EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 4, No. 1, 2011, 76-82
ISSN 1307-5543 – www.ejpam.com

On Certain Sufficient Conditions for Analytic Univalent Functions

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Abstract. In this paper, we introduce a new class $B_m^l(\alpha,\delta)$ of functions which is defined by hypergeometric function and obtain its relations with some well-known subclasses of analytic univalent functions. Furthermore, as a special case, we show that convex functions of order $1/2$ are also members of the family $B_m^l(\alpha,\delta)$.

2000 Mathematics Subject Classifications: 30C45

Key Words and Phrases: Univalent functions, starlike functions, convex functions, Hadamard product, generalized hypergeometric functions.

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1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in the open disc $U = \{z : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. We denote by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of $\mathscr{A}$ consisting of all functions which are, respectively starlike and convex of order α. Thus,

$$S^*(\alpha) = \left\{ f \in \mathscr{A} : Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ 0 \leq \alpha < 1,\ z \in U \right\}$$

and

$$K(\alpha) = \left\{ f \in \mathscr{A} : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ 0 \leq \alpha < 1,\ z \in U \right\}.$$

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We notice that $K(\alpha) \subset S^*(\alpha) \subset \mathscr{A}$. Further,

$$R(\alpha) = \left\{f \in \mathscr{A} : Re\left(f\prime(z)\right) > \alpha,\ 0 \leq \alpha < 1,\ z \in U\right\}.$$

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

For functions $\Phi \in \mathscr{A}$ given by $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \phi_n z^n$ and $\Psi \in \mathscr{A}$ given by $\Psi(z) = z + \sum\limits_{n=2}^{\infty} \psi_n z^n$, we define the Hadamard product (or Convolution) of Φ and Ψ by

$$(\Phi * \Psi)(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n,\ z \in U. \tag{2}$$

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ $(\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$ the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \quad &:= \quad \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \\ (l \leq m+1;\ l, m \in N_0 \quad &:= \quad N \cup \{0\}; z \in U) \end{aligned} \tag{3}$$

where N denotes the set of all positive integers and $(\alpha)_n$ is the Pochhammer symbol defined by

$$(\alpha)_n = \begin{cases} 1, & n = 0 \\ \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1), & n \in N. \end{cases} \tag{4}$$

Let $H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) : \mathscr{A} \to \mathscr{A}$ be a linear operator defined by

$$\begin{aligned} [(H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m))(f)](z) \quad &:= \quad z\ {}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2 \ldots, \beta_m; z) * f(z) \\ &= \quad z + \sum_{n=2}^{\infty} \Gamma_n\ a_n z^n \end{aligned} \tag{5}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(n-1)!(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}}\ . \tag{6}$$

For notational simplicity, we can use a shorter notation $H_m^l[\alpha_1]$ for $H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m)$ in the sequel. The linear operator $H_m^l[\alpha_1]$ is called Dziok-Srivastava operator (see [3]), includes (as its special cases) various other linear operators introduced and studied by Bernardi [1], Carlson and Shaffer [2], Libera [6], Livingston [7], Ruscheweyh [8] and Srivastava-Owa [9].

For $0 \leq \alpha < 1$ and $\delta \geq 0$, let $B_m^l(\alpha, \delta)$ consisting of functions of the form (1) and satisfying the condition

$$\left| \frac{H_m^l[\alpha_1+1]f(z)}{z} \left(\frac{z}{H_m^l[\alpha_1]f(z)}\right)^{\delta} - 1 \right| < 1 - \alpha,\ z \in U. \tag{7}$$

The class $B^l_m(\alpha,\delta)$ is a unified class of analytic functions which includes various new subclasses of analytic univalent functions. We observe that

Example 1. *If $l=2$ and $m=1$ with $\alpha_1=1$, $\alpha_2=1$, $\beta_1=1$ then*

$$B^2_1(\alpha,\delta):=\left\{f\in\mathscr{A}:\left|f\prime(z)\left(\frac{z}{f(z)}\right)^{\delta}-1\right|<1-\alpha,\ \delta\geq 0,\ 0\leq\alpha<1,\ z\in U.\right\}$$

The class $B^2_1(\alpha,\delta)$ has been studied by Frasin and Jahangiri [5]. Further $B^2_1(\alpha,2)$ has been studied by Frasin and Darus [4]. Also we note that $B^2_1(\alpha,1)\equiv S^*(\alpha)$ and $B^2_1(\alpha,0)\equiv R(\alpha)$.

Example 2. *If $l=2$ and $m=1$ with $\alpha_1=\eta+1\,(\eta>-1)$, $\alpha_2=1$, $\beta_1=1$, then*

$$B(\eta,\alpha,\delta):=\left\{f\in\mathscr{A}:\left|\frac{D^{\eta+1}f(z)}{z}\left(\frac{z}{D^{\eta}f(z)}\right)^{\delta}-1\right|<1-\alpha,\ \eta>-1,\ \delta\geq 0,\ 0\leq\alpha<1,\ z\in U.\right\},$$

where $D^{\eta}f(z)$ is called Ruscheweyh derivative operator [8] defined by

$$D^{\eta}f(z):=\frac{z}{(1-z)^{\eta+1}}*f(z)\equiv H^2_1(\eta+1,1;1)f(z).$$

Also we observe that $B(0,\alpha,1)\equiv K(\alpha)$.

Example 3. *If $l=2$ and $m=1$ with $\alpha_1=\mu+1(\mu>-1)$, $\alpha_2=1$, $\beta_1=\mu+2$, then*

$$B(\mu,\alpha,\delta):=\left\{f\in\mathscr{A}:\left|\frac{J_{\mu+1}f(z)}{z}\left(\frac{z}{J_{\mu}f(z)}\right)^{\delta}-1\right|<1-\alpha,\ \mu>-1,\ \delta\geq 0,\ 0\leq\alpha<1,\ z\in U\right\},$$

where J_μ is a Bernardi operator [1] defined by

$$J_{\mu}f(z):=\frac{\mu+1}{z^{\mu}}\int_0^z t^{\mu-1}f(t)dt\equiv H^2_1(\mu+1,1;\mu+2)f(z).$$

Note that the operator J_1 was studied earlier by Libera [6] and Livingston [7].

Example 4. *If $l=2$ and $m=1$ with $\alpha_1=a\,(a>0)$, $\alpha_2=1$, $\beta_1=c\,(c>0)$, then*

$$B(a,c,\alpha,\delta):=\left\{f\in\mathscr{A}:\left|\frac{L(a+1,c)f(z)}{z}\left(\frac{z}{L(a,c)f(z)}\right)^{\delta}-1\right|<1-\alpha,\ \delta\geq 0,\ 0\leq\alpha<1,\ z\in U\right\},$$

where $L(a,c)$ is a well-known Carlson-Shaffer linear operator [2] defined by

$$L(a,c)f(z):=\left(\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}z^{k+1}\right)*f(z)\equiv H^2_1(a,1;c)f(z).$$

The object of the present paper is to investigate the sufficient condition for functions to be in the class $B^l_m(\alpha,\delta)$. Furthermore, as a special case, we show that convex functions of order 1/2 are also members of the family $B^l_m(\alpha,\delta)$.

2. Main Results

To prove our results we need the following lemma.

Lemma 1. *[5] Let p be analytic in U with $p(0) = 1$ and suppose that*

$$Re \left\{1 + \frac{zp\prime(z)}{p(z)}\right\} > \frac{3\alpha - 1}{2\alpha}. \tag{8}$$

Then Re $\{p(z)\} > \alpha$ for $z \in U$ and $\frac{1}{2} \leq \alpha < 1$.

Using Lemma 1, we first prove the following theorem.

Theorem 1. *Let $f(z)$ be the functions of the form (1), $\delta \geq 0$ and $\frac{1}{2} \leq \alpha < 1$. If*

$$(\alpha_1 + 1)\frac{H^l_m[\alpha_1 + 2]f(z)}{H^l_m[\alpha_1 + 1]f(z)} - \delta\alpha_1\frac{H^l_m[\alpha_1 + 1]f(z)}{H^l_m[\alpha_1]f(z)} + \alpha_1(\delta - 1) \prec 1 + \beta z, \tag{9}$$

where $\beta = \frac{3\alpha-1}{2\alpha}$, then $f(z) \in B^l_m(\alpha, \delta)$.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{H^l_m[\alpha_1 + 1]f(z)}{z}\left(\frac{z}{H^l_m[\alpha_1]f(z)}\right)^{\delta} \tag{10}$$

Then the function $p(z)$ is analytic in U and $p(0) = 1$. Therefore, differentiating (10) logarithmically and the simple computation yields

$$\frac{zp\prime(z)}{p(z)} = (\alpha_1 + 1)\frac{H^l_m[\alpha_1 + 2]f(z)}{H^l_m[\alpha_1 + 1]f(z)} - \delta\alpha_1\frac{H^l_m[\alpha_1 + 1]f(z)}{H^l_m[\alpha_1]f(z)} + \alpha_1(\delta - 1) - 1.$$

By the hypothesis of the theorem, we have

$$\text{Re}\left\{1 + \frac{zp\prime(z)}{p(z)}\right\} > \frac{3\alpha - 1}{2\alpha}.$$

Hence by Lemma 1, we have

$$\text{Re}\left\{\frac{H^l_m[\alpha_1 + 1]f(z)}{z}\left(\frac{z}{H^l_m[\alpha_1]f(z)}\right)^{\delta}\right\} > \alpha,\ z \in U.$$

Therefore in view of definition $f(z) \in B^l_m(\alpha, \delta)$.

For $l = 2$ and $m = 1$ with $\alpha_1 = a\,(a > 0)$, $\alpha_2 = 1$, $\beta_1 = c\,(c > 0)$, we obtain the following corollary.

Corollary 1. *Let* $\frac{1}{2} \leq \alpha < 1$. *If* $f \in \mathscr{A}$ *and*

$$Re\left\{(a+1)\frac{L[a+2,c]f(z)}{L[a+1,c]f(z)} - \delta a\frac{L[a+1,c]f(z)}{L[a,c]f(z)} + a(\delta-1)\right\} > \frac{3\alpha-1}{2\alpha},\ z \in U,$$

then

$$Re\left\{\frac{L[a+1,c]f(z)}{z}\left(\frac{z}{L[a,c]f(z)}\right)^{\delta}\right\} > \alpha,\ z \in U.$$

Therefore $f(z) \in B(a,c,\alpha,\delta)$.

Taking $l=2$ and $m=1$ with $\alpha_1 = \mu+1 (\mu > -1)$, $\alpha_2 = 1$, $\beta_1 = \mu+2$, we get

Corollary 2. *Let* $\frac{1}{2} \leq \alpha < 1$. *If* $f \in \mathscr{A}$ *and*

$$Re\left\{(\mu+2)\frac{J_{\mu+2}f(z)}{J_{\mu+1}f(z)} - \delta(\mu+1)\frac{J_{\mu+1}f(z)}{J_{\mu}f(z)} + (\mu+1)(\delta-1)\right\} > \frac{3\alpha-1}{2\alpha},\ z \in U,$$

then

$$Re\left\{\frac{J_{\mu+1}f(z)}{z}\left(\frac{z}{J_{\mu}f(z)}\right)^{\delta}\right\} > \alpha,\ z \in U.$$

Therefore $f(z) \in B(\mu,\alpha,\delta)$.

Choosing $l=2$ and $m=1$ with $\alpha_1 = \eta+1\,(\eta > -1)$, $\alpha_2 = 1$, $\beta_1 = 1$, we have

Corollary 3. *Let* $\frac{1}{2} \leq \alpha < 1$. *If* $f \in \mathscr{A}$ *and*

$$Re\left\{(\eta+2)\frac{D^{\eta+2}f(z)}{D^{\eta+1}f(z)} - \delta(\eta+1)\frac{D^{\eta+1}f(z)}{D^{\eta}f(z)} + (\eta+1)(\delta-1)\right\} > \frac{3\alpha-1}{2\alpha},\ z \in U,$$

then

$$Re\left\{\frac{D^{\eta+1}f(z)}{z}\left(\frac{z}{D^{\eta}f(z)}\right)^{\delta}\right\} > \alpha,\ z \in U.$$

Therefore $f(z) \in B(\eta,\alpha,\delta)$.

Choosing $l=2$ and $m=1$ with $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$, we have

Corollary 4. *[5] If* $f \in \mathscr{A}$ *and*

$$Re\left\{1+\frac{zf''(z)}{f'(z)} + \delta\left(1-\frac{zf'(z)}{f(z)}\right)\right\} > \frac{3\alpha-1}{2\alpha},\ z \in U,$$

then

$$Re\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\delta}\right\} > \alpha,\ z \in U.$$

Therefore $f(z) \in B_1^2(\alpha,\delta)$.

Choosing $l = 2$ and $m = 1$ with $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 1$, $\delta = 1$ and $\alpha = \frac{1}{2}$ we have

Corollary 5. *If $f \in \mathscr{A}$ and*

$$Re\left\{\frac{z^2 f''' + 6zf''(z) + 6f'(z)}{2f'(z) + zf''} - \frac{zf''(z)}{f'(z)}\right\} > \frac{3}{2},\ z \in U,$$

then

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0,\ z \in U.$$

That is, $f(z) \in K$.

Choosing $l = 2$ and $m = 1$ with $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 1$, $\delta = 0$ and $\alpha = \frac{1}{2}$ we have

Corollary 6. *If $f \in \mathscr{A}$ and*

$$Re\left\{\frac{z^2 f''' + 4zf''(z) + 2f'(z)}{2f'(z) + zf''}\right\} > \frac{1}{2},\ z \in U,$$

then

$$Re\left\{f'(z) + \frac{zf''(z)}{2}\right\} > \frac{1}{2},\ z \in U.$$

Choosing $l = 2$ and $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$, $\delta = 1$ and $\alpha = \frac{1}{2}$ we have

Corollary 7. *If $f \in \mathscr{A}$ and*

$$Re\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > \frac{-3}{2},\ z \in U,$$

then

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2},\ z \in U.$$

That is, $f(z)$ is starlike of order $1/2$.

Choosing $l = 2$ and $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$, $\delta = 0$ and $\alpha = \frac{1}{2}$ we have

Corollary 8. *If $f \in \mathscr{A}$ and*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2},\ z \in U$$

then

$$Re\left\{f'(z)\right\} > \frac{1}{2},\ z \in U.$$

That is $f(z) \in B(0, 1/2) = R_{1/2}$.

ACKNOWLEDGEMENTS The authors would like to thank the referee(s) for their insightful comments and suggestions.

References

[1] S. D. Bernardi. Convex and starlike univalent functions. *Trans. Amer. Math. Soc.*, 135:429–Ű446, 1969.

[2] B. C. Carlson and S. B. Shaffer. Starlike and prestarlike hypergeometric functions. *SIAM J. Math. Anal.*, 15:737–Ű745, 1984.

[3] J. Dziok and H. M. Srivastava. Certain subclasses of analytic functions associated with the generalized hypergeometric function. *Intergral Transform Spec. Funct.*, 14:7–Ű18, 2003.

[4] B.A. Frasin and M. Darus. On certain analytic univalent functions. *Internat. J. Math. and Math. Sci.*, 25(5):305–Ű310, 2001.

[5] B.A. Frasin and Jay M. Jahangiri. A new and comprehensive class of analytic functions. *Analele Univ. Oradea*, XV:61Ű–64, 2008.

[6] R. J. Libera. Some classes of regular univalent functions. *Proc. Amer. Math. Soc.*, 16:755–758, 1965.

[7] A. E. Livingston. On the radius of univalence of certain analytic functions. *Proc. Amer. Math. Soc.*, 17:352–357, 1966.

[8] St. Ruscheweyh. New criteria for univalent functions. *Proc. Amer. Math. Soc.*, 49:109–115, 1975.

[9] H. M. Srivastava and S. Owa. Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, hadamard products, linear operators and certain subclasses of analytic functions. *Nagoya Math. J.*, 106:1–28, 1987.