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# Exploring the Associated Groups of Quasi-Free Groups

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**Abstract.** Let $G$ is a cyclic group. Then $H(G)$ is a trivial group and if $G = G_1 * \ldots * G_n$ is the free product of the groups $G_1, \ldots, G_n$, then $H(G) = H(G_1 * \ldots * G*) \approxeq H(G_1) \times \ldots \times H(G_n)$. Furthermore, if the groups $G_1, G_2, \ldots, G_n$are cyclic groups, then $H(G)$ is a trivial group. In this paper we show that for every group $G$ there exists a group denoted $H(G)$ and is called the associated group of $G$ satisfying some important properties that as application we show that if $F$ is a quasi-free group and $G$ is any group, then $H(F)$is trivial and $H(F * G) \approxeq H(G)$, where a group is termed a quasi-free group if it is a free product of cyclic groups of any order.

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## 1. Introduction

We introduce the following basic concepts needed for the definition of associated groups of given groups [1].

(1) Let $G$ be a group.

i. If $A$ and $B$ are two subsets of $G$, let $[A, B]$ be the subgroup of $G$ generated by the elements

$[a, b] = aba^{-1}b^{-1}$ with $a \in A$ and $b \in B$. Define $G' = [G, G]$ to be the derived subgroup of $G$ generated by the elements $[x, y] = xyx^{-1}y^{-1}$ with $x, y \in G$. It is clear that $G'$ is a normal subgroup of $G$. For more details see [8].

ii. If $R$ is a subset of $G$, let $R^G$ to be the intersection of all normal subgroups of $G$ containing $R$. It is clear that $R \subseteq R^G$ and $R^G$ is a normal subgroup of $G$ [7].

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(2) Let $X$ be a set and let $F_X$ be the free group of the reduced words of $X$ generated by $X$. Then any element $f \in F_X, f \neq 1$ is uniquely written as
$f = x_1{}^{\alpha_1}\chi_2{}^{\alpha 2} \dots x_n{}^{\alpha n}, x_{i+1}^{\alpha+1} \neq -\alpha_i, x_i \in X, \alpha_i = \pm 1, i = 1, 2, \dots, n.$

A group $H$ is called a free group of base $S \subseteq H$ if $H$ is isomorphic to $F_S$ that is, $H \cong F_S$. The universal property of the free group $F$ of base $S$ [4], states that given any function $f : S \to G$ from $S$ to a group $G$, there exists a unique homomorphism $\varphi : F_S \to G$ called the universal extension of $f$, that is, if $a \in S$, then $\varphi(a) = f(a)$. Also, $\varphi$ is an epimorphism if and only if $f(S)$ generates $G$.

(3) Let $X$ be a set and let $R \subseteq F_X$ a subset of $F_X$. Let $\langle x \mid R\rangle$ stand for the quotient group, such that

$\langle x \mid R\rangle = F_X/\bar{R}$, where $\bar{R} = R^{F_X}$ is the normal closure of $R$ in $F_X$.$\langle x \mid R\rangle$ is called a presentation. We say that the group $G$ has the presentation $\langle x \mid R\rangle$ if $G \cong \langle x \mid R\rangle$. From above we see that a group $G$ has the presentation $\langle x \mid R\rangle$ if and only if there exists an onto

function $f : X \to G$, such that $f(X)$ generates $G$ and the normal closure $R^{F_X}$ of $R$ in $F_X$ satisfies the condition that $R^{F_X} = ker(\varphi)$, where $\varphi : F_X \to G$ is the universal extension of $f$.

## 2. The Associated Groups

The concept of the associated group of a given group is introduced in [3] and [6] is defined as follows. Let $G$ denote an arbitrary group. For $x, y \in G$, let $\langle x, y\rangle$ and let $\langle G, G\rangle = \{\langle x, y\rangle : x, y \in G\}$ and $F_{\langle G,G\rangle}$ be the free group freely generated by all pairs $\langle x, y\rangle$ with $x, y \in G$. Then any element $\alpha \in F_{\langle G,G\rangle}, \alpha \neq 1$ is uniquely written as
$\alpha = \langle x_1, y_1\rangle^{\alpha_1} \langle x_2, y_2\rangle^{\alpha 2} \dots \langle x_n, y_n\rangle^{\alpha n} \langle x_{i+1}, y_{i+1}\rangle^{\alpha i+1} \neq \langle x_i, y_i\rangle^{-\alpha i}$,
where $x_i, y_i \in G, \alpha_I = \pm 1, i = 1, 2, \dots, n$.

**Proposition 1.** *For any group $G$ there is a unique epimorphism from $F_{\langle G,G\rangle}$ to $[G, G]$ taking each element $\langle x, y\rangle \in F_{\langle G,G\rangle}, x, y \in G$ to the element $[x, y] = xyx^{-1}y^{-1} \in [G, G]$.*

Proof. Let $f : \langle G, G\rangle \to [G, G]$ be the function given by $f(\langle x, y\rangle) = [x, y] = xyx^{-1}y^{-1}$ with $x, y \in G$. Since $F\langle G, G\rangle$ is a free group on $\langle G, G\rangle$, the universal property shows that there exists a unique homomorphism the function $\varphi_G : F_{\langle G,G\rangle} \to [G, G]$ satisfying the condition that

$\langle x_1, y_1\rangle\, \varphi_G(\langle x, y\rangle) = [x, y] = xyx^{-1}y^{-1}$ with $x, y \in G$. This shows that $f$ is the restriction of $\varphi_G$ on $f$. That is, $\varphi_G \mid \langle G, G\rangle = f$, or $\varphi_G(\langle x, y\rangle) = f(\langle x, y\rangle)$ for all,$y \in G$. So for any element $\alpha \in F_{\langle G,G\rangle}, \alpha \neq 1, \alpha$ can be written uniquely as
$\alpha = \langle x_1, y_1\rangle^{\alpha_1} \langle x_2, y_2\rangle^{\alpha 2} \dots \langle x_n, y_n\rangle^{\alpha n}$. and the value of $\alpha$ under $\varphi_G$ is given by
$\varphi_G(\alpha) = \langle x_1, y_1\rangle^{\alpha_1} \langle x_2, y_2\rangle^{\alpha_2} \dots \langle x_n, y_n\rangle^{\alpha n}$.

Now if $\Phi : F_{\langle G,G\rangle} \to [G, G]$ is a homomorphism, such that
$\Phi(\langle x, y\rangle) = [x, y] = xyx^{-1}y^{-1}$ with $x, y \in G$, then $\Phi = \varphi_G$. Consequently, $\varphi_G$ is the unique required homomorphism.

Since $f(\langle G,G\rangle) = [G,G]$ generates $[G,G]$, this implies that $\varphi_G$ is an epimorphism. This complete the proof.

**Definition 1.** *For any group $G$, let $\varphi_G{:}F_{\langle G,G\rangle} \to [G,G]$ be the unique epimorphism of Proposition 1 satisfying the condition that $\varphi_G(\langle x,y\rangle) = [x,y] = xyx^{-1}y^{-1}$ with $x,y \in G$.*

We denote by $C(G) = Ker\varphi_G$ ) the kernel of $\varphi_G$. Then it is clear that $C(G)$ is a normal subgroup of $F_{\langle G,G\rangle}$.

**Definition 2.** *For any group $G$, let $R(G) \subseteq F_{\langle G,G\rangle}$ be the set of the following elements of*

$$\left.\begin{array}{l}\langle x,x\rangle\\ \langle x,y\rangle\langle y,x\rangle\\ \langle y,z\rangle^x\langle x,z\rangle\langle xy,z\rangle^{-1}\\ \langle y,z\rangle^x\langle y,z\rangle^{-1}\langle x,[y,z]\rangle^{-1}\end{array}\right\}\text{ for } x,y,z\in G,\text{ where } \langle y,z\rangle^x = \langle y^x,z^x\rangle = \left\langle xyx^{-1}, xzx^{-1}\right\rangle.$$

**Lemma 1.** *Let $G$ be a group of presentation $\langle X \mid R\rangle$. Then $H(G) \cong \bar{R}\cap[F_X,F_X] \,/\, \left[F_X,\bar{R}\right]$.*

Proof. See [3].

**Theorem 1.** *For any group $G$, the normal closure $[R(G)]^{F\langle G,G\rangle}$ of $R(G)$ in $F_{\langle G,G\rangle}$ is contained in $C(G)$.*

Proof. First we show that $R(G) \subseteq C(G)$. This is equivalent of showing that the value of any element $\alpha \in R(G)$ under the epimorphism $\varphi_G : F_{\langle G,G\rangle} \to [G,G]$ equals $\varphi_G(\alpha) = 1$, the identity element of $G$.

(1) Let $x \in G$. Then $\langle x,x\rangle \in F\langle G,G\rangle$ and $\varphi G(\langle x,x\rangle) = [x,x] = xx^{-1}x^{-1} = 1$.

(2) Let $x,y,z \in G$. Then the elements $\langle y,z\rangle^x, \langle x,z\rangle, \langle xy,z\rangle^{-1}$ and $\left(\langle y,z\rangle^x\langle x,z\rangle\langle xy,z\rangle^{-1}\right.$ are in $F_{\langle G,G\rangle}$ and $\varphi G\left(\langle y,z^x\langle x,z\rangle\langle xy,z\rangle^{-1}\right) = \varphi_G\left(\langle y,z\rangle^X\right)\varphi_G(\langle x,z\rangle)\varphi_G\left(\langle xy,z\rangle^{-1}\right)''$

(3) Let $x,y,z \in G$. Then the elements $\langle y,z\rangle^x, \langle y,z\rangle^{-1}, \langle x,[y,z]\rangle^{-1}$ and $\langle y,z\rangle^x\langle y,z\rangle^{-1}\langle x,[y,z]\rangle^{-1}$ are in $F_{\langle G,G\rangle}$ and $\varphi_G\left(\langle y,z\rangle^x\langle y,z\rangle^{-1}\langle x,[y,\ z]\rangle^{-1}\right) = \varphi_G\left(\langle y,z\rangle^x\right)\varphi_G\left(\langle y,z\rangle^{-1}\right)\varphi_G\left(\langle x,[y,z]\rangle^{-1}\right)$

From above we have $R(G) \subseteq C(G)$. Since $C(G)$ is a normal subgroup of $F_{\langle G,G\rangle}$, this implies that the normal closure $[R(G)]^{\,F\langle G,G\rangle}$ of $R(G)$ in $F_{\langle G,G\rangle}$ is contained in $C(G)$. This complete the proof.

**Definition 3.** *[3] For any group $G$, let $B(G) = [R(G)]^{F\langle G,G\rangle}$ be the normal closure of $R(G)$ in $F_{\langle G,G\rangle}$ and $H(G)$ be the group $H(G) = C(G)/B(G) = \{\alpha B(G) : \alpha \in C(G)\}$, the quotient group of the set of left cosets of $B(G)$ in $C(G).H(G)$ is called the associated group of the group $G$.*

**Proposition 2.** *The associated group of any infinite cyclic group is trivial.*

Proof. If $G$ is an infinite cyclic, then $G$ is generated by a single element $g$ and $G$ has the presentation $G = \langle x \mid \varnothing \rangle = \langle X \mid R \rangle$, where $X = \{x\}$ and $R = \varnothing$, the empty set. Then the normal closure $R^{F_X}$ of $R$ in $F_X$ is $\{1\}$, the identity subgroup of $G$. By Lemma 1, $H(G) \cong \bar{R} \cap [F_X, F_X] \,/\, [F_X, \bar{R}] = \{1\}/\{1\} = \{1\}$. Consequently, $H(G) \cong \{1\}$. This completes the proof.

**Theorem 2.** *The associated group of the free product of two groups is the direct product of associated groups of the two groups. That is, if $K$ and $L$ are two groups, then*

$H(K * L) \cong H(K) \times H(L)$.

Proof. See [3].

**Remark 1.** *We have the following notes regarding Theorem 2, let $K = \langle X \mid R \rangle$ and $L = \langle Y \mid S \rangle$ be presentations of the groups $K$ and $L$, such that $X \cap Y = \varnothing$. By [5], $K * L$ has the presentation $K * L = \langle X \cup Y \mid R \cup S \rangle$. The definition of the presentation implies that $K = \langle X \mid R \rangle = F_X/\bar{R}$, where $\bar{R}$ is the normal closure of $R$ in $F_X, L = \langle Y \mid S \rangle = F_Y/\bar{S}$, where $\bar{S}$ is the normal closure of $S$ in $F_Y$, and $K * L = \langle X \cup Y \mid R \cup S \rangle = F_{X \cup Y / X \cup Y}$, where $\overline{X \cup Y} = [R \cup S] F_{X \cup Y}$, normal closure of $R \cup S$ in $F_{X \cup Y}$. Lemma 1, implies that $H(K) \cong \bar{R} \cap [F_X, F_X] \,/\, [F_X, \bar{R}]$,*

$H(L) \cong \bar{S} \cap [F_Y, F_Y] \,/\, [F_Y, \bar{S}]$ and $H(K * L) \cong (\overline{R \cup S}) \cap [F_{X \cup Y}, F_{X \cup Y}] \,/\, [F_{X \cup Y}, \overline{R \cup S}]$.

**Corollary 1.** *Consider the groups $K$ and $L$ of presentations $K = \langle x_0, x_1, \ldots, x_{n+1} \mid r_1, \ldots, r_n, x_0 \rangle$, and $L = \langle x_0, x_1, \ldots, x_{n+1} \mid r_1, \ldots, r_n \rangle$. Then $H(K) \cong H(L)$.*

Proof. By [5], $L = K * P$ is the free product of $K$ and $P$, where $P$ is an infinite cyclic group. By Theorem 1, $H(L) \cong H(K) \times H(P)$ and by Proposition 2 $H(P)$ is trivial. That is, $H(P) \cong \{1\}$. This implies that $H(L) \cong H(K) \times \{1\} \cong H(K)$. This complete the proof.

## 3. The Associated Groups of Quasi-Free Groups

Recall that a group is termed a quasi-free group if it is a free product of cyclic groups of any order. In this section we show that if $F$ is a quasi-free group and $G$ is any group then the associated group $H(F)$ of $F$ is trivial and the associated group $H(F * G)$ of the free product $F * G$ of $F$ and $G$ satisfies the condition $H(F * G) \cong H(G)$. First we introduce the following concept. If $G$ is a finite group, the Schur multiplier of $G$ introduced in [8, p. 14] is denoted by $M(G)$ and is defined to be the second cohomology group $H^2\ (G, C^*)$ of $G$, where $C^*$ is the set of nonzero complex numbers.

**Proposition 3.** *Let $G$ be a finite group. Then $H(G) \cong M(G)$. Furthermore, if $G$ has the presentation $\langle X \mid R \rangle$, where $X$ has cardinality $m$ and $R$ has cardinality $n$, then $H(G) \cong \{1\}$ if $m = n$ and $H(G)$ is cyclic if $n = m + 1$.*

Proof. If $G$ is finite, then by [2] we have $H(G) = M(G)$. If $m = n$, then

by [3], $M(G) = 1$. Consequently, $H(H) = 1$. If $n = m + 1$, then $M(G)$ is cyclic. This implies that $H(G)$ is cyclic. This complete the proof.

**Lemma 2.** *The associated group of any cyclic group is trivial.*

Proof. Let $G$ be any cyclic group. We need to show that $H(G)$ is trivial.

If $G$ is an infinite cyclic group, then by Proposition 2, $H(G) \cong \{1\}$. If $G$ is a finite cyclic group of order $n$ then $G$ has the presentation $G = \langle x \mid x^n \rangle$ of one generator $x$ and one relater $x^n = 1$. So the number of the generators of $G$ is the same number of relaters $= 1$. Then Proposition 3, shows that $H(G) \cong \{1\}$. This complete the proof.

**Theorem 3.** *The associated group of a quasi-free group is trivial.*

Proof. Let $F$ be a quasi-free group and $G$ be any group. Then
$H(F) \cong \{1\}$ and $H(F * G) \cong H(G)$.

Let $F$ be a quasi-free group. Then $F$ can be written as
$F = \underbrace{C_\infty * C_\infty * \ldots * C_\infty}_{p-\text{ factors}} * \underbrace{C_{\alpha_1} * C_{\alpha_2} * \ldots * C_{\alpha_n}}_{q-\text{ factors}}$, where $C_\infty$ stands for an infinite cyclic group and $C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_n}$ stand for finite cyclic groups of orders $\alpha_1, \alpha_2, \ldots, \alpha_\mathrm{n}$ respectively. From Proposition 1, we have

Lemma 2, implies that $H(F) \cong \underbrace{(1) \times (1) \times \ldots \times (1)}_{p-\text{ factors}} \times \underbrace{(1) \times (1) \times \ldots \times (1)}_{q-\text{ factors}} \cong 1.$

This completes the proof.

**Corollary 2.** *Let* $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ *be the group of integers and*

$Z_n = \{0, 1, \ldots, n-1\}$ be the group of integers modulo $n$. Then
$H(Z) \cong \{1\}$ and $H(Z_n) \cong \{1\}$.

**Corollary 3.** *If $K$ is a free group and $G$ is any group, then*
$H(K) \cong \{1\}, H(F\langle G, G\rangle) \cong \{1\}$ *and* $H\left(F_{\langle G,G\rangle} * G\right) \cong H(G)$.

As an example of a quasi-free group we have the following.

**Example 1.** *Let $Z$ be the group of integers and $G = PSL(2, Z)$ be the projective special linear group of degree 2 over $Z$. It is well known that[1] , $G = A * B$, the free product of the cyclic groups $A$ of order 2 , and the cyclic group $B$ of order 3 defined*

$G$ is a quasi-free group, and Theorem 3, shows that $H(F * G) \cong \{1\}$.

## 4. The Associated Groups of Dihedral Groups and Quaternion Groups

For the structures of dihedral groups and quaternion groups we refer the readers to [8].

**Proposition 4.** *Let $G$ be a dihedral group. Then*

(i) If $G = D_\infty$ is infinite, then $H(D_\infty) \cong \{1\}$.

(ii) If $G = D_n$ is finite, then $H(D_n)$ is cyclic.

Proof. (1) $G = D_\infty$ is defined as the group of two-by-two matrices with entries from the group of integers $Z$ of the form
$\begin{pmatrix} \varepsilon & k \\ 0 & 1 \end{pmatrix}$ where $\mathcal{E}$ is 1 or -1 , and $k$ is any integer.

$D_\infty = A * B$, the free product of the groups $A = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$,
$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

$A$and $B$ are of order 2 which implies $A$ and $B$ are cyclic groups, then $D_\infty$ is a quasi-free group and by Theorem 3.1, $H(D_\infty) \cong \{1\}$. (2) If $G = D_n$ is a finite dihedral group, then $G$ is of order $2n$ and is defined as the group of two-by-two matrices, with entries from the ring of integers $Z_n$ mod $n$ of the form $\begin{pmatrix} \varepsilon & k \\ 0 & 1 \end{pmatrix}$, where $\varepsilon$ is 1 or -1 , and $k$ is any integer mod $n$. Then by [8], $D_n$ has the presentation
$D_n = \langle x, y \mid x^n, y^2, (xy)^2 \rangle$ of 2 generators and 3 relations. Since $D_n$ is finite and $3 = 2+1$, Proposition 3 shows that $H(D_n)$ is cyclic. This complete the proof.

**Example 2.** *Let $G = PSL(2, F)$ be the projective special linear group of degree 2 over the Galois field $F$ consisting of 5 elements. Then $H(G) \cong C_2$ and for every free group $K$ have $H(K * G) \cong C_2$, where $C_2$ is a cyclic group of order 2, because it is well known that [3]. Now by [2], $G$ has the presentation $G = \langle x, y \mid x^5, y^3, (xy)^2 \rangle$ of two generators and three relaters. By Proposition 3, $H(G)$ is cyclic.*

**Proposition 5.** *Let $Q_n$ be the quaternion group of order $4n$. Then $H(Q_n)$ is cyclic.*

Proof. It is well known in [4] that $Q_n$ has the presentation
$Q_n = \langle a, b \mid a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle$.

So the presentation of $Q_n$ is of 2 generators 3 relations so, by Proposition 3, $H(Q_n)$ is cyclic. This complete the proof.

## 5. Conclusion

It is stated and proven associated group of any cyclic group is trivial.

and associated group of a quasi-free group is trivial. In future work, we must reach facts related to the following

(i) Let $G = G_1 *_{\mathrm{A}} G_2$ be the free product of the groups $G_1$ and $G_2$ with an amalgamation subgroup A introduced in [5]. Find $H(G)$ in terms of $H(G_1), H(G_2)$ and $H(A)$.

(ii) Let $G$ be the HNN-group $G = \langle H, t_i \mid rel(H), t_i A_i t_{i-1} = B_i,\ i \in I \rangle$ of base $H$ and associated pairs $(A_i, B_i)$, $i \in I$ of subgroups of $H$ introduced in [1]. Find $H(G)$ in terms of $H(H)$, $H(A_i)$ and $H(B_i)$, $i \in I$.

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