

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