EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 5360
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Bounds on Spectral Radius and Signless Laplacian Spectral Radius for Generalized Core-Satellite Graphs

Malathy V.[1], Kalyani Desikan[1,*]

[1] *Department of Mathematics, SAS, Vellore Institute of Technology, Chennai, India*

Abstract. A Generalized core-satellite graph $\Theta\left(c,\mathbf{S},\eta^*\right)$ belongs to the family of graphs of diameter two. It has a central core of nodes connected to a few satellites, where all satellite cliques are not identical and might be of different sizes. These graphs can be used to model any real-world complex network. Using core-satellite graphs, properties like hierarchical structure can be conveniently modeled for large complex networks. In this paper, we obtain the lower and upper bounds for the spectral radius and signless Laplacian spectral radius of the Generalized core-satellite graph, in terms of number of vertices, number of edges, and the graph parameters associated with the structure of the graph in both satellites and the core.

2020 Mathematics Subject Classifications: 05C50

Key Words and Phrases: Spectral radius, signless Laplacian spectral radius, generalized core-satellite graphs

1. Introduction

Generalized core-satellite graphs can be used to model any complex real-world network. Using core-satellite graphs, properties like hierarchical structure can be conveniently modeled for large, complex networks. A hierarchical design model provides a reference topology that separates a network into distinct layers. Each of these layers has a series of functions that define its role in the network. This model facilitates making the network scalable, stable, deterministic, and reliable, provides better security, is effortless to manage and design, provides enhanced performance, and is also cost-efficient. Factors like these combine to make core-satellite graphs a dynamic model design for certain types of real-world networks [1, 2]. Generalized core-satellite graphs are denoted as $\Theta\left(c,\mathbf{S},\eta^*\right)$ and belong to the family of graphs of diameter two. It has a central core of vertices connected to a few satellites, where all satellite cliques are not identical and might be of different sizes. There is no restriction on having the same number of vertices or nodes.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.5360

Email addresses: malathy.viswanathan2015@vit.ac.in (M. V.),
kalyanidesikan@vit.ac.in (K. Desikan)

Estrada and Benzi [1] introduced the Generalized core-satellite graphs, where they generalized both the windmill and complete split graphs and analyzed certain features like clustering, assortativity, and spectral properties. The authors characterized the eigenstructure of these graphs' adjacency and Laplacian matrices, observed that their Laplacian eigenvalues are integral, and commented on the asymptotic behavior of quantities such as the synchronizability index and infection threshold. Also, they determined the general and spectral properties of core-satellite graphs and provided the bounds of the spectral radius of the Generalized core-satellite graph.

Generalized core-satellite class of graphs is found to resolve most of the social network issues where the network's graphs are analyzed. Due to the expanding usage of social networks such as LinkedIn, Facebook, Instagram, Twitter, and Google+, malicious users seek to impinge on the privacy of other users and exploit their credentials by creating fraudulent accounts. This has become a cause of concern for users. Hence, social network providers are attempting to identify these users and their fake accounts to remove them from social networking environments using graph analysis. Graph similarity measures are used in graph analysis. Some significant similarity measures namely Jaccard, cosine, and L_1 norm are a few measures where Friendship graphs (core-satellite graph) are utilized to detect suspicious accounts [3].

Many results on bounds for the spectral radius and signless Laplacian spectral radius have been given as functions of graph parameters such as the number of vertices, edges, degree sequence, average 2-degree, diameter, covering number, domination number, independence number, and others [4–7]. In [8], Nair Abreu et al. mentioned that this class of graphs is equivalent to chordal graphs having only one minimal vertex separator and the subclass of a quasi-threshold graph. They demonstrated that this class of graphs belongs to the hierarchical structure of chordal graphs. Moreover, in [9], Das has proved a conjecture on the complete split-like graph. In [10], Liu et al. presented several upper and lower bounds on the k-th largest eigenvalue of A_α - matrix and characterized the extremal graphs corresponding to the bounds obtained.
We find many recent articles on spectral radius and signless Laplacian spectral radius. In [11], Wang and Guo investigated the upper bounds of the spectral radius of the coalescense of two graphs generalizing some results by Passbani and Salemi in 2019 and as an application provided a new sharp upper bound on the spectral radius of a tree. Ghorbani and Amraei [12], investigated the spectra of certain classes of vertex or edge-transitive graphs for extended adjacency matrix and obtained some new bounds for both the smallest and the largest eigenvalues examining the behaviour of A_{ex}-energy of a graph. In [13], authors characterized irregular bipartite graphs with maximum spectral radius and presented an upper bound on the spectral radius in terms of the order and maximum degree.
Das and Liu in [14] proved complete split graph $CS(n, \alpha)$, the graph on n vertices consisting of a clique on $(n - \alpha)$ vertices and an independent set on the remaining α where $(1 \leq \alpha \leq n - 1)$ vertices in which each vertex of the clique is adjacent to each vertex of

the independent set. They determined its Laplacian spectrum when $1 \leq \alpha \leq n-1$ and the signless Laplacian spectrum when $1 \leq \alpha \leq n-1, \alpha \neq 3$. In [15], the authors obtained sharp upper bounds on the Q-index of (minimally) 2-connected graphs with given size, and characterize the corresponding extremal graphs.

In this article, we have considered Generalized core-satellite graph $\Theta(c, \mathbf{S}, \eta^*)$ and obtained results on the upper and lower bounds for the spectral radius and signless Laplacian spectral radius in terms of the parameters related to the structure of the graph through a different approach.

2. Preliminaries

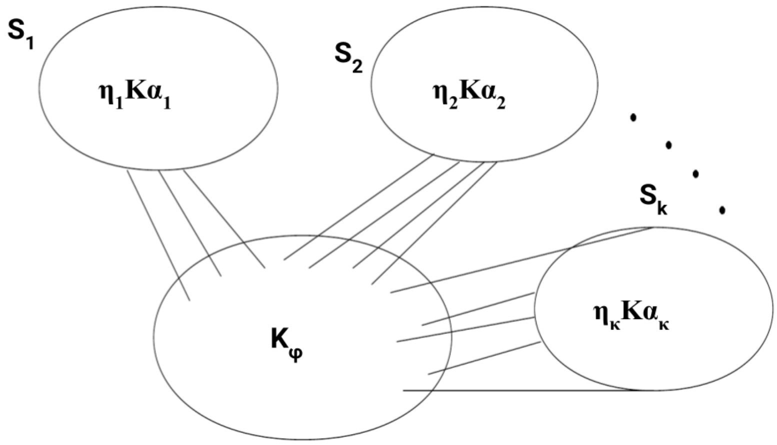

Figure 1: Generalized core-satellite graph $(\eta_1 K_{\alpha_1}) \bigtriangledown K_\phi \cup (\eta_2 K_{\alpha_2}) \bigtriangledown K_\phi \cup ... \cup (\eta_k K_{\alpha_k}) \bigtriangledown K_\phi$

In this section, we discuss some preliminary findings that will be required to support our main results.

Let $G = (V(G), E(G))$ be a simple, connected, undirected, and finite graph. Let the order, $|V(G)|$, be n and let the size of the graph, $|E(G)|$, be m, respectively. Let $A(G)$ denote the (0,1) - adjacency matrix and $D(G)$ the diagonal matrix whose diagonal entries are degree sequence of G. Let $Q(G) = D(G) + A(G)$ be the signless Laplacian matrix of the graph G.
According to Geršgorin's Theorem, if C is an $n \times n$ real symmetric matrix then its eigenvalues are non-negative real numbers. Since $A(G)$ and $Q(G)$ are real symmetric matrices, their eigenvalues are non-negative real numbers. The eigenvalues of $A(G)$ and $Q(G)$ are ordered as $\rho(G) = \rho_1(G) \geq \rho_2(G) \geq ... \geq \rho_n(G)$ and $\mu(G) = \mu_1(G) \geq \mu_2(G) \geq ... \geq \mu_n(G)$, respectively. The largest eigenvalue of the adjacency matrix of the graph (known as spectral radius) and signless Laplacian matrix (known as signless Laplacian spectral radius)

are denoted as $\rho_1(G)$ and $\mu_1(Q(G))$, respectively.

Definition 1. *[1] The join (or complete product) $G_{\psi_1} \bigtriangledown G_{\psi_2}$ of graphs G_{ψ_1} and G_{ψ_2} is obtained from $G_{\psi_1} \cup G_{\psi_2}$ by joining every vertex of G_{ψ_1} with every vertex of G_{ψ_2}.*

The Generalized core-satellite graph consists of core clique $c = K_\phi$ with ϕ nodes and satellites $S_1, S_2, S_3, ..., S_k$ with $K_{\alpha_1}, K_{\alpha_2}, ..., K_{\alpha_k}$ cliques. Let η_1, η_2, η_3,...,η_k be the number of copies of the cliques $K_{\alpha_1}, K_{\alpha_2}$,...,$K_{\alpha_k}$ having degrees $d_{\alpha_1}, d_{\alpha_2}, ..., d_{\alpha_k}$, respectively. Let $d_{\alpha_1} < d_{\alpha_2} < .... < d_{\alpha_k}$.
Let

$$\mathbf{S} = (S_1, S_2, ..., S_k)$$

where each S_i is the i^{th} satellites in the graph having η_i copies of K_{α_i} cliques and let

$$\eta^* = (\eta_1, \eta_2, ..., \eta_k).$$

Hence, the Generalized core-satellite graph is denoted as $\Theta(c, \mathbf{S}, \eta^*)$.

Theorem 1. *[1]* The spectral radius (Perron eigenvalue) $\rho_1(G)$ is given by the largest root of

$$(\lambda - \phi + 1)\prod_{i=1}^{k}(\lambda - \alpha_i + 1) = \phi\sum_{i=1}^{k}\eta_i\alpha_i\prod_{j\neq i}(\lambda - \alpha_j + 1)$$

and satisfies the bounds

$$(\phi - 1) + \max_{1\leq i\leq k}(\alpha_i) < \rho_1(G) < (\phi - 1) + \sum_{i=1}^{k}\eta_i\alpha_i. \tag{1}$$

Theorem 2. *[16]* Let $G = (V, E)$ be a graph. Then

$$\min_{v\in V(G)}\left(\sum_{uv\in E} d(u)\right)^{1/2} \leq \rho_1(G) \leq \max_{v\in V(G)}\left(\sum_{uv\in E} d(u)\right)^{1/2}. \tag{2}$$

Moreover, if G is connected then either of the equalities holds iff $\sum_{uv\in E} d(u)$ is the same $\forall$ $v \in V$.

Remark 1. (a) The number of edges in S_i alone is

$$\eta_i\left(\frac{\alpha_i(\alpha_i - 1)}{2}\right)$$

for $i = 1, 2....k$.

(b) The number of edges joining the satellite graphs with vertices of the core graph is

$$\phi\eta_i\alpha_i$$

for $i = 1, 2....k$.

(c) The number of edges in the core graph K_ϕ alone is

$$\frac{\phi\left(\phi-1\right)}{2}.$$

Hence, the total number of edges in the core-satellite graph G is

$$m = \left\{ \sum_{i=1}^{k} \left(\frac{\eta_i \alpha_i (\alpha_i - 1)}{2} + \phi \eta_i \alpha_i \right) + \frac{\phi(\phi-1)}{2} \right\}$$

and the number of vertices of the graph G is

$$n = \phi + \sum_{i=1}^{k} \eta_i \alpha_i.$$

Theorem 3 and Theorem 4 derived by Duan and Zhou, provide the lower bound and upper bound for the largest eigenvalue of any general non-negative matrix.

Theorem 3. *[17]* Let $A = (a_{ij})$ be an $n \times n$ non-negative matrix with row sums $r_1, r_2, ..., r_n$ where $r_1 \geq r_2 \geq .... \geq r_n$. Let S and T be the smallest diagonal and the smallest non-diagonal elements of A, respectively. Let

$$\varphi_n = \frac{(r_n + S - T) + \sqrt{(r_n - S + T)^2 + 4T \sum_{i=1}^{n-1} (r_i - r_n)}}{2} \tag{3}$$

Then $\rho_1(A(G)) \geq \varphi_n$. Moreover, if A is irreducible, $\rho_1(A(G)) = \varphi_n$ iff $r_1 = r_2 = ... = r_n$ or $T > 0$, and for some $2 \leq t \leq n$, A satisfies the following conditions:
(1) $a_{ii} = S$ for $1 \leq i \leq t-1$.
(2) $a_{ik} = T$ for $1 \leq i \leq n-1$, $1 \leq k \leq t-1$ with $k \neq i$.
(3) $r_t = r_{t+1} .... = r_n$.
(4) $a_{nk} = T$ for $1 \leq k \leq t-1$.

Theorem 4. *[17]* Let $A = (a_{ij})$ be an $n \times n$ non-negative matrix with row sums r_1,r_2,...,r_n where $r_1 \geq r_2 \geq .... \geq r_n$. Let M and N be the largest diagonal and the largest non-diagonal elements of A, respectively. Suppose $N > 0$. For $1 \leq l \leq n$, let

$$\varphi_l = \frac{(r_l + M - N) + \sqrt{(r_l - M + N)^2 + 4N \sum_{i=1}^{l-1} (r_i - r_l)}}{2} \tag{4}$$

Then $\rho_1(A(G)) \leq \varphi_l$ for $1 \leq l \leq n$. Moreover, if A is irreducible, $\rho_1(A(G)) = \varphi_l$ iff $r_1 = r_2 = ... = r_n$ or for some $2 \leq t \leq l$, A satisfies the following conditions:
(1) $a_{ii} = M$ for $1 \leq i \leq t-1$.

(2) $a_{ik} = N$ for $1 \leq i \leq l-1$, $1 \leq k \leq t-1$ with $k \neq i$.
(3) $r_t = r_{t+1} ... = r_n$.
(4) $a_{ik} = N$ for $1 \leq i \leq n$, $1 \leq k \leq t-1$.

Theorem 5. *[16]* Let M be a real symmetric $n \times n$ matrix, and let β be an eigenvalue of M with eigenvector $\mathbf{x}$ all of whose entries are non-negative. Denote the i^{th} row sum of M by $R_i(M)$. Then

$$\min_{1 \leq i \leq n} R_i(M) \leq \beta(Q(G)) \leq \max_{1 \leq i \leq n} R_i(M). \tag{5}$$

Moreover, if all entries of $\mathbf{x}$ are positive then either of the equalities holds if and only if the row sums of M are all equal.

Theorem 6. *[16]* Let M be a real symmetric $n \times n$ matrix, and let β be an eigenvalue of M with eigenvector $\mathbf{x}$ all of whose entries are non-negative. Denote the i^{th} row sum of M by $R_i(M)$. Let p be a polynomial. Then

$$\min_{1 \leq i \leq n} R_i(p(M)) \leq p(\beta(Q(G))) \leq \max_{1 \leq i \leq n} R_i(p(M)). \tag{6}$$

Moreover, if all entries of $\mathbf{x}$ are positive then either of the equalities holds if and only if the row sums of M are all equal.

Lemma 1. *[5, 16]* Let $G = (V, E)$ be a simple graph. Then

$$\sqrt{2} \min_{v \in V(G)} \sqrt{d^2(v) + \sum_{uv \in E(G)} d(u)} \leq \mu_1(Q(G))$$

$$\leq \sqrt{2} \max_{v \in V(G)} \sqrt{d^2(v) + \sum_{uv \in E(G)} d(u)}.$$

Moreover, if G is connected, both the equalities hold iff $2d^2(v) + 2 \sum_{uv \in E(G)} d(u)$ is the same $\forall v \in V(G)$.

3. Main Results

3.1. Bounds on spectral radius

In this section, we derive tight upper bound and lower bound of $\rho_1(G)$ for Generalized core-satellite graphs.

Theorem 7. Let $G = \Theta(c, \mathbf{S}, \eta^*)$ be the Generalized core-satellite graph with n vertices and m edges. Then the lower bound and upper bound for the spectral radius of G are

$$\sum_{j=1}^{k}\left\{\phi\sum_{i=1}^{k}\Big(\eta_i\alpha_i+(\phi-1)\Big)+(\alpha_j-1)(\phi+\alpha_j-1)\right\}^{1/2}\leq k\rho_1(G) \tag{7}$$

$$\rho_1(G)\leq\left\{\phi\Big(\sum_{i=1}^{k}\eta_i\alpha_i(\phi+\alpha_i-1)\Big)\right\}^{1/2}. \tag{8}$$

Proof. We derive the lower bound given in equation (7) using the lower bound of equation (2). From Theorem 2 we observe that the lower bound is obtained by considering vertices having mimimum degree. In $S_i \bigtriangledown K_\phi$ we see that vertices with minimum degree are in the satellite S_i.
For the S_l^{th} clique, we have

$$\min_{v\in V(G)}\left(\sum_{u\sim v\in E(G)} d(u)\right)^{1/2}=\left(\sum_{u\sim v\in S_l} d(u)\right)^{1/2}$$

$$=\left\{\phi\sum_{i=1}^{k}\Big(\eta_i\alpha_i+(\phi-1)\Big)+(\alpha_l-1)(\phi+\alpha_l-1)\right\}^{1/2}\leq\rho_1(G).$$

$$\left(\sum_{u\sim v\in S_1} d(u)\right)^{1/2}+\left(\sum_{u\sim v\in S_2} d(u)\right)^{1/2}+...+\left(\sum_{u\sim v\in S_k} d(u)\right)^{1/2}\leq k\rho_1(G).$$

Hence, we have

$$\left\{\phi\sum_{i=1}^{k}\Big(\eta_i\alpha_i+(\phi-1)\Big)+(\alpha_1-1)(\phi+\alpha_1-1)\right\}^{1/2}+$$

$$\left\{\phi\sum_{\{i=1}^{k}\Big(\eta_i\alpha_i+(\phi-1)\Big)+(\alpha_2-1)(\phi+\alpha_2-1)\right\}^{1/2}+...+$$

$$\left\{\phi\sum_{i=1}^{k}\Big(\eta_i\alpha_i+(\phi-1)\Big)+(\alpha_k-1)(\phi+\alpha_k-1)\right\}^{1/2}\leq k\rho_1(G).$$

Hence,

$$\sum_{l=1}^{k}\left\{\phi\sum_{i=1}^{k}\Big(\eta_i\alpha_i+(\phi-1)\Big)+(\alpha_l-1)(\phi+\alpha_l-1)\right\}^{1/2}\leq k\rho_1(G). \tag{9}$$

From Theorem 2 we observe that the upper bound is obtained by considering vertices having maximum degree. In $S_i \bigtriangledown K_\phi$, we see that vertices with a maximum degree are in the core K_ϕ. Hence, from Theorem 2, we have,

$$\max_{v\in V(G)}\left(\sum_{uv\in E(G)} d(u)\right)^{1/2} = \left(\sum_{u\sim v\in K_\phi} d(u)\right)^{1/2}$$

$$= \left\{\phi\Big(\sum_{i=1}^{k}\eta_i\alpha_i(\phi+\alpha_i-1)\Big)\right\}^{1/2} \geq \rho_1(G). \tag{10}$$

Hence proved.

Remark 2. We observe that

$$\left\{\phi\Big(\sum_{i=1}^{k}\eta_i\alpha_i(\phi+\alpha_i-1)\Big)\right\}^{1/2} \geq \left\{\phi\left(\frac{\sum_{i=1}^{k}\eta_i\alpha_i\left(\phi+\alpha_i-1\right)}{k}\right)\right\}^{1/2} \geq \rho_1(G) \tag{11}$$

Theorem 8. Let G be the Generalized core-satellite graph of order n with m edges and $S_1, S_2, S_3, ..., S_k$ be the k satellites connected to the core graph K_ϕ, then

$$\sqrt{\frac{2m}{k}} < \rho_1(G) < \sqrt{2m}.$$

Proof. The sum of the degrees of the vertices of satellites $S_1, S_2, S_3, ..., S_k$ and the sum of the degrees of the vertices of a core graph K_ϕ is $2m$, that is,

$$\sum_{i=1}^{k}\left(\sum_{u\sim v\in S_i} d(u)\right) + \left(\sum_{u\sim v\in K_\phi} d(u)\right) = 2m.$$

Considering the lower bound of equation (2), we have for S_l

$$\min_{v\in V(G)}\left(\sum_{u\sim v\in E(G)} d(u)\right)^{1/2} = \left(\sum_{u\sim v\in S_l} d(u)\right)^{1/2}$$

$$= \left\{\phi\sum_{i=1}^{k}\Big(\eta_i\alpha_i + (\phi-1)\Big) + (\alpha_l-1)(\phi+\alpha_l-1)\right\}^{1/2} \leq \rho_1(G).$$

Hence for the S_l^{th} clique, we have

$$\left(\sum_{u\sim v\in S_l} d(u)\right) = \left\{\phi\sum_{i=1}^{k}\Big(\eta_i\alpha_i + (\phi-1)\Big) + (\alpha_l-1)(\phi+\alpha_l-1)\right\} \leq \rho_1(G)^2.$$

We have

$$\sum_{l=1}^{k}\left(\sum_{u\sim v\in S_l} d(u)\right)=\sum_{l=1}^{k}\left\{\phi\sum_{i=1}^{k}\Big(\eta_i\alpha_i+(\phi-1)\Big)+(\alpha_l-1)(\phi+\alpha_l-1)\right\}\leq k\rho_1(G)^2.$$

Also, we have

$$\sum_{l=1}^{k}\left(\sum_{u\sim v\in S_l} d(u)\right)=\sum_{l=1}^{k}\left\{\phi\sum_{i=1}^{k}\Big(\eta_i\alpha_i+(\phi-1)\Big)+(\alpha_l-1)(\phi+\alpha_l-1)\right\}<2m.$$

We now show that $k(\rho_1(G))^2>2m$, for when $\phi\geq 1$ and $k\geq 2$.

We have

$$\begin{aligned}2m&=2\left\{\sum_{i=1}^{k}\left(\frac{\eta_i\alpha_i(\alpha_i-1)}{2}+\phi\eta_i\alpha_i\right)+\frac{\phi(\phi-1)}{2}\right\}\\&=\sum_{i=1}^{k}\eta_i\alpha_i^2+(2\phi-1)\sum_{i=1}^{k}\eta_i\alpha_i+\phi(\phi-1).\end{aligned}$$

Consider $(k(\rho_1(G))^2-2m)\geq$

$$\sum_{l=1}^{k}\left\{\phi\sum_{i=1}^{k}\Big(\eta_i\alpha_i+(\phi-1)\Big)+(\alpha_l-1)(\phi+\alpha_l-1)\right\}$$

$$-\left\{\sum_{i=1}^{k}\eta_i\alpha_i^2+(2\phi-1)\sum_{i=1}^{k}\eta_i\alpha_i+\phi(\phi-1)\right\}$$

$$=((k-2)\phi+1)\sum_{i=1}^{k}\eta_i\alpha_i-\sum_{i=1}^{k}\eta_i\alpha_i^2+(k^2-1)\phi(\phi-1)+\sum_{l=1}^{k}(\alpha_l-1)(\phi+\alpha_l-1)>0.$$

Therefore

$$2m<k(\rho_1(G))^2.$$

This implies

$$\sqrt{\frac{2m}{k}}<\rho_1(G) \tag{12}$$

Considering the upper bound of equation (2), we have

$$\left(\sum_{u\sim v\in K_\phi} d(u)\right)^{1/2}=\left\{\phi\sum_{i=1}^{k}\left(\eta_i\alpha_i\left(\phi+\alpha_i-1\right)\right)\right\}^{1/2}\geq\rho_1(G).$$

From the above inequality, we get

$$2m > \left\{ \phi \sum_{i=1}^{k} \left(\eta_i \alpha_i \left(\phi + \alpha_i - 1 \right) \right) \right\} \geq (\rho_1(G))^2.$$

Therefore

$$\sqrt{2m} > \rho_1(G).$$

Hence

$$\sqrt{\frac{2m}{k}} < \rho_1(G) < \sqrt{2m}. \tag{13}$$

Hence proved.

In the following theorem, we make use of Theorem 4 to obtain the upper bound for the Generalized core-satellite graph G. In order to make use of Theorem 4 we require the row sums of the adjacency matrix. In the adjacency matrix $A(G)$ of graph G, the vertices are arranged such that the top ϕ rows correspond to the vertices in the core K_ϕ, followed by the vertices of η_k copies of the cliques $K_{\alpha_k} \in S_k$. The remaining rows correspond to vertices of S_{k-1}, S_{k-2},...,S_1. Let $R_1^\wedge$, $R_2^\wedge$,...,$R_k^\wedge$ and $R_\phi^\wedge$ be the row sums corresponding to the vertices of the core K_ϕ and the satellites S_k, S_{k-1},...,S_1, i.e.

$$R_{\phi,1}^\wedge = R_{\phi,2}^\wedge = ... = R_{\phi,\phi}^\wedge = \sum_{i=1}^{k} \eta_i \alpha_i + (\phi - 1) = R_1^\wedge$$

$$R_{k,1}^\wedge = R_{k,2}^\wedge = ... = R_{k,\eta_k \alpha_k}^\wedge = (\phi + \alpha_k - 1) = R_2^\wedge$$
$$R_{k-1,1}^\wedge = R_{k-1,2}^\wedge = ... = R_{k-1,\eta_{k-1}\alpha_{k-1}}^\wedge = (\phi + \alpha_{k-1} - 1) = R_3^\wedge$$
$$R_{k-2,1}^\wedge = R_{k-2,2}^\wedge = ... = R_{k-2,\eta_{k-2}\alpha_{k-2}}^\wedge = (\phi + \alpha_{k-2} - 1) = R_4^\wedge$$

...
...
In general, we have

$$R_{k-(j-2),1} = R_{k-(j-2),2} = ... = R_{k-(j-2),\eta_{k-(j-2)}\alpha_{k-(j-2)}} = (\phi + \alpha_{k-(j-2)} - 1) = R_j^\wedge$$

...

$$R_{1,1}^\wedge = R_{1,2}^\wedge = ... = R_{1,\eta_1 \alpha_1}^\wedge = (\phi + \alpha_1 - 1) = R_{k+1}^\wedge.$$

Theorem 9. For the Generalized core-satellite graph G of order n and size m, let $S_1, S_2, S_3, .., S_k$ be the k satellites connected to the core graph K_ϕ. Let $R_1^\wedge, R_2^\wedge, ..., R_{k+1}^\wedge$

be the row sums corresponding to the core K_ϕ and the satellites S_k, S_{k-1},...,S_1. The row sums are such that $R_1^\wedge \geq R_2^\wedge \geq ... \geq R_{k+1}^\wedge$, then

$$\rho_1(G) \leq \frac{(\phi+\alpha_L-2)}{2} + \frac{\sqrt{(\phi+\alpha_L)^2 + 4\sum_{i=1}^{L-1}(R_i^\wedge - R_L^\wedge)}}{2}$$

where $1 \leq i \leq k$ and $1 \leq L \leq k+1$.

Proof. Using Theorem 4 we have

$$\rho_1(G) \leq \frac{(R_L^\wedge + M - N) + \sqrt{(R_L^\wedge - M + N)^2 + 4N\sum_{i=1}^{L-1}(R_i^\wedge - R_L^\wedge)}}{2}.$$

From the adjacency matrix of G, we have $M = 0$ and $N = 1$. Substituting the value for $N = 1$, the smallest non-diagonal number in the above equation, we get

$$\rho_1(G) \leq \frac{(R_L^\wedge - 1) + \sqrt{(R_L^\wedge + 1)^2 + 4\sum_{i=1}^{L-1}(R_i^\wedge - R_L^\wedge)}}{2}. \tag{14}$$

Here we discuss the cases corresponding to the row sums. We have three cases depending on the choice of $R_L^\wedge$, i.e. when $R_L^\wedge = R_1^\wedge$, $R_L^\wedge = R_2^\wedge$ and $R_L^\wedge = R_j^\wedge$, where $3 \leq j \leq L$.

Case (i) When $R_L^\wedge = R_1^\wedge$, where $R_1^\wedge = \sum_{i=1}^{k}\eta_i\alpha_i + (\phi-1)$ is the row sum corresponding to the vertex $v \in K_\phi$, we get

$$\rho_1(G) \leq \frac{(R_1^\wedge - 1) + \sqrt{(R_1^\wedge + 1)^2}}{2} = R_1^\wedge. \tag{15}$$

Case (ii) When $R_L^\wedge = R_2^\wedge = (\phi+\alpha_k-1)$, the row sum of a vertex belonging to the satellite S_k, we get

$$\begin{aligned}(R_1^\wedge - R_2^\wedge) = \phi\left\{\sum_{i=1}^{k}\eta_i\alpha_i + (\phi-1) - (\phi+\alpha_k-1)\right\} &= \phi\left(\sum_{i=1}^{k}\eta_i\alpha_i - \alpha_k\right)\\ &= \phi\Big((n-\phi) - \alpha_k\Big).\end{aligned}$$

Substituting the values of $R_2^\wedge$ and $(R_1^\wedge - R_2^\wedge)$ in equation (14), we get

$$\rho_1(G) \leq \frac{(\phi+\alpha_k-1) + \sqrt{(\phi+\alpha_k)^2 + 4(\phi(n-\phi)-\alpha_k)}}{2}. \tag{16}$$

Case (iii) We now discuss the case when $R_L^\wedge = R_j^\wedge$ for some j where $3 \leq j \leq L$, the row sum of a vertex belonging to the satellite $S_{k-(j-2)}$ is $(\phi+\alpha_{k-(j-2)}-1)$ where $j \neq 2$.

Consider

$$\sum_{i=1}^{j-1}(R_i^{\wedge} - R_j^{\wedge}) = \phi(R_1^{\wedge} - R_j^{\wedge}) + \eta_k\alpha_k(\alpha_k - \alpha_{k-(j-2)}) + \eta_{k-1}\alpha_{k-1}(\alpha_{k-1} - \alpha_{k-(j-2)}) + ...$$

$$+\eta_{k-(j-3)}\alpha_{k-(j-3)}(\alpha_{k-(j-3)} - \alpha_{k-(j-2)})$$

$$= \left\{\phi\left(\sum_{i=1}^{k}\eta_i\alpha_i + (\phi-1) - (\phi + \alpha_{k-(j-2)} - 1)\right) + \sum_{i=0}^{j-3}\eta_{k-i}\alpha_{k-i}(\alpha_{k-i} - \alpha_{k-(j-2)})\right\}$$

where $3 \leq j \leq L$.
Substituting the above expression in equation (14), we obtain

$$\rho_1(G) \leq \frac{(\phi + \alpha_{k-(j-2)} - 2)}{2} +$$

$$\frac{\sqrt{(\phi + \alpha_{k-(j-2)})^2 + 4\left(\phi(\sum_{i=1}^{k}\eta_i\alpha_i - \alpha_{k-(j-2)}) + \sum_{i=0}^{j-3}\eta_{k-i}\alpha_{k-i}(\alpha_{k-i} - \alpha_{k-(j-2)})\right)}}{2}. \tag{17}$$

Hence proved.

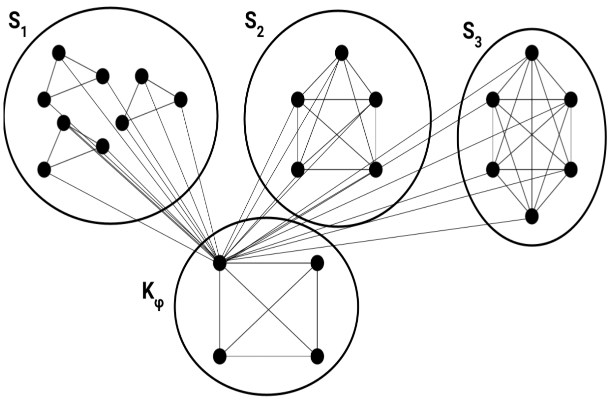

Figure 2: S_1: $3K_3$, S_2: K_5, S_3: K_6 join with one of the vertices of K_4 (core) as illustration.

Example 1. Consider the Core-satellite graph given in Figure 2. This graph has S_1, S_2 and S_3 as the satellite graphs and $K_\phi = K_4$ as the core graph. S_1 comprises of three copies of K_3, S_2 comprises of one copy of K_5, S_3 comprises of one copy K_6. As an illustration, the join of the cliques in the satellite with one vertex of the core graph is shown. In fact, the remaining vertices of the core are similarly joined with cliques of the satellites. In this example, the number of vertices is 24 and the number of edges is 120. The calculated value of the spectral radius is $\rho_1(G) = 12.2485$.

We have the following observations from the bounds obtained in Theorems 7, 8 and 9.

(i) From equation (7) of Theorem (7) we get the lower bound as 11.013 and from equation (8) of Theorem (7) we get the upper bound as 24.33.

(ii) Using equation (11) we obtain the improved upper bound as 14.043.

(iii) From equation (13) of Theorem 8, we get $8.944 < \rho_1(G) < 15.49$.

(iv) Using Theorem 9, three cases are discussed.

(a) From equation (15) of case (i) we obtain the upper bound for $\rho_1(G)$ as 23.

(b) From equation (16) of case (ii) we obtain the tight upper bounds as 13

(c) From equation (17) of the case (iii) we obtain the upper bound as 15.587.

From the above discussions, we observe that the tight lower and upper bounds are 11.013 and 13, respectively.

3.2. Bounds on Signless Laplacian Spectral Radius

In this section, we derive the upper and lower bounds for the signless Laplacian matrix $Q(G) = D(G) + A(G)$ for the graph G. Let $\mu_1(Q(G))$ be the signless Laplacian spectral radius.

Theorem 10. Let G be a Generalized core-satellite graph with $V(G)$ and $E(G)$ as the set of vertices and edges of the graph G. Then the upper and lower bounds for signless Laplacian spectral radius of $\mu_1(Q(G))$ are

$$\sqrt{2}\left[(d_{\alpha_k})^2 + \phi\left(\sum_{i=1}^{k} \eta_i\alpha_i + (\phi-1)\right) + (\alpha_k - 1)(\phi + \alpha_k - 1)\right]^{1/2} \le \mu_1(Q(G))$$

$$\le \sqrt{2}\Bigg[\sum_{i=1}^{k} \eta_i\alpha_i + (\phi-1)^2 + 2\left(\sum_{i=1}^{k}(\eta_i\alpha_i C_2 + \phi\eta_i\alpha_i) + \phi C_2\right) - \left(\sum_{i=1}^{k} \eta_i\alpha_i + (\phi-1)\right)\Bigg]^{1/2} \tag{18}$$

Proof. Since $Q(G) = D(G) + A(G)$, for vertex v its row sum is denoted as $R_v(Q(G))$ and we have $R_v(Q(G)) = 2d(v)$. Also we have $R_v(AD) = R_v(A^2) = \sum_{u\sim v} d(v)$ is true for the graph G. Then

$$R_v(Q(G))^2 = R_v(D(D+A) + AD + A^2) = d_v R_v(Q) + 2(\sum_{u\sim v} d(v))$$

$$= 2(d_v^2 + \sum_{u\sim v} d(v)) = 2(d_v^2 + (2m - d(v) - \sum_{uv\notin E(G)} d(u)) \tag{19}$$

From Lemma 1, we have

$$\sqrt{2} \min_{v\in V(G)} \sqrt{d^2(v) + \sum_{uv\in E(G)} d(u)} \le \mu_1(Q(G)) \le \sqrt{2} \max_{v\in V(G)} \sqrt{d^2(v) + \left(2m - d(v) - \sum_{uv\notin E(G)} d(u)\right)} \tag{20}$$

In this graph G, consider the vertex v taken from satellite graph S_k, since $d_{\alpha_k} = \max(d_{\alpha_i})$, for $i = 1, 2, ..., k$. The sum of the degrees of vertices adjacent to v is the sum of the degrees of the vertices of the core graph and the remaining $(\alpha_k - 1)$ vertices of the clique $\alpha_k \in S_k$

$$\sum_{u\sim v} d(u)) = \left[\phi\left(\sum_{i=1}^{k} \eta_i\alpha_i + (\phi - 1)\right) + (\alpha_k - 1)(\phi + \alpha_k - 1)\right].$$

We have

$$\mu_1(Q(G)) \ge \sqrt{2}\sqrt{\left[(d_{\alpha_k})^2 + \phi\left(\sum_{i=1}^{k} \eta_i\alpha_i + (\phi - 1)\right) + (\alpha_k - 1)(\phi + \alpha_k - 1)\right]} \tag{21}$$

Similarly consider the vertex v from core K_ϕ, the sum of the degrees of the vertices adjacent to v is the sum of the degrees of the vertices of the satellites, and the remaining $(\phi - 1)$ vertices of K_ϕ.

We have, for $v \in K_\phi$

$$d(v) = \sum_{i=1}^{k} \eta_i\alpha_i + (\phi - 1).$$

Also,

$$\sum_{u\sim v\in K_\phi} d(u) = \left(2m - d(v) - \sum_{u\sim v\in K_\phi} d(u)\right)$$

Therefore,

$$\mu_1(Q(G)) \le$$

$$\sqrt{2}\sqrt{\left(\sum_{i=1}^{k} \eta_i\alpha_i + (\phi - 1)\right)^2 + \left[2\left(\sum_{i=1}^{k}(\eta_i\alpha_i C_2 + \phi\eta_i\alpha_i) + \phi C_2\right) - \left(\sum_{i=1}^{k} \eta_i\alpha_i + (\phi - 1)\right)\right]}.$$

(22)

In the following theorem, we make use of Theorem 3 to obtain the lower and Theorem 4 to obtain the upper bounds for the Generalized core-satellite graph G. In the signless Laplacian matrix $Q(G) = A(G) + D(G)$ of the graph G, the vertices are arranged such that the top ϕ rows correspond to the vertices in the core K_ϕ, followed by the vertices of η_k copies of the cliques $K_{\alpha_k} \in S_k$. The remaining rows correspond to vertices of S_{k-1}, S_{k-2},...,S_1. Let $R_1^\wedge$, $R_2^\wedge$,...,$R_k^\wedge$ and $R_\phi^\wedge$ be the row sums corresponding to the vertices of the core K_ϕ and the satellites S_k, S_{k-1},...,S_1, i.e.

$$R^\wedge_{\phi,1} = R^\wedge_{\phi,2} = ... = R^\wedge_{\phi,\phi} = 2\sum_{i=1}^{k}\eta_i\alpha_i + (\phi - 1) = R_1^\wedge$$

$$R^\wedge_{k,1} = R^\wedge_{k,2} = ... = R^\wedge_{k,\eta_k\alpha_k} = 2(\phi + \alpha_k - 1) = R_2^\wedge$$
$$R^\wedge_{k-1,1} = R^\wedge_{k-1,2} = ... = R^\wedge_{k-1,\eta_{k-1}\alpha_{k-1}} = 2(\phi + \alpha_{k-1} - 1) = R_3^\wedge$$
$$R^\wedge_{k-2,1} = R^\wedge_{k-2,2} = ... = R^\wedge_{k-2,\eta_{k-2}\alpha_{k-2}} = 2(\phi + \alpha_{k-2} - 1) = R_4^\wedge$$

...
...
In general for the j^{th} term, we have

$$R_{k-(j-2),1} = R_{k-(j-2),2} = ... = R_{k-(j-2),\eta_{k-(j-2)}\alpha_{k-(j-2)}} = 2(\phi + \alpha_{k-(j-2)} - 1) = R_j^\wedge$$

..
..

$$R_{1,1} = R_{1,2} = ... = R_{1,\eta_k\alpha_k} = 2(\phi + \alpha_1 - 1) = R_{k+1}^\wedge$$

Theorem 11. For the Generalized Core-satellite graph G of order n and size m, let $S_1, S_2, S_3, ..., S_k$ be the k satellites connected to the core graph K_ϕ. Let $R_1^\wedge, R_2^\wedge, ..., R_{k+1}^\wedge$ be the row sums corresponding to the core K_ϕ and the satellites S_k, S_{k-1},...,S_1. The row sums are such that $R_1^\wedge \geq R_2^\wedge \geq .... \geq R_{k+1}^\wedge$, then

$$\mu_1(Q(G)) \geq \frac{(3\phi + 2\alpha_{k-(j-2)} + \alpha_1 - 3)}{2} + \frac{\sqrt{(\phi + 2\alpha_{k-(j-2)} - \alpha_1)^2}}{2} \tag{23}$$

where $1 \leq i \leq k$ and $3 \leq j \leq k+1$.

Proof. To prove this theorem, applying Theorem 3 for the signless Laplacian matrix $Q(G) = A(G) + D(G)$ of the graph G, we have

$$\mu_1(Q(G)) \geq \frac{(R_L^\wedge + S - T) + \sqrt{(R_L^\wedge - S + T)^2 + 4T\sum_{i=1}^{L-1}(R_i^\wedge - R_L^\wedge)}}{2}. \tag{24}$$

For the matrix $Q(G)$, we observe that the smallest diagonal element $S = (\phi + \alpha_1 - 1)$, and the smallest non-diagonal element $T = 0$.
Here we discuss the cases corresponding to the row sums $R_L^\wedge = R_2^\wedge$ and $R_j^\wedge$. Also $R_L^\wedge \neq R_1^\wedge$ as the lower bound is obtained by considering the row sum of any vertex $v_i \in S_i$. The row sums of the vertices belonging to the satellite S_k are greater compared to the row sums of the vertices belonging to the satellites S_{k-1}, S_{k-2},...,S_1.

Case (i) We now consider $R_L^\wedge = R_2^\wedge = 2(\phi + \alpha_k - 1)$ and by substituting the values of $R_L^\wedge$ and T in equation (24), we get

$$\mu_1(Q(G)) \geq \frac{(R_2^\wedge + S) + \sqrt{(R_2^\wedge - S)^2}}{2}$$

Now substituting for $R_2^\wedge$ and S in the above equation, we get

$$\mu_1(Q(G)) \geq \frac{(3\phi + 2\alpha_k + \alpha_1 - 3)}{2} + \frac{\sqrt{(\phi + 2\alpha_k - \alpha_1 - 1)^2}}{2}. \tag{25}$$

Case (ii) We discuss the case for $R_L^\wedge = R_j^\wedge = 2(\phi + \alpha_{k-(j-2)} - 1)$, the row sum of the vertex belonging to the satellite $S_{k-(j-2)}$, where $j \neq 2$ and $3 \leq j \leq L$.
Equation (24) reduces to

$$\mu_1(Q(G)) \geq \frac{(R_j^\wedge + S) + \sqrt{(R_j^\wedge - S)^2}}{2}.$$

We see that the smallest diagonal element of $Q(G)$ is $S = (\phi + \alpha_1 - 1)$ and the smallest non-diagonal element $T = 0$. Substituting for $R_j^\wedge$ and S in the above equation, we get

$$\mu_1(Q(G)) \geq \frac{(3\phi + 2\alpha_{k-(j-2)} + \alpha_1 - 3)}{2} + \frac{\sqrt{(\phi + 2\alpha_{k-(j-2)} - \alpha_1 - 1)^2}}{2} \tag{26}$$

Hence proved.

Theorem 12. For the Generalized core-satellite graph G of order n and size m, let $S_1, S_2, S_3, ..., S_k$ be the k satellites connected to the core graph K_ϕ. Let $R_1^\wedge, R_2^\wedge, .., R_{k+1}^\wedge$ be the row sums corresponding to the core K_ϕ and the satellites S_k, S_{k-1},...,S_1. The row sums are such that $R_1^\wedge \geq R_2^\wedge \geq ... \geq R_{k+1}^\wedge$, then

$$\mu_1(Q(G)) \leq \frac{(n + 2(\phi + \alpha_{k-(j-2)} - 2))}{2}$$

$$+ \frac{\sqrt{2(\phi + \alpha_{k-(j-2)} - n)^2 + 8\left\{\phi(\sum_{i=1}^{k} \eta_i \alpha_i - \alpha_{k-(j-2)}) + \sum_{i=0}^{j-3} \eta_{k-i}\alpha_{k-i}(\alpha_{k-i} - \alpha_{k-(j-2)})\right\}}}{2}$$

where $3 \leq j \leq k+1$.

Proof. We apply Theorem 4 to prove our result. We apply the values of $M = (\sum_{i=1}^{k} \eta_i \alpha_i + (\phi - 1))$, the largest diagonal element and $N = 1$, the largest non-diagonal element.

Consider

$$\mu_1(Q(G)) \leq \frac{(R_L^\wedge + M - N) + \sqrt{(R_L^\wedge - M + N)^2 + 4N \sum_{i=1}^{L-1} (R_i^\wedge - R_L^\wedge)}}{2}.$$

Here we apply the value of $N = 1$, the largest non-diagonal element in the above equation, we get

$$\mu_1(Q(G)) \leq \frac{(R_L^\wedge + M - 1) + \sqrt{(R_L^\wedge - M + 1)^2 + 4 \sum_{i=1}^{L-1} (R_i^\wedge - R_L^\wedge)}}{2}. \tag{27}$$

We discuss the cases corresponding to the row sums $R_L^\wedge = R_1^\wedge$, $R_2^\wedge$ and $R_j^\wedge$.

Case (i) Consider $R_L^\wedge = R_1^\wedge$, where $R_1^\wedge = 2\left(\sum_{i=1}^{k} \eta_i \alpha_i + (\phi - 1)\right)$. The vertex $v \in K_\phi$ has the maximum degree. Also, the row sum of any vertex $v_i \in S_i$ has the row sum $R_i^\wedge = 2(\phi + \alpha_i - 1)$.

$$\mu_1(Q(G)) \leq \frac{(R_1^\wedge + M - 1) + \sqrt{(R_1^\wedge - M + 1)^2 + 4 \sum_{i=1}^{L-1} (R_i^\wedge - R_1^\wedge)}}{2}. \tag{28}$$

We observe that the summation

$$\sum_{i=1}^{L-1} (R_i^\wedge - R_1^\wedge) = 0.$$

Also $(R_1^\wedge + M - 1) = (3n - 4)$ and $(R_1^\wedge - M + 1) = n$

Substituting the above expressions in equation (28), we get

$$\mu_1(Q(G)) \leq (2n - 1). \tag{29}$$

Case (ii) Consider $R_L^\wedge = R_2^\wedge = 2(\phi + \alpha_k - 1)$, the row sum of the vertex belonging to the satellite S_k, as the row sum of the vertex belonging to the satellite S_k is the maximum compared to the row sums of the vertices belonging to the satellite $S_1, S_2, ..., S_{k-1}$.

$$\mu_1(Q(G)) \leq \frac{(R_2^\wedge + M - 1) + \sqrt{(R_2^\wedge - M + 1)^2 + 4 \sum_{i=1}^{L-1} (R_i^\wedge - R_2^\wedge)}}{2}. \tag{30}$$

Consider

$$\sum_{i=1}^{L-1} (R_i^\wedge - R_2^\wedge) = \phi(R_1^\wedge - R_2^\wedge) = 2\phi\left(\sum_{i=1}^{k} \eta_i \alpha_i + (\phi - 1) - (\phi + \alpha_k - 1)\right)$$

$$= 2\phi\left((n-\phi)-\alpha_k\right).$$

Substituting the values of $R_2^{\wedge}$, M and $\sum_{i=1}^{L-1}(R_i^{\wedge}-R_2^{\wedge})$ in equation (30), we get

$$\mu_1(Q(G)) \le \frac{(n+2(\phi+\alpha_k-2))}{2} + \frac{\sqrt{2(\phi+2\alpha_k-n)^2+8\left\{\phi\left(n-\phi\right)-\alpha_k\right\}}}{2}. \tag{31}$$

Case (iii) Consider the case $R_L^{\wedge} = R_j^{\wedge}$ for some general j, where $3 \le j \le L$. Here $R_j^{\wedge} = 2(\phi+\alpha_{k-(j-2)}-1)$, the row sum of the vertex belonging to the satellite $S_{k-(j-2)}$ where $j \neq 2$.

$$\mu_1(Q(G)) \le \frac{(R_j^{\wedge}+M-1)+\sqrt{(R_j^{\wedge}-M+1)^2+4\sum_{i=1}^{L-1}(R_i^{\wedge}-R_j^{\wedge})}}{2}. \tag{32}$$

Consider

$$\sum_{i=1}^{j-1}\left(R_i^{\wedge}-R_j^{\wedge}\right) = \Bigg(2\phi\sum_{i=1}^{k}(\eta_i\alpha_i-\alpha_{k-(j-2)})+2\eta_k\alpha_k(\alpha_k-\alpha_{k-(j-2)})+2\eta_{k-1}\alpha_{k-1}(\alpha_{k-1}-\alpha_{k-(j-2)})+\dots$$

$$+2\eta_{k-(j-3)}\alpha_{k-(j-3)}(\alpha_{k-(j-3)}-\alpha_{k-(j-2)})\Bigg)$$

$$= \left\{2\phi\left(\sum_{i=1}^{k}\eta_i\alpha_i-\alpha_{k-(j-2)}\right)+2\sum_{i=0}^{j-3}\eta_{k-i}\alpha_{k-i}(\alpha_{k-i}-\alpha_{k-(j-2)})\right\}$$

where $3 \le j \le L$.

Substituting for $R_j^{\wedge}$, M and $\sum_{i=1}^{L-1}\left(R_i^{\wedge}-R_j^{\wedge}\right)$ in equation (32), we get

$$\mu_1(Q(G)) \le \frac{(n+2(\phi+2\alpha_{k-(j-2)}-2))}{2}.$$

$$+\frac{\sqrt{2(\phi+\alpha_{k-(j-2)}-n)^2+8\{\phi\left((n-\phi)-\alpha_{k-(j-2)}\right)+\sum_{i=0}^{j-3}\eta_{k-i}\alpha_{k-i}(\alpha_{k-i}-\alpha_{k-(j-2)})\}}}{2} \tag{33}$$

Hence proved.

Remark 3. For the graph G in Example 1 the calculated value of the signless Laplacian spectral radius is $\mu_1(Q(G)) = 30.2016$. We have the following observations from Theorem 10, Theorem 11, and Theorem 12.

(i) We obtain the lower and upper bounds from equation (18) of Theorem 10 as $18 \leq \mu_1(Q(G)) \leq 38.626$.

(ii) We obtain the lower bound from Theorem 11 and to determine the tight lower bound, we discuss the following cases.

(a) When $R_L^{\wedge} = R_2^{\wedge}$, we have from equation (25) the lower bound as $\mu_1(Q(G)) \geq 18$.

(b) When $R_L^{\wedge} = R_3^{\wedge}$, we have from equation (26) the lower bound as $\mu_1(Q(G)) \geq 16$, for any j, where $3 \leq j \leq L$ we obtain this value by taking $j = 3$.

(iii) We obtain the upper bound from Theorem 12 and to determine the tight upper bound, we discuss the following cases.

(a) When $R_L^{\wedge} = R_1^{\wedge}$ we get from equation (29), the upper bound as 46.

(b) When $R_L^{\wedge} = R_2^{\wedge}$ we get from equation (31), the upper bound as $\mu_1(Q(G)) \leq 30.77$.

(c) When $R_L^{\wedge} = R_3^{\wedge}$ we get from equation (33), the upper bound as $\mu_1(Q(G)) \leq 30.79$.

From the above discussions, we infer that the closest lower and upper bounds are 18 and 30.77, respectively.

4. Conclusion

Generalized core-satellite graphs are hierarchical network structures that are much more suitable to model real-world complex networks. This structural design has been applied as this has a wide range of benefits, for developing a network that is reliable, resilient, scalable, flexible, cost-effective, has better security, easier management design, enhanced performance, and improved cost-efficiency.

In this article, we have provided varied results on bounds for the spectral radius in terms of m and k (number of satellites). We have arrived at the tight bounds from among the bounds derived. We have also derived bounds on the signless Laplacian spectral radius by applying a few conditions on the general bounds derived by Duan and Zhou, using row sums. We discussed various cases and identified the optimal conditions for the bounds to be proximate to $\mu_1(Q(G))$.

References

[1] Ernesto Estrada and Michele Benzi. Core–satellite graphs: Clustering, assortativity and spectral properties. *Linear Algebra and its Applications*, 517:30–52, 2017.

[2] Erzsébet Ravasz and Albert-László Barabási. Hierarchical organization in complex networks. *Physical review E*, 67(2):026–112, 2003.

[3] Mohammadreza Mohammadrezaei, Mohammad Ebrahim Shiri, Amir Masoud Rahmani, et al. Identifying fake accounts on social networks based on graph analysis and classification algorithms. *Security and Communication Networks*, 2018, 2018.

[4] Yanqing Chen and Ligong Wang. Sharp bounds for the largest eigenvalue of the signless laplacian of a graph. *Linear algebra and its applications*, 433(5):908–913, 2010.

[5] Shuchao Li and Yi Tian. Some bounds on the largest eigenvalues of graphs. *Applied Mathematics Letters*, 25(3):326–332, 2012.

[6] Vladimir Nikiforov. More spectral bounds on the clique and independence numbers. *Journal of Combinatorial Theory, Series B*, 99(6):819–826, 2009.

[7] Kamal Lochan Patra and Binod Kumar Sahoo. Bounds for the laplacian spectral radius of graphs. *Electronic Journal of Graph Theory & Applications*, 5(2), 2017.

[8] Nair Abreu, Claudia Marcela Justel, and Lilian Markenzon. Integer laplacian eigenvalues of chordal graphs. *Linear Algebra and its Applications*, 614:68–81, 2021.

[9] Kinkar Chandra Das. Proof of a conjecture on the complete split-like graphs. *Utilitas Mathematica*, 117, 2020.

[10] Shuting Liu, Kinkar Chandra Das, and Jinlong Shu. On the eigenvalues of aα-matrix of graphs. *Discrete Mathematics*, 343(8):111917, 2020.

[11] Zhi-Wen Wang and Ji-Ming Guo. Some upper bounds on the spectral radius of a graph. *Linear Algebra and Its Applications*, 601:101–112, 2020.

[12] Modjtaba Ghorbani and Najaf Amraei. A note on eigenvalue, spectral radius and energy of extended adjacency matrix. *Discrete Applied Mathematics*, 322:102–116, 2022.

[13] Jie Xue, Ruifang Liu, Jiaxin Guo, and Jinlong Shu. The maximum spectral radius of irregular bipartite graphs. *Advances in Applied Mathematics*, 142:102433, 2023.

[14] Kinkar Ch Das and Muhuo Liu. Complete split graph determined by its (signless) laplacian spectrum. *Discrete Applied Mathematics*, 205:45–51, 2016.

[15] Ji-Ming Guo and Meng-Ni Yu. A sharp lower bound of the spectral radius with application to the energy of a graph. *Discrete Applied Mathematics*, 293:59–63, 2021.

[16] Lingsheng Shi. Bounds on the (laplacian) spectral radius of graphs. *Linear algebra and its applications*, 422(2-3):755–770, 2007.

[17] Xing Duan and Bo Zhou. Sharp bounds on the spectral radius of a nonnegative matrix. *Linear Algebra and Its Applications*, 439(10):2961–2970, 2013.