EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 5203
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On Apostol-Type Multi Poly-Genocchi Polynomials with Parameters a, b, and c

Mark P. Laurente[1,*], AZ D. Ababa[2]

[1] *Department, Faculty, Davao De Oro State College – Main Campus (Compostela), Compostela City, Davao de Oro, Philippines*
[2] *Institute of Mathematics, Arts and Sciences, Faculty, Davao del Sur State College, Digos City, Davao del Sur, Philippines*

Abstract. In this paper, we investigate and analyze the Apostol numbers and polynomials, extending these properties by integrating them with the multi-polylogarithm function. Through this approach, we establish new properties and introduce a novel concept, which we refer to as the Apostol-type multi-poly Genocchi polynomials with parameters a, b, and c. Several properties of these polynomials are established including identities, the relation to Bernoulli polynomials, including some recurrence relations, addition and explicit formulas which are parallel on generalized poly-Genocchi polynomials.

2020 Mathematics Subject Classifications: 11B68, 11M35, 33C45

Key Words and Phrases: Genocchi numbers and polynomials, apostol Genocchi numbers and polynomials, poly-Genocchi numbers and polynomials, multi poly-Genocchi numbers and polynomials, apostol-type multi poly-Genocchi numbers and polynomials, poly Bernoulli and generating function

---

1. Introduction

The Genocchi numbers, denoted by G_n, are an important sequence of integers that arise in various areas of mathematics, including combinatorics, number theory, and graph theory. They were named after the Italian mathematician Angelo Genocchi (1817–1889), who studied it extensively. The Genocchi polynomials are related to the well-known Bernoulli and Euler polynomials a famous work of Jakob Bernoulli (1654-1705) when he studied the sums of the *pth* power of the first $n-1$ integers $1^p + 2^p + 3^p + (n-1)^p$. One of the outgrowths on the generalization of classical Bernoulli polynomials $B_n(x)$ is the development of the poly-Bernoulli numbers $B_n^{(k)}$ and polynomials $B_n^{(k)}(x)$. Many researchers in recent decades provided relation to Euler numbers E_n and polynomials

---

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.5203

Email addresses: mark.laurente@ddosc.edu.ph (M. Laurente), az.ababa@dssc.edu.ph (AZ D. Ababa)

$E_n(x)$, Genocchi numbers $\mathcal{G}_n$ and polynomials $\mathcal{G}_n(x)$, poly-Euler numbers $E_n^{(k)}$ and poly-Euler polynomials $E_n^{(k)}(x)$, poly-Genocchi numbers $\mathcal{G}_n^{(k)}$ and poly-Genocchi polynomials $\mathcal{G}_n^{(k)}(x)$.

In 1970 J.M. Gandhi [**1**] presents the conjectured of Genocchi Numbers by

$$G_{2N} = (-1)^{N \sum 1^2 \sum 2^2 \sum 3^2 \cdots \sum (N-1)^2}$$

where $\sum$ notation is defined by

$$\sum k^2 = k^2 - (k-1)^2 \sum k^2 \, (k-1)^2$$

$$\sum k^2 \sum (k-1)^2 = k^2 \sum (k+1)^2 - (k-1)^2 \sum k^2$$

$$= k^2 \left\{(k+1)^2 - k^2\right\} - (k-1)^2 \left\{k^2 - (k-1)^2\right\}$$

which generalizes to the recurrence form

$$\sum k^2 \sum (k-1)^2 \cdots \sum (k+N)^2 = k^2 \sum (k+1)^2 \sum (k+2)^2 \cdots \sum (k+N)^2 - (k-1)^2 \sum k^2 \sum (k+1)^2$$

The recurrence relation was proved by Riordan and Stien [**2**] by means of Analytic Calculus. There are several ways to define the Genocchi numbers. In this paper, we adopt the definition of [**3–8**]which are a sequence of integers that are defined by the exponential generating function

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} \mathcal{G}_n \frac{t^n}{n!}, \quad |t| < \pi. \tag{1}$$

with the usual convention about replacing G^n by G_n, is used. These are few Genocchi numbers: $G_0 = 0$, $G_1 = 1$, $G_2 = -1$, $G_3 = 0$, $G_4 = 1$, $G_5 = 0$, $G_6 = -3$, $G_7 = 0$, $G_8 = 17$,and etc.

The classical definition of Genocchi Polynomials, denoted by $\mathcal{G}_n(x)$, is usually defined by means of the exponential generating function

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n(x) \frac{t^n}{n!}, \quad |t| < \pi \tag{2}$$

where $\mathcal{G}_n(x)$ is the Genocchi polynomials of degree n and is given by

$$\mathcal{G}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_k x^{n-k}.$$

The first few Genocchi polynomials are [**9**]: $\mathcal{G}_1(x) = 1$, $\mathcal{G}_2(x) = 2x - 1$, $\mathcal{G}_3(x) = 3x^2 - 3x$, $\mathcal{G}_4(x) = 4x^3 - 6x^2 + 1$, $\mathcal{G}_5(x) = 5x^4 - 10x^3 + 5x$

2. Preliminaries

2.1. Generating Function and Polylogarithm and Genocchi Polynomials

The geometric series [10] $\frac{1}{1-x} = 1+x+x^2+x^3+\dots$, has formal power series $\sum_{n\geq 0} x^n$, say for example the generating function of $a+ab+ab^2+ab^3+\dots = \sum_{n\geq 0} ab^n x^n$ is $\frac{a}{1-bx}$. Also the exponential generating function for the sequence of numbers (a_r) is defined to be the power series $a_0 + a_1\frac{x}{1!} + a_2\frac{x}{2!} + a_3\frac{x}{3!} + \dots + a_r\frac{x}{r!} + \dots = \sum_{r=0}^{\infty} a_r \frac{x^r}{r!}$. Consider the sequence $(a_r) = (1, k, k^2, k^3, \dots, k^r, \dots)$ where k is a nonzero constant. The exponential generating function for (a_r) is $1 + \frac{kx}{1!} + \frac{k^2x^2}{2!} + \dots = \sum_{r=0}^{\infty} \frac{(kx)^r}{r!} = e^{kx}$.

The classical polylogarithmic function **[11]** is defined by

$$L_{i_k}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} \tag{3}$$

which is the k-th polylogarithm if $k \geq 1$ and a rational function if $k \leq 0$.

The multiple polylogarithms **[12]** are defined by

$$L_{i_{k_1,k_2,k_3,\cdots,k_r}}(z) = \sum_{0<m_1<m_2\dots<m_r} \frac{z^{m_r}}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}}. \tag{4}$$

where $k_1, k_2, \cdots, k_r$ are positive integers and z a complex number in the unit disk. For $r = 1$, this is the classical polylogarithm $L_{i_k}(\mathrm{z})$.

2.2. Variation of Genocchi Polynomial

Different authors who studied the work of Apostol, used generating functions with a twist—inserting a parameter λ—which gave them a greater flexibility and allowed connections to new types of zeta functions, when they were studying in the generalization the Bernoulli polynomials. Then several mathematician was intrigue of this grand work and further do more research on it which now resulted to the closed relation of and several variation of Bernoulli, Euler, and Genocchi polynomials. One of the interesting result that arise from those researches is the Apostol-Genocchi polynomial.

$$\sum_{n=0}^{\infty} \mathcal{G}(x,\lambda)\frac{t^n}{n!} = \frac{2t}{\lambda e^t + 1} e^{xt}, \tag{5}$$

Other variations of Genocchi polynomials that appeared in the literature is the Apostol-Genocchi polynomials of higher order, which defined by

$$\sum_{n=0}^{\infty} \mathcal{G}^{(k)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t + 1}\right)^k e^{xt}, \tag{6}$$

(see [13], [14], [3, 15, 16],[8]).

It is also important to note that the notation of Araci in equation (7) of higher order denoted by k has a different meaning in the notation of k the poly-Logarithm other researchers like Corcino, Lou, Srivastava, who study the higher order of Genocchi polynomial is using α to denote the higher order.

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k)}(x)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^k e^{xt}, \tag{7}$$

Araci [15] and Kim et al. [14] did some research on equation (7), the Genocchi polynomials of higher order arising from Genocchi basis. The main objective of their studies is to derive interesting identities on (5) using a new method constructed by Kim et al. [17].

Moreover, Araci and He [3, 15, 16] introduced (7) an extension of the classic Genocchi polynomials called the Apostol-Genocchi polynomials from which Araci [8] introduced (6) the higher order of such polynomials as the generalized Apostol-Genocchi polynomials $\mathcal{G}_n^{(k)}(x,\lambda)$ of order $k \in \mathbb{C}$,

Another variation of Genocchi polynomials, is also known as poly-Genocchi polynomials, was introduced by Kim et al. [18] using the concept of kth poly-logarithm, denoted by $\mathrm{Li}_k(z)$, which is given by

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k)}(x)\frac{t^n}{n!} = \frac{2L_{i_k}(1-e^{-t})}{e^t+1}e^{xt}. \tag{8}$$

Moreover, a modified poly-Genocchi polynomials, denoted by $\mathcal{G}_{n,2}^{(k)}(x)$, were defined by Kim et al. [19] as follows

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(k)}(x)\frac{t^n}{n!} = \frac{L_{i_k}(1-e^{-2t})}{e^t+1}e^{xt}. \tag{9}$$

Note that, when $k=1$, equations (8) and (9) give the Genocchi polynomials in (2). That is,

$$\mathcal{G}_n^{(1)}(x) = \mathcal{G}_{n,2}^{(1)}(x) = \mathcal{G}_n(x).$$

Kim et. al [18] obtained several properties of these polynomials.

On the other hand, Kurt [20] defined two forms of generalized poly-Genocchi polynomials with parameters a, b, and c, as follows

$$\frac{2L_{i_k}(1-(ab)^{-t})}{a^{-t}+b^t}e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(k)}(x;a,b,c)\frac{t^n}{n!} \quad \left(|t| < \frac{\pi}{|\ln a + \ln b|}\right) \tag{10}$$

$$\frac{2L_{i_k}(1-(ab)^{-2t})}{a^{-t}+b^t}e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(k)}(x;a,b,c)\frac{t^n}{n!}, \quad \left(|t| < \frac{\pi}{|\ln a + \ln b|}\right) \tag{11}$$

In [21], Lim defined the degenerate Genocchi polynomials $\mathcal{G}_n^{(r)}(\lambda, x)$ of order r as

$$\left(\frac{2t}{(1+\lambda t)^{1/\lambda}}\right)^r (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(r)}(\lambda, x)\frac{t^n}{n!}.$$

Besides those generalizations, Araci [5], Duran et al. [22] and Agyuz et al. [23] also introduced the q-analogue of the Genocchi polynomials $\mathcal{G}_{n,q}(x)$ as follows:

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,q}(x)\frac{t^n}{n!} = t\int_{\mathbb{Z}_p} q^{-y} e^{t[y+x]_q} d\mu_{-q}(y),$$

where

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}.$$

This definition is constructed by p-adic fermionic q-integral on $\mathbb{Z}_p$ with respect to μ_{-q}. It can also be defined by

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,q}(x)\frac{t^n}{n!} = [2]_q t \sum_{m=0}^{\infty} (-1)^m e^{t[m+x]_q},$$

which becomes $\mathcal{G}_{n,q}(0) := \mathcal{G}_{n,q}$ when $x = 0$, and referred to as the nth q-Genocchi number.

Also, Corcino C.B. and Corcino R.B [24]study regarding Higher Order Apostol-Type Poly-Genocchi Polynomials with Parameters a, b and c, they define this by

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x)\frac{t^n}{n!} = \left(\frac{L_{i_k}(1-(ab)^{-2t})}{a^{-t}+b^t} e^{xt}\right)^{\alpha}, \quad |t| < \frac{\sqrt{(ln\lambda)^2+\pi^2}}{|lna+lnb|},$$

they also study the other variation [25], which the concept is from modified degenerate polylexponential function.

3. Close Relation of Bernoulli polynomial and Euler polynomial to Genocchi Polynomial

A generalization of Bernoulli polynomials introduced by Kaneko [26] is defined in terms of the following kth polylogarithm $\mathrm{Li}_k(z)$:

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}, \quad (|z| < 1) \tag{12}$$

where $k \in \mathbb{Z}$. When $z = 1$, the kth polylogarithm gives the Riemann zeta function. That is,

$$\mathrm{L}i_k(1) = \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

Also, when $k = 1$, the $1st$ polylogarithm yields the natural logarithmic function as follows:

$$Li_1(z) = -\ln(1-z).$$

This special case of the polylogarithm motivates the construction of poly-Bernoulli numbers in the sense that

$$\mathrm{Li}_1(1-e^{-x}) = x.$$

The poly-Bernoulli polynomials and numbers were defined in [19] by means of the following generating functions, respectively:

$$\frac{L_{i_k}(1-e^{-t})}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!} \qquad (|t|<2\pi) \tag{13}$$

$$\frac{L_{i_k}(1-e^{-t})}{e^t-1} = \sum_{n=0}^{\infty} B_n^{(k)}\frac{t^n}{n!}, \qquad (|t|<2\pi) \tag{14}$$

If $k = 1$, from equations (13) and (14), respectively, we have

$$B_n^{(1)}(x) = B_n(x), B_n^{(1)} = B_n. \tag{15}$$

We present again equation (8), since this a poly-Genoccchi Polynomials and this work of Kim et al. [19] defined poly-Genocchi polynomials $\mathcal{G}_n^{(k)}(x)$ as follows

$$\frac{2L_{i_k}(1-e^{-t})}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(k)}(x)\frac{t^n}{n!}, \qquad (|t|<\pi).$$

Note that when $x = 0$, (8) reduces to

$$\frac{2L_{i_k}(1-e^{-t})}{e^t+1} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(k)}\frac{t^n}{n!}, \qquad (|t|<\pi). \tag{16}$$

where $\mathcal{G}_n^{(k)}$ are called the poly-Genocchi numbers.

Note also that when $k = 1$, equation (8) gives the Genocchi polynomials

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{G}_n^{(1)}(x)\frac{t^n}{n!} &= \frac{2L_{i_1}(1-e^{-t})}{e^t+1}e^{xt} \\
&= \frac{2t}{e^t+1}e^{xt} \\
&= \sum_{n=0}^{\infty} \mathcal{G}_n(x)\frac{t^n}{n!}.
\end{aligned}$$

Moreover, they defined a modified poly-Genocchi polynomials, denoted by $\mathcal{G}_{n,2}^{(k)}(x)$, as follows:

$$\frac{L_{i_k}(1-e^{-2t})}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(k)}(x)\frac{t^n}{n!}. \tag{17}$$

Equation (17) also gives the Genocchi polynomials (**??**)

$$\begin{aligned}\frac{L_{i_1}(1-e^{-2t})}{e^t+1}e^{xt} &= \sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(1)}(x)\frac{t^n}{n!}\\ &= \frac{2t}{e^t+1}e^{xt}\\ &= \sum_{n=0}^{\infty} \mathcal{G}_{n}(x)\frac{t^n}{n!}.\end{aligned}$$

Consequently, Kim et al. [19] obtained several properties of the poly-Genocchi polynomials.

On the other hand Kurt [20] ivestigated most creative stage of poly-Genocchi of equation (10) and (11) which are following

$$\frac{2L_{i_k}(1-(ab)^{-t})}{a^{-t}+b^t}c^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_{n}^{(k)}(x;a,b,c)\frac{t^n}{n!}$$

$$\frac{2L_{i_k}(1-(ab)^{-2t})}{a^{-t}+b^t}c^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(k)}(x;a,b,c)\frac{t^n}{n!}$$

which are motivated by the definitions in (8) and (9), respectively.

If we put $x = 0, a = 1, b = c = e$ in (10), we have

$$G_n^{(k)}(0;1,e,e)\frac{t^n}{n!} = \frac{2L_{i_k}(1-e^{-t})}{e^t+1}$$

which gives the definition of Kurt [20] for the poly-Genocchi numbers $\mathcal{G}_n^{(k)}$ which is also the definition of Kim et al. in [19] the poly-Genocchi numbers.

Further, in [12] they defined the Generalized Poly-Genocchi Polynomials with Parameters a, b, and c by

$$\sum_{n=0}^{\infty} \mathcal{G}_{n}^{(k)}(x;a,b,c)\frac{t^n}{n!} = \frac{L_{i_k}(1-(ab)^{-2t})}{a^{-t}+b^t}c^{xt}, \qquad |t| < \frac{\pi}{|\ln a + \ln b|}. \tag{18}$$

4. On Apostol-Type Multi Poly-Genocchi Polynomials with Parameters a, b and c

Definition 4.1. An **Apostol-type of multi poly Genocchi polynomials** with parameters a, b and c is defined by

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,k_3,\cdots,k_r)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \frac{L_{i_{k_1,k_2,k_3,\cdots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}c^{rxt}, \tag{19}$$

where x is any real number with $k_1, k_2, \ldots, k_r \in \mathbb{Z}^+$, $\lambda \in \mathbb{C}$ and a, b, c are any positive real numbers.

When $c = e$ equation(19) reduces to

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}(x;\lambda,a,b,e)\frac{t^n}{n!} = \frac{L_{i_{k_1,k_2,k_3,\cdots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}e^{rxt} \tag{20}$$

We use $\mathcal{G}_n^{(k_1,k_2,k_3,\cdots,k_r)}(x;\lambda,a,b)$ to denote $\mathcal{G}_n^{(k_1,k_2,k_3,\cdots,k_r)}(x;\lambda,a,b,e)$. That is,

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,k_3,\cdots,k_r)}(x;\lambda,a,b)\frac{t^n}{n!} = \frac{L_{i_{k_1,k_2,k_3,\cdots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}e^{rxt}. \tag{21}$$

For instance, if we put $a = 1, b = e$ in equation (21), then this will reduce to

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,k_3,\cdots,k_r)}(x;\lambda,1,e)\frac{t^n}{n!} = \frac{L_{i_{k_1,k_2,k_3,\cdots,k_r}}(1-e^{-2t})}{(1+\lambda e^t)^r}e^{rxt}. \tag{22}$$

We may use the notation $\mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}(x;\lambda) = \mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}(x;\lambda,1,e)$ and call them Apostol-type multi poly-Genocchi polynomials. When $\lambda = 1$ equation (22) gives

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}(x;1)\frac{t^n}{n!} = \frac{L_{i_{k_1,k_2,k_3,\cdots,k_r}}(1-e^{-2t})}{(1+e^t)^r}e^{rxt}. \tag{23}$$

When $x = 0$ equation (21) gives

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,k_3,\cdots,k_r)}(0;\lambda,a,b)\frac{t^n}{n!} = \frac{L_{i_{k_1,k_2,k_3,\cdots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}. \tag{24}$$

We use $\mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}(a,b)$ to denote $\mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}(0;a,b)$ the Apostol-type multi poly-Genocchi Numbers.

Theorem 4.2. An Apostol-type of multi poly-Genocchi polynomials with parameters a, b, and c satisfy the relation.

$$\mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}(x;\lambda,a,b,c) = (\ln a+\ln b)^n \mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}\left(\frac{rx\ln c+r\ln a}{\ln ab};\lambda\right), ab\neq 1. \tag{25}$$

Proof. By definition of an Apostol-type of multi poly-Genocchi polynomials in equation (19) we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^n}{n!} &= \frac{L_{i_{k_1,k_2,k_3,\cdots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}c^{rxt}\\ &= \frac{L_{i_{k_1,k_2,k_3,\ldots,k_r}}(1-(ab)^{-2t})}{\left((a^{-t})(1+\lambda\frac{b^t}{a^{-t}})\right)^r}c^{rxt}\\ &= \frac{L_{i_{k_1,k_2,k_3,\ldots,k_r}}(1-(ab)^{-2t})}{(1+\lambda(ab)^t)^r}c^{rxt}a^{rt}\\ &= \frac{L_{i_{k_1,k_2,k_3,\ldots,k_r}}(1-(e)^{-2t\ln ab})}{(1+\lambda e^{t\ln ab})^r}e^{rxt\ln c}e^{rt\ln a}\\ &= \frac{L_{i_{k_1,k_2,k_3,\ldots,k_r}}(1-(e)^{-2t\ln ab})}{(1+\lambda e^{t\ln ab}t)^r}e^{t(rx\ln c+r\ln a)}\end{aligned}$$

Let $z = t\ln ab$, then t$= \dfrac{z}{lnab}$. Thus we have,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,k_3,\cdots,k_r)}(x;\lambda,a,b,c)\frac{t^n}{n!} &= \frac{L_{i_{k_1,k_2,k_3,\ldots,k_r}}(1-(e)^{-2z})}{(1+\lambda e^z)^r}e^{\frac{z}{\ln ab}(rx\ln c+r\ln a)}\\ &= \frac{L_{i_{k_1,k_2,k_3,\ldots,k_r}}(1-(e)^{-2z})}{(1+\lambda e^z)^r}e^{(\frac{rx\ln c+r\ln a}{\ln ab})z}\\ &= \sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}\left(\frac{rx\ln c+r\ln a}{\ln ab};\lambda\right)\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}\left(\frac{rx\ln c+r\ln a}{\ln ab};\lambda\right)\frac{(t\ln ab)^n}{n!}\\ &= \sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}\left(\frac{rx\ln c+r\ln a}{\ln ab};\lambda\right)\frac{(t)^n(\ln ab)^n}{n!}\\ &= \sum_{n=0}^{\infty}(\ln a+\ln b)^n\mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}\left(\frac{rx\ln c+r\ln a}{\ln ab};\lambda\right)\frac{(t)^n}{n!}\end{aligned}$$

Comparing the coeffiecient of $\dfrac{t^n}{n!}$ we obtain the desired result. Therefore, $\mathcal{G}_n^{(k_1,k_2,k_3,\ldots,k_r)}(x;\lambda,a,b,c) = (\ln a+\ln b)^n G_n^{(k_1,k_2,k_3,\ldots,k_r)}\left(\dfrac{rx\ln c+r\ln a}{\ln ab};\lambda\right)$.

The next theorem contains a kind of recurrence relation of an Apostol-type of multi poly-Genocchi polynomials.

Theorem 4.3. An Apostol-type of multi poly-Genocchi polynomials satisfy the recurrence relation

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x+1;\lambda,a,b,c)=\sum_{m=0}^{n}\binom{n}{m}(r\ln c)^m\mathcal{G}_{n-m}^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c).$$

Proof. By the equation (19) in Definition (4.1)

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x+1;\lambda,a,b,c)\frac{t^n}{n!}&=\frac{Li_{k_1,k_2,\ldots,k_r}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}c^{rt(x+1)}\\&=\frac{Li_{k_1,k_2,\ldots,k_r}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}c^{rxt}c^{rt}\\&=\frac{Li_{k_1,k_2,\ldots,k_r}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}e^{rxt\ln c}e^{rt\ln c}\end{aligned}$$

Rewriting $e^{rt\ln c}=\sum_{n=0}^{\infty}\frac{(rt\ln c)^n}{n!}$ as exponential generating function form so we have,

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x+1;\lambda,a,b,c)\frac{t^n}{n!}=\left(\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(rt\ln c)^n}{n!}\right)$$

Using the product of two generating function, so we have,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x+1;\lambda,a,b,c)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\mathcal{G}_{n-m}^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^{n-m}}{(n-m)!}\frac{(r\ln c)^m t^m}{m!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\mathcal{G}_{n-m}^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^{n-m}}{(n-m)!}\frac{(r\ln c)^m t^m}{m!}\frac{n!}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(r\ln c)^m\mathcal{G}_{n-m}^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coeffiecient of $\frac{t^n}{n!}$ we obtain the desired result. Therefore, $\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x+1;\lambda,a,b,c)=\sum_{m=0}^{n}\binom{n}{m}(r\ln c)^m\mathcal{G}_{n-m}^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)$.

Theorem 4.4. An Apostol-type of multi poly-Genocchi polynomials satisfy the relation

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)=\sum_{i=0}^{n}\binom{n}{i}(r\ln c)^{n-i}\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)x^{n-i},\qquad x\neq 0.$$

Proof. Using equation (19) of Definition (4.1) can be written as

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^n}{n!} &= \frac{L_{i_{k_1,k_2,\ldots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}e^{rxt\ln c}\\ &= e^{rxt\ln c}\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{(rxt\ln c)^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(rxt\ln c)^{n-i}}{(n-i)!}\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)\frac{t^i}{i!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}(xr\ln c)^{n-i}\frac{t^{n-i}}{(n-i)!}\mathcal{G}_r^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)\frac{t^i}{i!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}(xr\ln c)^{n-i}\frac{t^{n-i}}{(n-i)!}\mathcal{G}_r^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)\frac{t^i}{i!}\frac{n!}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}(r\ln c)^{n-i}\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)x^{n-i}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. Therefore, $\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c) = \sum_{i=0}^{n}\binom{n}{i}(r\ln c)^{n-i}\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)x^{n-i}$.

The following theorem contains a differential equation that can be used to classify an Apostol-type of multi poly-Genocchi polynomials as Appell polynomials [12].

Theorem 4.5. An Apostol-type of multi poly-Genocchi polynomials satisfy the relation

$$\frac{d}{dx}\mathcal{G}_{n+1}^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c) = (n+1)(r\ln c)\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c). \tag{26}$$

Proof. By applying the first derivative to equation (4.1), with respect to x we have

$$\sum_{n=0}^{\infty}\frac{d}{dx}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^n}{n!} = rt(\ln c)\frac{L_{i_{k_1,k_2,\ldots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}e^{rxt\ln c}$$

this means that,

$$\sum_{n=0}^{\infty}\frac{d}{dx}\frac{\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)}{r\ln c}\frac{t^{n-1}}{n!} = \frac{L_{i_{k_1,k_2,\ldots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}e^{rxt\ln c}(*)$$

Rewriting the left hand side of(*) we get

$$\sum_{n=0}^{\infty}\frac{d}{dx}\frac{\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)}{r\ln c}\frac{t^{n-1}}{n!}=\frac{d}{dx}\frac{\mathcal{G}_0^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)}{r\ln c}\frac{t^{-1}}{0!}$$
$$+\sum_{n=1}^{\infty}\frac{d}{dx}\frac{\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)}{r\ln c}\frac{t^{n-1}}{n!}$$
$$=\sum_{n=1}^{\infty}\frac{d}{dx}\frac{\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)}{r\ln c}\frac{t^{n-1}}{n!}$$

Changing the index,

$$\frac{L_{i_{k_1,k_2,\ldots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}e^{rxt\ln c}=\sum_{n=0}^{\infty}\frac{d}{dx}\frac{\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)}{r\ln c}\frac{t^n}{(n+1)!}$$
$$=\sum_{n=0}^{\infty}\frac{d}{dx}\frac{\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)}{r\ln c}\frac{t^n}{(n+1)n!}$$

so we have,

$$\sum_{n=0}^{\infty}\frac{d}{dx}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{d}{dx}\frac{\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)}{r\ln c}\frac{t^n}{(n+1)n!}$$
$$\sum_{n=0}^{\infty}(r\ln c)(n+1)\frac{d}{dx}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{d}{dx}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)(x;\lambda,a,b,c)}\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result.Therefore, $\frac{d}{dx}\mathcal{G}_{n+1}^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)=(n+1)(r\ln c)\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)$.

When $c=e^{1/r}$, the next corollary is obtained from (4.5)

Corollary 4.6. An Apostol-type multi poly-Genocchi polynomials satisfy the relation

$$\frac{d}{dx}\mathcal{G}_{n+1}^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,e^{1/r})=(n+1)\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,e^{1/r}).$$

Corollary (4.6) is one the properties to be classified as Appell polynomial. Being classified as Appell polynomials, the Apostol-type multi poly-Genocchi polynomials must possess the following properties

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)=\sum_{i=0}^{\infty}\binom{n}{i}c_ix^{n-i}$$

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b) = \left(\sum_{i=0}^{\infty}\frac{c_i}{i!}D^i\right)x^n$$

for some scalar $c_i \neq 0$ and where D is the operator $\dfrac{d}{dx}$. It is necessary to find the sequence (c_n). However, using Theorem (4.4) with $c = e$,

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b) = \sum_{i=0}^{\infty}\binom{n}{i}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)\,x^{n-i}$$

And letting $c_i = \mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)$ we obtain the next corollary.

Corollary 4.7. An Apostol-type of multi poly-Genocchi polynomials satisfy the following formula:

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,e^{1/r}) = \sum_{i=0}^{\infty}\frac{\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)}{i!}D^i x^n.$$

The following theorem contains the addition formula for $\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)$.

Theorem 4.8. An Apostol of type multi poly-Genocchi polynomials satisfy the following addition formula:

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x+y;\lambda,a,b,c) = \sum_{i=0}^{n}\binom{n}{i}(r\ln c)^{n-i}\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)y^{n-i}, \qquad y \neq 0.$$

Proof. Using equation 19 of Definition 4.1,

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x+y;\lambda,a,b,c)\frac{t^n}{n!} &= \frac{L_{i_{k_1,k_2,\ldots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}c^{(x+y)rt} \\
&= \frac{L_{i_{k_1,k_2,\ldots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+b^t)^r}c^{xrt}c^{yrt} \\
&= \left(\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^n}{n!}\right)e^{yrt\ln c}
\end{aligned}$$

Since $e^{yrt\ln c} = \displaystyle\sum_{n=0}^{\infty}\frac{(yrt\ln c)^n}{n!} = \sum_{n=0}^{\infty}(yr\ln c)^n\frac{(t)^n}{n!}$, so have,

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x+y;\lambda,a,b,c)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(yr\ln c)^n\frac{t^n}{n!}\right)$$

Using the product of two generating function,

$$= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\mathcal{G}_{n-i}^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^{n-i}}{(n-i)!}\right)(\ln c)^i\frac{t^i}{i!}$$

and we multiply it by $\frac{i!}{i!}$, so we have,

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\mathcal{G}_r^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^{n-i}}{(n-i)!}\right)(\ln c)^i\frac{t^i}{i!}\frac{i!}{i!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}(yr\ln c)^{n-i}\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\right)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}(r\ln c)^{n-i}\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)y^{n-i}\right)\frac{t^n}{n!}
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the desired result.

When $c = e^{1/r}$, the next corollary is obtained.

Corollary 4.9. An Apostol-type of multi poly-Genocchi polynomials satisfy the following addition formula:

$$
\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x+y;\lambda,a,b,e^{1/r}) = \sum_{i=0}^{n}\binom{n}{i}\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,e^{1/r})y^{n-i}, \qquad y \neq 0.
$$

Theorem 4.10. An Apostol-type multi of poly-Genocchi polynomials satisfy the following explicit formulas:

$$
\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c) = \sum_{m=0}^{\infty}\sum_{l=m}^{n}\left\{ {l \atop m} \right\}\binom{n}{l}(r\ln c)^l\mathcal{G}_{n-l}^{(k_1,k_2,\ldots,k_r)}(-m\ln c;\lambda,a,b)(x)^{(m)} \tag{27}
$$

$$
\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c) = \sum_{m=0}^{\infty}\sum_{l=m}^{n}\left\{ {l \atop m} \right\}\binom{n}{l}(r\ln c)^l\mathcal{G}_{n-l}^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)(x)_m \tag{28}
$$

$$
\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c) = \sum_{l=0}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\left\{ {l+s \atop s} \right\}\frac{\binom{n-l}{m}}{\binom{l+s}{s}}\mathcal{G}_{n-l-m}^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)B_m^{(s)}(rx\ln c) \tag{29}
$$

where $(x)^{(n)} = x(x+1)\cdots(x+n-1), (x)_n = x(x-1)\cdots(x-n+1)$,
and

$$
\left(\frac{t}{e^t-1}\right)^s e^{xt} = \sum_{n=0}^{\infty}B_n^{(s)}(x)\frac{t^n}{n!}
$$

Proof. First, we prove 27. Using equation 19 of Definition (4.1) and rewrite $e^{rxt\ln c} = (e^{-rt\ln c})^{-x} = \left[1-(1-e^{-rt\ln c})\right]^{-x}$ we have,

$$
\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \frac{L_{i_k}(1-(ab)^{-2t})}{(a^{-t}+b^t)^r}\left[1-(1-e^{-rt\ln c})\right]^{-x}.
$$

Using generalized Binomial Theorem and the definition of Stirling numbers of second kind we have,

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \frac{L_{i_{k_1,k_2,\ldots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}\sum_{m=0}^{\infty}\binom{x+m-1}{m}(1-e^{-rt\ln c})^m$$
$$= \sum_{m=0}^{\infty}(x)^{(m)}\frac{(e^{rt\ln c}-1)^m}{e^{rt\ln c}m!}\frac{L_{i_{k_1,k_2,\ldots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}$$

Note that $(x)^{(m)} = x(x+1)\cdots(x+m-1)\dfrac{(x-1)!}{(x-1)!} = \dfrac{(x+m-1)!}{(x-1)!}$, so we have

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}x;\lambda,a,b,c)\frac{t^n}{n!} = \sum_{m=0}^{\infty}\frac{(x+m-1)!}{(x-1)!m!}\left(\frac{e^{rt\ln c}-1}{e^{rt\ln c}}\right)^m\frac{L_{i_k}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}$$

$$= \sum_{m=0}^{\infty}\frac{(x+m-1)!}{(x-1)!}\frac{(e^{t\ln c}-1)^m}{m!}\frac{L_{i_k}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}e^{-rmt\ln c}$$
$$= \sum_{m=0}^{\infty}(x)^{(m)}\sum_{n=0}^{\infty}\left\{ {n \atop m} \right\}(r\ln c)^n\frac{t^n}{n!}\left(\sum_{n=0}^{\infty}\mathcal{G}_n^{(k)}(-mr\ln c;\lambda,a,b)\frac{t^n}{n!}\right)$$
$$= \sum_{m=0}^{\infty}(x)^{(m)}\left(\sum_{n=0}^{\infty}\left\{ {n \atop m} \right\}\frac{(tr\ln c)^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(-m\ln c;\lambda,a,b)\frac{t^n}{n!}\right)$$
$$= \sum_{m=0}^{\infty}(x)^{(m)}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\left\{ {l \atop m} \right\}(r\ln c)^l\frac{t^l}{l!}\mathcal{G}_{n-l}^{(k_1,k_2,\ldots,k_r)}(-m\ln c;\lambda,a,b)\frac{t^{n-l}}{(n-l)!}$$
$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{\infty}\sum_{l=m}^{n}\left\{ {l \atop m} \right\}\binom{n}{l}r\ln c)^l\mathcal{G}_{n-l}^{(k_1,k_2,\ldots,k_r)}(-m\ln c;\lambda,a,b)(x)^{(m)}\right\}\frac{t^n}{n!}$$

Comparing the coefficients of $\dfrac{t^n}{n!}$ completes the proof of equation 27.

To prove identity of equation 28, express $e^{rxt\ln c}$ as

$$\left[1+(e^{rt\ln c}-1)\right]^x = \sum_{m=0}^{\infty}\binom{x}{m}(e^{rt\ln c}-1)^m$$

and take note that

$$(x)_m = x(x-1)\cdots(x-m+1) = \frac{x!}{(x-m)!}$$

applying it to definition 19, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;\lambda,a,b,c)\frac{t^n}{n!} &= \frac{L_{i_{k_1,k_2,\ldots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}e^{rxt\ln c}\\
&= \frac{L_{i_{k_1,k_2,\ldots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}\sum_{m=0}^{\infty}\binom{x}{m}(e^{rt\ln c}-1)^m\\
&= \sum_{m=0}^{\infty}(x)_m\frac{(e^{rt\ln c}-1)^m}{m!}\frac{L_{i_{k_1,k_2,\ldots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}\\
&= \sum_{m=0}^{\infty}(x)_m\left(\sum_{n=0}^{\infty}\begin{Bmatrix} n\\ m\end{Bmatrix}\frac{(rt\ln c)^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(0;\lambda,a,b)\frac{t^n}{n!}\right)\\
&= \sum_{m=0}^{\infty}(x)_m\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{Bmatrix} l\\ m\end{Bmatrix}(r\ln c)^l\frac{t^l}{l!}\mathcal{G}_{n-l}^{(k_1,k_2,\ldots,k_r)}(a,b)\frac{t^{n-l}}{(n-l)!}\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{\infty}\sum_{l=m}^{n}\begin{Bmatrix} l\\ m\end{Bmatrix}\binom{n}{l}(r\ln c)^l\mathcal{G}_{n-l}^{(k_1,k_2,\ldots,k_r)}(a,b)(x)_m\right\}\frac{t^n}{n!}.
\end{aligned}$$

Again,comparing the coefficients of $\dfrac{t^n}{n!}$ completes the proof of (28).

For relation (29), from equation 19 of Definition 4.1 can be written as

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k)}(x;\lambda,a,b,e)\frac{t^n}{n!} = \frac{L_{i_k}(1-(ab)^{-2t})}{a^{-t}+b^t}e^{xt\ln c}(*)$$

and multiplying the right-hand side by $(*)\dfrac{(e^t-1)^s}{(e^t-1)^s}\dfrac{t^s}{t^s}\dfrac{s!}{s!}$, we get

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \left(\frac{(e^t-1)^s}{s!}\right)\left(\frac{t^s e^{xt\ln c}}{(e^t-1)^s}\right)\left(\frac{L_{i_{k_1,k_2,\ldots,k_r}}(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)^r}\right)\frac{s!}{t^s}$$

$$\begin{aligned}
&= \left(\sum_{n=0}^{\infty}\begin{Bmatrix} n+s\\ s\end{Bmatrix}\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{m=0}^{\infty}B_m^{(s)}(x\ln c)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\mathcal{G}_n^{k_1,k_2,\ldots,k_r}(\lambda,a,b)\frac{t^m}{m!}\right)\frac{s!}{t^s}\\
&= \left(\sum_{n=0}^{\infty}\begin{Bmatrix} n+s\\ s\end{Bmatrix}\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}B_m^{(s)}(x\ln c)\mathcal{G}_{n-m}^{k_1,k_2,\ldots,k_r}(\lambda,a,b)\frac{t^n}{n!}\right)\frac{s!}{t^s}\\
&= \left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{Bmatrix} l+s\\ s\end{Bmatrix}\frac{t^{l+s}}{(l+s)!}\sum_{m=0}^{n-l}\binom{n-l}{m}B_m^{(s)}(x\ln c)\mathcal{G}_{n-l-m}^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)\frac{t^{n-l}}{(n-l)!}\right)\frac{s!}{t^s}\\
&= \left(\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}\sum_{m=0}^{n-l}\begin{Bmatrix} l+s\\ s\end{Bmatrix}\frac{l!s!}{(l+s)!}\binom{n-l}{m}B_m^{(s)}(x\ln c)\mathcal{G}_{n-l-m}^{(k_1,k_2,\ldots,k_r)}(\lambda,a,b)\frac{n!}{(n-l)!l!}\frac{t^n}{n!}\right)
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \left\{ \begin{matrix} l+s \\ s \end{matrix} \right\} \frac{\binom{n-l}{m}}{\binom{l+s}{s}} B_m^{(s)}(x \ln c) \mathcal{G}_{n-l-m}^{(k_1,k_2,\ldots,k_r)}(\lambda, a, b) \right) \frac{t^n}{n!}.$$

Again,comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of equation (29).

5. Conclusion

In this paper, we define Apostol-type of multi poly Genocchi polynomials with parameters a, b and c through the concept of multi-poly logarithm. We present the special cases when $c = e$ that gives as Equation (22), and when $a = 1$, and $b = e$ we obtain as Equation (23). Furthermore, we establish the relation in Theorem 4.2 and the recurrence relations given in Theorems 4.3 and 4.4, derived from the concept of poly-Bernoulli polynomials. We also provide Theorem 4.5 together with corollaries that involve a differential equation, which allows us to classify the Apostol-type multi poly-Genocchi polynomials as a family of Appell polynomials. Finally, we derive the addition formula in Theorem 4.8 and the explicit formulas in Theorem 4.10. The researcher recommends the following for possible investigation, by letting indices to non-positive. The modified Apostol-type of multi poly-Genocchi polynomials with parameters a, b, and c.

Acknowledgements

The authors sincerely extend their gratitude to Davao de Oro State College–Main Campus (Compostela) and Davao del Sur State College for their unwavering support, encouragement, and valuable contributions, which greatly facilitated the successful completion of this study.

References

[1] J.M. Gandhi. A conjectured representation of genocchi numbers. *The American Mathematical Monthly*, 77:505–506, 1970.

[2] J. Riordan and P. Stien. Proof of a conjecture on genocchi numbers. *Discrete Mathematics*, 5:381–388, 1973.

[3] Y. He. Some new results on products of the apostol-genocchi polynomials. *Journal of Computational Analysis and Applications*, 22(4):591–600, 2017.

[4] T. Agoh. Convolution identities for bernoulli and genocchi polynomials. *Electronic Journal of Combinatorics*, 21(1):Article ID P1.65, 2014.

[5] S. Araci, M. Acikgoz, H. Jolany, and J.J. Seo. A unified generating function of the q-genocchi polynomials with their interpolation functions. *Proceedings of the Jangjeon Mathematical Society*, 15(20):227–233, 2012.

[6] S. Araci. Novel identities for q-genocchi numbers and polynomials. *Journal of Function Spaces and Applications*, 2012:Article ID 214961, 2012.

[7] S. Araci, M. Acikgoz, and E. Sen. Some new formulae for genocchi numbers and polynomials involving bernoulli and euler polynomials. *International Journal of Mathematics and Mathematical Sciences*, page Article ID 760613, 2014.

[8] S. Araci, W.A. Khan, M. Acikgoz, C. Ozel, and P. Kumam. A new generalization of apostol type hermite-genocchi polynomials and its applications. *SpringerPlus*, 5:Article ID 860, 2016.

[9] H. Jolany, H. Sharifi, and R.E. Alikelaye. Some results for the apostol-genocchi polynomials of higher order. *Bulletin of the Malaysian Mathematical Sciences Society*, 36(2):465–479, 2010.

[10] M. Zabrocky. *An Introduction to Ordinary Generating Functions.* 2015.

[11] H. Jolany and R. Corcino. Explicit formula for generalization of poly-bernoulli numbers and polynomials with a, b, c parameters. *Journal of Classical Analysis*, 6:119–135, 2015.

[12] M.P. Laurente, R.B. Corcino, and M.P. Vega. On multi poly-genocchi polynomials with parameters a, b, and c. *European Journal of Pure and Applied Mathematics*, 13:444–458, 2020.

[13] T. Kim, D.S. Kim, D.V. Dolgy, and S.H. Rim. Some formula for the product of two bernoulli and euler polynomials. *Abstract and Applied Analysis*, 2012:Article ID 784307, 15 pages.

[14] T. Kim, S.H. Rim, D.V. Dolgy, and S.H. Lee. Some identities of genocchi polynomials arising from genocchi basis. *Journal of Inequalities and Applications*, pages Article ID 43, 6 pages, 2013.

[15] S. Araci, E. Sen, and M. Acikgoz. Theorems on genocchi polynomials of higher order arising from genocchi basis. *Taiwanese Journal of Mathematics and Mathematical Sciences*, 18(2):473–482, 2014.

[16] Y. He, S. Araci, H.M. Srivastava, and M. Acikgoz. Some new identities for the apostol-bernoulli polynomials and the apostol-genocchi polynomials. *Applied Mathematics and Computation*, 262:31–41, 2015.

[17] T. Kim and D.S. Kim. Some identities of higher-order euler polynomials arising from euler basis. *Integral Transforms and Special Functions*, 2013.

[18] M. Kaneko. Multiple zeta values and poly-bernoulli numbers. Tokyo Metropolitan University Seminar Report, 1997.

[19] T. Kim, Y.S. Jang, and J.J. Seo. A note on poly-genocchi numbers and polynomials. *Applied Mathematical Sciences*, 8:4775–4781, 2014.

[20] B. Kurt. Some identities for the generalized poly-genocchi polynomials with the parameters a, b, and c. *Journal of Mathematical Analysis*, 8(1):156–163, 2017.

[21] D. Lim. Some identities of degenerate genocchi polynomials. *Bulletin of the Korean Mathematical Society*, 53(2):569–579, 2016.

[22] U. Duran, M. Acikgoz, and S. Araci. Symmetric identities involving weighted q-genocchi polynomials under s_4. *Proceedings of the Jangjeon Mathematical Society*, 18(4):455–465, 2015.

[23] E. Agyuz, M. Acikgoz, and S. Araci. A symmetric identity on the q-genocchi polynomials of higher-order under third dihedral group d_3. *Proceedings of the Jangjeon*

Mathematical Society, 18(2):177–187, 2015.

[24] C. B. Corcino and R. B. Corcino. Higher order apostol-type poly-genocchi polynomials with parameters a, b and c. *Communications of the Korean Mathematical Society*, 36(3):423–445, 2021.

[25] R. B. Corcino and C. B. Corcino. Higher order bivariate bell-based apostol-frobenius-type poly-genocchi polynomials with parameters a and b. *European Journal of Pure and Applied Mathematics*, 17(3):1471–1489, 2024.

[26] M. Kaneko. Poly-bernoulli numbers. *Journal de Théorie des Nombres*, 9:221–228, 1997.