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On a Hybrid Class of p-Laplacian Initial Value Problems with Modified Mittag-Leffler Kernel

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Abstract. In this work, we establish key results on the existence theory for a category of initial value problems (IVPs) involving hybrid fractional integro-differential equations (HFIDEs) with a p-Laplacian operator, utilizing the modified Mittag-Leffler kernel. By employing Krasnoselskii and Banach fixed point theorems (FPTs), we determine the conditions required for the existence of solutions. Additionally, we examine the Hyers-Ulam (H-U) stability of the problem. Lastly, we present an example to confirm our theoretical results.

2020 Mathematics Subject Classifications: 26A33, 34A08, 34D20

Key Words and Phrases: Fractional-order, $m\mathcal{ABC}$ fractional derivative, Existence and uniqueness, Stability, FPTs

1. Introduction

This paper addresses the existence of solutions for a hybrid class of $m\mathcal{ABC}$-HFIDEs with a p-Laplacian operator given by the abstract form:

$$ {}^{C}\mathcal{D}^{\beta}\left(\Psi_p\left({}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}\left(\frac{w(\Upsilon)-h(\Upsilon,w(\Upsilon))}{Q(\Upsilon)+\frac{1}{\Gamma(\gamma)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\gamma-1}f(\sigma,w(\sigma))d\sigma}\right)\right)\right)=g(\Upsilon,w(\Upsilon)), \tag{1}$$
$$ w(0)=h(0,w(0))+Q(0)^{\mathcal{AB}}I^{\rho}_{0+}\Theta, $$

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where $\rho, \beta, \gamma \in (0,1)$ and ${}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}$ represents the $m\mathcal{ABC}$ derivative, ${}^{\mathcal{AB}}I^{\rho}_{0+}$ is the Atangana-Baleanu fractional integral, ${}^{C}\mathcal{D}^{\beta}$ signifies the Caputo fractional derivative, $\Psi_p, p > 1$ is a p-Laplacian operator, $\Theta \in \mathbb{R}$, $Q : \Omega \to \mathbb{R}$, where $\Omega = [0, Z]$, $f, g, h \in C(\Omega \times \mathbb{R}, \mathbb{R})$ with $Q(\Upsilon) + I^{\gamma}_{0+} f(\Upsilon, w(\Upsilon)) \neq 0$.

Fractional differential equations extend traditional differential equations by incorporating derivatives of non-integer orders. This extension allows for the modeling of processes that involve complex dynamics, such as systems with memory and hereditary characteristics. Unlike standard derivatives, which are local operators, fractional derivatives consider the entire history of the function, making them ideal for modeling phenomena where past states influence the present and future behavior. This non-local nature of fractional derivatives has made them increasingly popular in various scientific and engineering disciplines, where they offer a more nuanced understanding of systems exhibiting non-traditional dynamics [6, 9, 15, 22, 26].

The use of fractional differential equations has become widespread across different fields due to their ability to model processes more accurately than traditional differential equations. In physics, they are used to describe anomalous diffusion processes, where the movement of particles does not follow the standard pattern seen in classical diffusion [24]. In biology, fractional differential equations help model complex biological processes, such as the diffusion of substances across cellular membranes and the dynamics of cell potentials [19–21]. In engineering, these equations are crucial for modeling materials with viscoelastic properties, where the relationship between stress and strain is not instantaneous but depends on the material's history [27]. In finance, fractional models are employed to capture memory effects in stock prices and to model the dynamics of financial instruments over time. These applications highlight the versatility and effectiveness of fractional differential equations in providing deeper insights into various complex systems.

The realm of fractional calculus has expanded remarkably with the introduction of various fractional derivative definitions, each bringing its own advantages and specific uses. Among the pioneering contributions is the Caputo derivative, introduced by Michele Caputo [7] in 1967, which has gained widespread recognition for its practical utility. Despite its widespread adoption, the Caputo derivative's reliance on a single kernel presents certain constraints, particularly when modeling diverse phenomena. To overcome these limitations, Caputo and Fabrizio [8] proposed a new approach by introducing a non-singular derivative based on the exponential function. This innovation effectively addresses the issue of singularity, although it encounters difficulties when applied to systems that do not naturally follow exponential patterns. Seeking to further expand the modeling potential of fractional derivatives, Atangana and Baleanu [4] introduced a derivative based on the extended Mittag-Leffler function. This derivative allows for a more flexible description of non-local and non-singular kernels, thereby extending the range of phenomena that can be accurately represented. Building on these significant developments, Refai and Baleanu recently introduced the $m\mathcal{ABC}$-derivative, a novel operator that merges the strengths of both the Caputo and Atangana-Baleanu derivatives [2]. This new tool offers a robust solution for tackling complex problems that were previously challenging to address with existing methodologies, marking a substantial advancement in the field of fractional calculus.

Initial value problems (IVPs) play a pivotal role in the mathematical modeling of real-world systems, where the state of a system at a given initial time dictates its future behavior. In the context of p-Laplacian equations, IVPs involve determining the evolution of a system governed by a nonlinear differential operator. The analysis of IVPs for p-Laplacian equations is particularly challenging due to the nonlinearity of the operator, which can lead to complex dynamics, including the existence of multiple solutions, bifurcations, and sensitivity to initial conditions. Understanding these aspects is crucial for accurately predicting the behavior of the modeled systems, whether they pertain to physical processes, biological systems, or engineering applications. In recent times, a novel category known as hybrid boundary value problems has gained prominence, integrating aspects from both linear and nonlinear theories. This hybrid approach facilitates a more thorough comprehension of intricate systems, where conventional methods might be inadequate. A notable contribution to this field is the research conducted by Dhage [10, 13], which highlights the significance of hybrid differential equations (HDEs) in the analysis of dynamical systems. Dhage meticulously categorized HDEs based on different types of perturbations, emphasizing their importance in refining perturbation techniques within the expansive domain of differential and integral equations. His work underscores the potential of HDEs to provide deeper insights and more robust solutions to complex mathematical problems. Following Dhage's pioneering contributions, numerous researchers in mathematics and related disciplines have focused on exploring various hybrid differential equations (HDEs). A key discovery from this extensive research is that fractional-order hybrid differential equations (FHDEs) offer a more detailed representation of hereditary and memory effects, particularly in fields such as biology, chemistry, and physics. This enhanced capability allows FHDEs to outperform traditional integer-order HDEs, capturing the interest of many scholars and prompting deeper investigations into their properties. Building on the foundational work of Dhage et al. [11, 12], who explored the conditions for the existence and uniqueness of solutions in FHDEs, Baleanu et al. [5] integrated Caputo fractional derivatives within a hybrid framework. Their study of a thermostat model demonstrated the effectiveness of this approach in revealing complex dynamical behaviors. The exploration of FHDEs has since led to a wealth of contributions, with researchers examining various derivatives such as the Hadamard derivative [1], the Riemann derivative [32], the Hilfer derivative [30], and the $\mathcal{ABC}$ derivative [3, 18, 20, 28, 29, 31]. Additionally, innovative formulations like the $m\mathcal{ABC}$ derivative have been proposed [19, 21], further broadening the scope of fractional calculus in the context of HDEs. Despite the growing interest in fractional calculus, the application of the $m\mathcal{ABC}$ fractional derivative in FHDEs involving the p-Laplacian operator with IVPs remains largely unexplored in the current literature. This intriguing gap presents a unique opportunity for further investigation and serves as the primary motivation for this work.

This work makes the following key contributions to the field.

(i) This research represents the first known attempt to address $m\mathcal{ABC}$-HFIDEs in conjunction with the p-Laplacian operator for the system described in (1). It thoroughly examines the existence, uniqueness, and stability of the proposed system.

(ii) By utilizing the properties of the $m\mathcal{ABC}$ derivative, we have derived the solution for the system outlined in (1), as detailed in Lemma 2.

(iv) Expanding upon the seminal contributions of previous research [17, 20, 21, 31], this study offers novel insights and significantly extends the applicability of prior findings.

This paper is organized in a carefully structured way. Section 2 establishes the necessary foundation by providing the reader with essential background information. This includes the definition of the $m\mathcal{ABC}$ fractional derivative, relevant results, and the fundamental concept of fixed-point theory that underpins our analysis. Section 3 addresses the core issues of existence and uniqueness of solutions for system (1). We effectively utilize both the Banach and Krasnoselskii FPTs to accomplish these objectives. To deepen our understanding, Section 4 thoroughly examines the stability properties of the system, highlighting its behavior in response to perturbations. Finally, Section 5 offers a numerical example to demonstrate the application and importance of our main findings.

2. Preliminaries and Hypotheses

In this section, we provide a comprehensive overview of the Caputo-type Mittag-Leffler fractional derivative (CMLFD) and integral operator and several fundamental properties.

Definition 1. *[4] Let $0 < \rho < 1$ and $y \in H^1(0, Z)$, where $Z > 0$, the CMLFD of order ρ of y is defined as*

$$\left({}^{\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}y\right)(\Upsilon) = \frac{\mathscr{B}(\rho)}{1-\rho}\int_0^{\Upsilon} E_{\rho}\left(-\mu_{\rho}(\Upsilon-\sigma)^{\rho}\right) y'(\sigma)d\sigma, \quad 0 < \Upsilon < Z \tag{2}$$

where $\mu_{\rho} = \dfrac{\rho}{1-\rho}$, $\mathscr{B}(\rho) = 1-\rho+\dfrac{\rho}{\Gamma(\rho)}$ is a normalization function satisfying $\mathscr{B}(0) = \mathscr{B}(1) = 1$. Let E_{ρ} denote the ML function, defined by

$$E_{\rho}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k+1)}, \quad \rho > 0, \quad z \in \mathbb{C}$$
$$H^1(0, Z) = \left\{v \in L^2(0, Z) \mid v' \in L^2(0, Z)\right\}.$$

Additionally, L^m denotes the space of functions for which the m-th power of their absolute value is Lebesgue integrable.

Definition 2. *[4] Let $0 < \rho < 1$ and $y \in L^1(0, Z)$, where $Z > 0$, the fractional integral associated with the above CMLFD of order ρ of y is described as:*

$$\left({}^{\mathcal{AB}}I^{\rho}_{0+}y\right)(\Upsilon) = \frac{1-\rho}{\mathscr{B}(\rho)}y(\Upsilon) + \frac{\rho}{\mathscr{B}(\rho)}\left({}^{\mathrm{RL}}I^{\rho}_{0+}y\right)(\Upsilon), \quad 0 < \Upsilon < Z \tag{3}$$

where ${}^{\mathrm{RL}}I^{\rho}_{0+}$ denotes the Riemann-Liouville fractional integral operator, given by:

$${}^{\mathrm{RL}}I^{\rho}_{0+}y(\Upsilon) = \frac{1}{\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\rho-1}y(\sigma)d\sigma, \quad \Upsilon > 0. \tag{4}$$

Remark 1. *In* (2), *the kernel* $E_\rho\left(-\mu_\rho(\curlyvee-\sigma)^\rho\right)$ *is non-singular. Since*

$$E_\rho\left(-\mu_\rho(\curlyvee-\sigma)^\rho\right)=E_\rho\left(-\frac{\rho}{1-\rho}(\curlyvee-\sigma)^\rho\right).$$

As $\curlyvee$ *approaches* σ, *the term* $(\curlyvee-\sigma)^\rho$ *approaches* 0. *Thus, we have*

$$E_\rho\left(-\mu_\rho(\curlyvee-\sigma)^\rho\right)=E_\rho(0)=1.$$

From this, we observe that $\left({}^{\mathcal{ABC}}\mathcal{D}^\rho_{0+}y\right)(0)=0$, *if* $y\in H^1(0,Z)$.

The aforementioned operators serve as fundamental tools in establishing various theoretical underpinnings. The efficacy of these operators in practical applications is significantly influenced by the underlying function spaces, as elucidated by Al-Refai et al. in [2]. For instance, we fix $y(\curlyvee)\in H^1(0,Z)$ then the system $\left({}^{\mathcal{ABC}}\mathcal{D}^\rho_{0+}y\right)(\curlyvee)-\omega y(\curlyvee)=0,\ \omega\in\mathbb{R}$, has only the trivial solution $y(\curlyvee)=0$. However, in this context, the space is restrictive for the Caputo derivative. If we fix the space $\chi(y)=\{y: y'\in L^1[0,1]\}$, then the following system

$$\left({}^{\mathcal{ABC}}\mathcal{D}^\rho_{0+}y\right)(\curlyvee)=\begin{cases}\omega y(\curlyvee),\ \curlyvee\in(0,Z);\\ y_0,\ \curlyvee=0\end{cases}$$

with $0<\rho<1$, we have the solution

$$y(\curlyvee)=y_0E_{\rho,1}(-\omega\curlyvee^\rho),$$

where

$$E_{\rho,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k+1)}.$$

This demonstrates the significant impact of space. To address this challenge, Al-Refai et al. [2] published a research work in which a larger space is selected to eliminate the need for additional conditions. Following [2], we presents a $m\mathcal{ABC}$ fractional derivative operator, which is applicable in a broader functional space to address the initialization problem effectively.

Definition 3. *[2, 25] Let* $y\in L^1(0,Z), Z>0$ *and* $\rho\in(0,1)$, *the* $m\mathcal{ABC}$ *derivative is described by*

$$\begin{aligned}\left({}^{m\mathcal{ABC}}\mathcal{D}^\rho_{0+}y\right)(\curlyvee)=&\frac{\mathscr{B}(\rho)}{1-\rho}\Big[y(\curlyvee)-E_\rho\left(-\mu_\rho\curlyvee^\rho\right)y(0)\\ &-\mu_\rho\int_0^{\curlyvee}(\curlyvee-\sigma)^{\rho-1}E_{\rho,\rho}\left(-\mu_\rho(\curlyvee-\sigma)^\rho\right)y(\sigma)d\sigma\Big],\quad 0<\curlyvee<Z\end{aligned}\tag{5}$$

where $\mu_\rho=\dfrac{\rho}{1-\rho}$, *the normalized function* $\mathscr{B}(\rho)$ *satisfies the property* $\mathscr{B}(0)=\mathscr{B}(1)=1$ *and* $E_{\rho,\rho}$ *is the two-parameters M-L function.*

We see that Definitions 1 and 3 are the same in the space $H^1(0,Z) \subseteq L^1(0,Z)$. However, if $y \in L^1(0,Z)$, it is not guaranteed that $\left({}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}y\right)(0) = 0$. To support this result, we have the following example:

Example 1. *Consider*

$$u(\Upsilon) = \begin{cases} \Upsilon^{-\frac{3}{4}}, & \Upsilon \neq 0 \\ B, & \Upsilon = 0 \end{cases}$$

where $B \in \mathbb{R}$ *and* $u \in L^1(0,Z)$*. For* $\mathscr{B}(\rho) = 1 - \rho + \dfrac{\rho}{\Gamma(\rho)}$ *and* $\mu_\rho = \dfrac{\rho}{1-\rho}$*, the modified Atangana-Baleanu derivative in Caputo sense is:*

$$\left({}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}u\right)(\Upsilon) = \frac{\mathscr{B}(\rho)}{1-\rho}\Bigg[u(\Upsilon) - E_\rho\left(-\mu_\rho \Upsilon^\rho\right)u(0) - \mu_\rho \int_0^\Upsilon (\Upsilon-\sigma)^{\rho-1} E_{\rho,\rho}\left(-\mu_\rho(\Upsilon-\sigma)^\rho\right)u(\sigma)\,d\sigma\Bigg].$$

If $\rho = \dfrac{3}{4}$*, then above expression becomes*

$$\left({}^{m\mathcal{ABC}}\mathcal{D}^{\frac{3}{4}}_{0+}u\right)(\Upsilon) = 3.25\Bigg[u(\Upsilon) - E_{\frac{3}{4}}\left(-3\Upsilon^{\frac{3}{4}}\right)u(0) - 3\int_0^\Upsilon (\Upsilon-\sigma)^{-\frac{1}{4}} E_{\frac{3}{4},\frac{3}{4}}\left(-3(\Upsilon-\sigma)^{\frac{3}{4}}\right)u(\sigma)\,d\sigma\Bigg]. \tag{6}$$

For $\Upsilon \neq 0$*:*

$$u(\Upsilon) = \Upsilon^{-\frac{3}{4}}.$$

For $\Upsilon = 0$*:*

$$u(0) = B.$$

Since

$$\int_0^\Upsilon (\Upsilon-\sigma)^{-\frac{1}{4}} E_{\frac{3}{4},\frac{3}{4}}\left(-3(\Upsilon-\sigma)^{\frac{3}{4}}\right)\sigma^{-\frac{3}{4}}d\sigma = \Gamma\left(\frac{1}{4}\right)E_{\frac{3}{4}}\left(-3\Upsilon^{\frac{3}{4}}\right)$$

using the fact that

$$\int_0^\Upsilon (\Upsilon-\sigma)^{\rho-1} E_{\rho,\rho}\left(-\mu_\rho(\Upsilon-\sigma)^\rho\right)\sigma^{-\rho}d\sigma = \Gamma(1-\rho)E_\rho\left(-\mu_\rho \Upsilon^\rho\right).$$

Thus (6) *becomes*

$$\left({}^{m\mathcal{ABC}}\mathcal{D}^{\frac{3}{4}}_{0+}u\right)(\Upsilon) = 3.25\left[\Upsilon^{-\frac{3}{4}} - E_{\frac{3}{4}}\left(-3\Upsilon^{\frac{3}{4}}\right)B - 3\Gamma\left(\frac{1}{4}\right)E_{\frac{3}{4}}\left(-3\Upsilon^{\frac{3}{4}}\right)\right].$$

Consequently

$${}^{m\mathcal{ABC}}\mathcal{D}^{\frac{3}{4}}_{0+}u(0) = -9.75\Gamma\left(\frac{1}{4}\right) \neq 0.$$

Remark 2. *In the preceding illustration, it is observed that the fractional derivative* ${}^{m\mathcal{ABC}}\mathcal{D}_{0+}^{\frac{3}{4}}u(0) \neq 0$ *at the point* $\Upsilon = 0$.

Finally, we recall the well-known related to Caputo fractional derivative and the properties of the p-Laplacian operator.

Definition 4. *[6, 26] For any* $\rho > 0$ *and* $u \in C(0,T) \cap L(0,Z)$*, we have*

$$I_{0+}^{\rho}\mathcal{D}_{0+}^{\rho}u(\Upsilon) = u(\Upsilon) + c_0 + c_1\Upsilon + \ldots + c_{n-1}\Upsilon^{n-1},$$

fore some $c_i \in \mathbb{R}, i = 1, 2, \ldots, n-1$. *where* $n = [\rho] + 1$. *In particular, when* $\rho \in (0,1)$, $I_{0+}^{\rho}\mathcal{D}_{0+}^{\rho}u(\Upsilon) = u(\Upsilon) + c_0$.

Definition 5. *[23] The p-Laplacian operator is given by*

$$\Psi_p(\bar{u}) = |\bar{u}|^{p-2}\bar{u} = \bar{u}^{p-1}, \ \bar{u} \geq 0, \quad p > 1, \tag{7}$$

at which $\Psi_p^{-1} = \Psi_q$ *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 1. *[23] Assume that* $\Psi_p(\bar{u}), p \geq 2$, *be p-Laplacian operator and* $|\bar{u}|, |\bar{v}| \leq \mathcal{M}$, *then*

$$|\Psi_p(\bar{u}) - \Psi_p(\bar{v})| \leq (p-1)\mathcal{M}^{p-2}|\bar{u} - \bar{v}|. \tag{8}$$

Definition 6. *A function* $w \in AC(\Omega, \mathbb{R})$ *is called a solution of the system* (1) *if function* $g \in L^1(\Omega, \mathbb{R})$, *and* w *fulfills* (1).

Lemma 2. *Let* $0 < \rho < 1$ *and* $g \in L^1(0, Z)$. *Then, the solution* $w \in AC(\Omega, \mathbb{R})$ *of the system* (1) *iff it is a solution to the subsequent integral equation:*

$$\begin{aligned} w(\Upsilon) = &\left(Q(\Upsilon) + \frac{1}{\Gamma(\gamma)}\int_0^{\Upsilon}(\Upsilon - \sigma)^{\gamma-1}f(\sigma, w(\sigma))d\sigma\right) \\ &(\times)\left[{}^{\mathcal{AB}}I_{0_+}^{\rho}(\Theta) + \frac{1-\rho}{\mathscr{B}(\rho)}\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon - \sigma)^{\beta-1}g(\sigma, w(\sigma))d\sigma\right)\right. \\ &\left.+ \frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon - \sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma - \tau)^{\beta-1}g(\tau, w(\tau))d\tau\right)\right)d\sigma\right] \\ &+ h(\Upsilon, w(\Upsilon)). \end{aligned} \tag{9}$$

Proof. Since

$${}^{C}\mathcal{D}^{\beta}\left(\Psi_p\left({}^{m\mathcal{ABC}}\mathcal{D}_{0+}^{\rho}\left(\frac{w(\Upsilon) - h(\Upsilon, w(\Upsilon))}{Q(\Upsilon) + I_{0+}^{\gamma}f(\Upsilon, w(\Upsilon))}\right)\right)\right) = g(\Upsilon, w(\Upsilon)).$$

Taking I_{0+}^{β} on both sides of the above equation, we have

$$\Psi_p\left({}^{m\mathcal{ABC}}\mathcal{D}_{0+}^{\rho}\left(\frac{w(\Upsilon) - h(\Upsilon, w(\Upsilon))}{Q(\Upsilon) + I_{0+}^{\gamma}f(\Upsilon, w(\Upsilon))}\right)\right) = I_{0_+}^{\beta}g(\Upsilon, w(\Upsilon)) + c_0,$$

for some $c_0 \in \mathbb{R}$. At $\Upsilon = 0$, we have

$$\Psi_p\left({}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}\left(\frac{w(0)-h(0,w(0))}{Q(0)}\right)\right) = c_0.$$

That is

$$\Psi_p\left({}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}\left(\frac{w(\Upsilon)-h(\Upsilon,w(\Upsilon))}{Q(\Upsilon)+I^{\gamma}_{0+}f(\Upsilon,w(\Upsilon))}\right)\right) - \Psi_p\left({}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}\left(\frac{w(0)-h(0,w(0))}{Q(0)}\right)\right)$$
$$= I^{\beta}_{0_+}g(\Upsilon,w(\Upsilon)).$$

By the properties of p-Laplacian operator, we have

$${}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}\left(\frac{w(\Upsilon)-h(\Upsilon,w(\Upsilon))}{Q(\Upsilon)+I^{\gamma}_{0+}f(\Upsilon,w(\Upsilon))}\right) - {}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}\left(\frac{w(0)-h(0,w(0))}{Q(0)}\right)$$
$$= \Psi_q\left(I^{\beta}_{0_+}g(\Upsilon,w(\Upsilon))\right).$$

Taking ${}^{\mathcal{AB}}I^{\rho}_{0+}$ on both sides of the above equation, we have

$$\frac{w(\Upsilon)-h(\Upsilon,w(\Upsilon))}{Q(\Upsilon)+I^{\gamma}_{0+}f(\Upsilon,w(\Upsilon))} - \frac{w(0)-h(0,w(0))}{Q(0)} = {}^{\mathcal{AB}}I^{\rho}_{0+}\left(\Psi_q\left(I^{\beta}_{0_+}g(\Upsilon,w(\Upsilon))\right)\right)$$

by using the fact that $\left({}^{\mathcal{AB}}I^{\rho}_{0+}\,{}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}w\right)(\Upsilon) = w(\Upsilon) - w(0)$.

Thus

$$w(\Upsilon) = \left(Q(\Upsilon)+I^{\gamma}_{0+}f(\Upsilon,w(\Upsilon))\right)\left({}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta) + {}^{\mathcal{AB}}I^{\rho}_{0+}\left(\Psi_q\left(I^{\beta}_{0_+}g(\Upsilon,w(\Upsilon))\right)\right)\right) + h(\Upsilon,w(\Upsilon))$$

or equivalently

$$\begin{aligned} w(\Upsilon) = &\left(Q(\Upsilon) + \frac{1}{\Gamma(\gamma)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\gamma-1}f(\sigma,w(\sigma))d\sigma\right) \\ &(\times)\Bigg[{}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta) + \frac{1-\rho}{\mathscr{B}(\rho)}\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}g(\sigma,w(\sigma))d\sigma\right) \\ &+ \frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}g(\tau,w(\tau))d\tau\right)\right)d\sigma\Bigg] \\ &+ h(\Upsilon,w(\Upsilon)). \end{aligned} \tag{10}$$

Conversely, we have

$$\frac{w(\Upsilon)-h(\Upsilon,w(\Upsilon))}{Q(\Upsilon)+I^{\gamma}_{0+}f(\Upsilon,w(\Upsilon))} = {}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta) + {}^{\mathcal{AB}}I^{\rho}_{0_+}\left(\Psi_q\left(I^{\beta}_{0_+}(g(\Upsilon,w(\Upsilon)))\right)\right).$$

Taking ${}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}$ derivative on both sides of the above equation, we have

$${}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}\left(\frac{w(\Upsilon)-h(\Upsilon,w(\Upsilon))}{Q(\Upsilon)+I^{\gamma}_{0+}f(\Upsilon,w(\Upsilon))}\right)$$

$$= {}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}\left({}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta) + {}^{\mathcal{AB}}I^{\rho}_{0_+}\left(\Psi_q\left(I^{\beta}_{0_+}g(\Upsilon, w(\Upsilon))\right)\right)\right)$$
$$= \Theta + \left(\Psi_q\left(I^{\beta}_{0_+}g(\Upsilon, w(\Upsilon))\right)\right)$$

by utilizing the fact that $\left({}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}\left({}^{\mathcal{AB}}I^{\rho}_{0+}w\right)\right)(\Upsilon) = w(\Upsilon)$. Taking p-Laplacian operator on both sides, we have

$$\Psi_p\left({}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}\left(\frac{w(\Upsilon) - h(\Upsilon, w(\Upsilon))}{Q(\Upsilon) + I^{\gamma}_{0+}f(\Upsilon, w(\Upsilon))}\right)\right) = \Psi_p(\Theta) + I^{\beta}_{0_+}g(\Upsilon, w(\Upsilon)).$$

Taking ${}^{C}\mathcal{D}^{\beta}$ on both sides, we get

$${}^{C}\mathcal{D}^{\beta}\left(\Psi_p\left({}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}\left(\frac{w(\Upsilon) - h(\Upsilon, w(\Upsilon))}{Q(\Upsilon) + I^{\gamma}_{0+}f(\Upsilon, w(\Upsilon))}\right)\right)\right) = g(\Upsilon, w(\Upsilon))$$

by utilizing the fact that ${}^{C}\mathcal{D}^{\beta}I^{\beta}_{0_+}w(\Upsilon) = w(\Upsilon)$ and further $w(0) = h(0, w(0)) + Q(0){}^{\mathcal{AB}}I^{\rho}_{0+}\Theta$.

Describe the operator $\Phi : AC(\Omega, \mathbb{R}) \to AC(\Omega, \mathbb{R})$ by

$$\begin{aligned}
&(\Phi w)(\Upsilon)\\
&= \left(Q(\Upsilon) + \frac{1}{\Gamma(\gamma)}\int_0^{\Upsilon}(\Upsilon - \sigma)^{\gamma-1}f(\sigma, w(\sigma))d\sigma\right)\\
&\quad(\times)\Bigg[{}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta) + \frac{1-\rho}{\mathscr{B}(\rho)}\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon - \sigma)^{\beta-1}g(\sigma, w(\sigma))d\sigma\right)\\
&\quad+ \frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon - \sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma - \tau)^{\beta-1}g(\tau, w(\tau))d\tau\right)\right)d\sigma\Bigg]\\
&\quad+ h(\Upsilon, w(\Upsilon)).
\end{aligned} \tag{11}$$

According to equation (11), each fixed point of the operator Φ is associated with the desired solution of the system (1).

Note 2.1. *For our convenience, we split the operator* (11) *as:*

$$(\Phi w)(\Upsilon) = (Aw)(\Upsilon) + (Bw)(\Upsilon),\ \Upsilon \in \Omega,$$

where

$$\begin{aligned}
&(Aw)(\Upsilon)\\
&= \left(Q(\Upsilon) + \frac{1}{\Gamma(\gamma)}\int_0^{\Upsilon}(\Upsilon - \sigma)^{\gamma-1}f(\sigma, w(\sigma))d\sigma\right)\\
&\quad(\times)\Bigg[{}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta) + \frac{1-\rho}{\mathscr{B}(\rho)}\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon - \sigma)^{\beta-1}g(\sigma, w(\sigma))d\sigma\right)\\
&\quad+ \frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon - \sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma - \tau)^{\beta-1}g(\tau, w(\tau))d\tau\right)\right)d\sigma\Bigg],
\end{aligned} \tag{12}$$

and

$$(Bw)(\Upsilon) = h(\Upsilon, w(\Upsilon)),\ \Upsilon \in \Omega. \tag{13}$$

Now

$$\begin{aligned}
&|(Aw)(\Upsilon) - (A\overline{w})(\Upsilon)| \\
&= \Bigg| \left(Q(\Upsilon) + \frac{1}{\Gamma(\gamma)} \int_0^{\Upsilon} (\Upsilon - \sigma)^{\gamma-1} f(\sigma, w(\sigma)) d\sigma \right) \\
&\quad (\times) \Bigg[{}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta) + \frac{1-\rho}{\mathscr{B}(\rho)} \Psi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\Upsilon} (\Upsilon - \sigma)^{\beta-1} g(\sigma, w(\sigma)) d\sigma \right) \\
&\quad + \frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)} \int_0^{\Upsilon} (\Upsilon - \sigma)^{\rho-1} \left(\Psi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\sigma} (\sigma - \tau)^{\beta-1} g(\tau, w(\tau)) d\tau \right) \right) d\sigma \Bigg] \\
&- \left(Q(\Upsilon) + \frac{1}{\Gamma(\gamma)} \int_0^{\Upsilon} (\Upsilon - \sigma)^{\gamma-1} f(\sigma, \overline{w}(\sigma)) d\sigma \right) \\
&\quad (\times) \Bigg[{}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta) + \frac{1-\rho}{\mathscr{B}(\rho)} \Psi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\Upsilon} (\Upsilon - \sigma)^{\beta-1} g(\sigma, \overline{w}(\sigma)) d\sigma \right) \\
&\quad + \frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)} \int_0^{\Upsilon} (\Upsilon - \sigma)^{\rho-1} \left(\Psi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\sigma} (\sigma - \tau)^{\beta-1} g(\tau, \overline{w}(\tau)) d\tau \right) \right) d\sigma \Bigg] \Bigg|
\end{aligned}$$

We denote

$$\begin{aligned}
A_w &= \left(Q(\Upsilon) + \frac{1}{\Gamma(\gamma)} \int_0^{\Upsilon} (\Upsilon - \sigma)^{\gamma-1} f(\sigma, w(\sigma)) d\sigma \right) \\
B_w &= {}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta) + \frac{1-\rho}{\mathscr{B}(\rho)} \Psi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\Upsilon} (\Upsilon - \sigma)^{\beta-1} g(\sigma, w(\sigma)) d\sigma \right) \\
&\quad + \frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)} \int_0^{\Upsilon} (\Upsilon - \sigma)^{\rho-1} \left(\Psi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\sigma} (\sigma - \tau)^{\beta-1} g(\tau, w(\tau)) d\tau \right) \right) d\sigma; \\
A_{\overline{w}} &= \left(Q(\Upsilon) + \frac{1}{\Gamma(\gamma)} \int_0^{\Upsilon} (\Upsilon - \sigma)^{\gamma-1} f(\sigma, \overline{w}(\sigma)) d\sigma \right) \\
B_{\overline{w}} &= {}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta) + \frac{1-\rho}{\mathscr{B}(\rho)} \Psi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\Upsilon} (\Upsilon - \sigma)^{\beta-1} g(\sigma, \overline{w}(\sigma)) d\sigma \right) \\
&\quad + \frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)} \int_0^{\Upsilon} (\Upsilon - \sigma)^{\rho-1} \left(\Psi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\sigma} (\sigma - \tau)^{\beta-1} g(\tau, \overline{w}(\tau)) d\tau \right) \right) d\sigma.
\end{aligned}$$

The right side of the above expression can be written as $|A_w B_w - A_{\overline{w}} B_{\overline{w}}|$. *Using the property of absolute values for products, we can express this as:*

$$\begin{aligned}
|A_w B_w - A_{\overline{w}} B_{\overline{w}}| &= |A_w B_w - A_w B_{\overline{w}} + A_w B_{\overline{w}} - A_{\overline{w}} B_{\overline{w}}| \\
&\leq |A_w (B_w - B_{\overline{w}}) + B_{\overline{w}} (A_w - A_{\overline{w}})|
\end{aligned}$$

$$\leq |A_w||B_w - B_{\overline{w}}| + |B_{\overline{w}}||A_w - A_{\overline{w}}|. \tag{14}$$

Now

$$\begin{aligned}
|B_w - B_{\overline{w}}| \leq & \Bigg| {}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta) + \frac{1-\rho}{\mathscr{B}(\rho)}\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}g(\sigma,w(\sigma))d\sigma\right) \\
& + \frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}g(\tau,w(\tau))d\tau\right)\right)d\sigma \\
& - {}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta) - \frac{1-\rho}{\mathscr{B}(\rho)}\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}g(\sigma,\overline{w}(\sigma))d\sigma\right) \\
& - \frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}g(\tau,\overline{w}(\tau))d\tau\right)\right)d\sigma\Bigg| \\
\leq & \frac{1-\rho}{\mathscr{B}(\rho)}\Bigg\{\Bigg|\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}g(\sigma,w(\sigma))d\sigma\right) \\
& - \Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}g(\sigma,\overline{w}(\sigma))d\sigma\right)\Bigg|\Bigg\} \\
& + \frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\rho-1}\Bigg|\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}g(\tau,w(\tau))d\tau\right) \\
& - \Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}g(\tau,w(\tau))d\tau\right)\Bigg|d\sigma,
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
|A_w - A_{\overline{w}}| \leq & \Bigg|\left(Q(\Upsilon) + \frac{1}{\Gamma(\gamma)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\gamma-1}f(\sigma,w(\sigma))d\sigma\right) \\
& - \left(Q(\Upsilon) + \frac{1}{\Gamma(\gamma)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\gamma-1}f(\sigma,\overline{w}(\sigma))d\sigma\right)\Bigg| \\
\leq & \left|\frac{1}{\Gamma(\gamma)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\gamma-1}[f(\sigma,w(\sigma)) - f(s,\overline{w}(s))]d\sigma\right|.
\end{aligned} \tag{16}$$

We will now outline the following assumptions:

(A1) For positive constants $L_f, L_g, L_h > 0$, it holds that for any elements $w, w_1, \overline{w}, \overline{w}_1 \in AC(\Omega)$

$$|f(\Upsilon,w(\Upsilon)) - f(\Upsilon,\overline{w}(\Upsilon))| \leq L_f|w(\Upsilon) - \overline{w}(\Upsilon)|,$$
$$|g(\Upsilon,w(\Upsilon)) - g(\Upsilon,\overline{w}(\Upsilon))| \leq L_g|w(\Upsilon) - \overline{w}(\Upsilon)|,$$

and

$$|h(\Upsilon,w(\Upsilon)) - h(\Upsilon,\overline{w}(\Upsilon))| \leq L_h|w(\Upsilon) - \overline{w}(\Upsilon)|.$$

(A2) There exist functions $F, G \in L^1(\Omega, \mathbb{R}_+)$ such that

$$|f(\Upsilon, w(\Upsilon))| \leq F(\Upsilon) \quad \text{and} \quad |g(\Upsilon, w(\Upsilon))| \leq G(\Upsilon),\ \Upsilon \in \Omega.$$

(A3) For any constant $L_Q > 0$, it follows that

$$|Q(\Upsilon_2) - Q(\Upsilon_1)| \leq L_Q|\Upsilon_2 - \Upsilon_1|,\ \Upsilon_1, \Upsilon_2 \in \Omega.$$

3. Existence Results

This section initiates a thorough analysis aimed at proving the existence of solutions for the system (1). To accomplish this, we effectively utilize two fundamental methodologies: the Banach contraction principle and Krasnoselskii FPTs [14, 16].

Theorem 1. *Given the assumptions* $(A1) - (A3)$, *the system* (1) *possesses a unique solution when*

$$\Delta = \Bigg[\left(L_Q Z + |Q(0)| + \frac{Z^\gamma \|F\|_{L^1}}{\Gamma(\gamma+1)} \right) \left(\frac{(q-1)\mathcal{M}^{q-2} Z^\beta}{\mathscr{B}(\rho)\Gamma(\beta+1)} \left\{ 1 - \rho + \frac{Z^\rho}{\Gamma(\rho)} \right\} L_g \right) + \left({}^{\mathcal{AB}}I^\rho_{0_+}|\Theta| + \frac{1}{\mathscr{B}(\rho)} \left(\frac{Z^\beta \|G\|_{L^1}}{\Gamma(\beta+1)} \right)^{q-1} \left\{ 1 - \rho + \frac{Z^\rho}{\Gamma(\rho)} \right\} \right) \left(\frac{L_f Z^\gamma}{\Gamma(\gamma+1)} \right) + L_h \Bigg] < 1. \tag{17}$$

Proof. Let $w, \overline{w} \in AC(\Omega)$. Then from Note 2.1, we have

$$\begin{aligned} \|\Phi w - \Phi \overline{w}\| &= \max_{\Upsilon \in \Omega} |((A+B)w)(\Upsilon) - ((A+B)\overline{w})(\Upsilon)| \\ &\leq \max_{\Upsilon \in \Omega} |(Aw)(\Upsilon) - (A\overline{w})(\Upsilon)| + \max_{\Upsilon \in \Omega} |(Bw)(\Upsilon) - (B\overline{w})(\Upsilon)|. \end{aligned} \tag{18}$$

From (14)-(16), we have

$$\|Aw - A\overline{w}\| \leq \max_{\Upsilon \in \Omega} \{ |A_w||B_w - B_{\overline{w}}| + |B_{\overline{w}}||A_w - A_{\overline{w}}| \}. \tag{19}$$

We can now evaluate the expression mentioned earlier in the following manner:

$$\begin{aligned} \max_{\Upsilon \in \Omega} |A_w| &= \max_{\Upsilon \in \Omega} \left| Q(\Upsilon) + \frac{1}{\Gamma(\gamma)} \int_0^\Upsilon (\Upsilon - \sigma)^{\gamma-1} f(\sigma, w(\sigma)) d\sigma \right| \\ &\leq \max_{\Upsilon \in \Omega} |Q(\Upsilon) - Q(0)| + |Q(0)| + \frac{1}{\Gamma(\gamma)} \int_0^\Upsilon (\Upsilon - \sigma)^{\gamma-1} \max_{\Upsilon \in \Omega} |f(\sigma, w(\sigma))| d\sigma \\ &\leq L_Q Z + |Q(0)| + \frac{1}{\Gamma(\gamma)} \int_0^\Upsilon (\Upsilon - \sigma)^{\gamma-1} \max_{\Upsilon \in \Omega} F(\sigma) d\sigma \\ &\leq L_Q Z + |Q(0)| + \frac{Z^\gamma \|F\|_{L^1}}{\Gamma(\gamma+1)}; \end{aligned}$$

$$\max_{\Upsilon\in\Omega}|B_w - B_{\overline{w}}| \leq \frac{(q-1)\mathcal{M}^{q-2}Z^{\beta}}{\mathscr{B}(\rho)\Gamma(\beta+1)}\left\{1-\rho+\frac{Z^{\rho}}{\Gamma(\rho)}\right\}L_g\|w-\overline{w}\|;$$

since

$$\begin{aligned}
&\frac{1-\rho}{\mathscr{B}(\rho)}\Bigg\{\max_{\Upsilon\in\Omega}\Bigg|\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}g(\sigma,w(\sigma))d\sigma\right)\\
&\quad-\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}g(\sigma,\overline{w}(\sigma))d\sigma\right)\Bigg|\Bigg\}\\
&\leq \frac{1-\rho}{\mathscr{B}(\rho)}(q-1)\mathcal{M}^{q-2}\Bigg|\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}g(\sigma,w(\sigma))d\sigma\\
&\quad-\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}g(\sigma,\overline{w}(\sigma))d\sigma\Bigg|\\
&\leq \frac{1-\rho}{\mathscr{B}(\rho)}(q-1)\mathcal{M}^{q-2}\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}|g(\sigma,w(\sigma))-g(\sigma,\overline{w}(\sigma))|d\sigma\\
&\leq \frac{1-\rho}{\mathscr{B}(\rho)}\frac{(q-1)\mathcal{M}^{q-2}Z^{\beta}}{\Gamma(\beta+1)}L_g\|w-\overline{w}\|,
\end{aligned}$$

and

$$\begin{aligned}
&\max_{\Upsilon\in\Omega}\Bigg|\frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}g(\tau,w(\tau))d\tau\right)\right)d\sigma\\
&-\frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}g(\tau,\overline{w}(\tau))d\tau\right)\right)d\sigma\Bigg|\\
&\leq \frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\rho-1}\max_{\Upsilon\in\Omega}\Bigg|\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}g(\tau,w(\tau))d\tau\right)\\
&\quad-\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}g(\tau,\overline{w}(\tau))d\tau\right)\Bigg|d\sigma\\
&\leq \frac{(q-1)\mathcal{M}^{q-2}Z^{\beta+\rho}}{\Gamma(\beta+1)\mathscr{B}(\rho)\Gamma(\rho)}L_g\|w-\overline{w}\|.
\end{aligned}$$

$$\begin{aligned}
\max_{\Upsilon\in\Omega}|B_{\overline{w}}| = \max_{\Upsilon\in\Omega}\Bigg|&{}^{\mathcal{AB}}I_{0_+}^{\rho}(\Theta)+\frac{1-\rho}{\mathscr{B}(\rho)}\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}g(\sigma,w(\sigma))d\sigma\right)\\
&+\frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}g(\tau,w(\tau))d\tau\right)\right)d\sigma\Bigg|.
\end{aligned}$$

Since from (7), we have

$$\left|\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}g(\sigma,w(\sigma))d\sigma\right)\right|$$

$$
\begin{aligned}
&\le \Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}G(s)d\sigma\right)\\
&\le \Psi_q\left(\frac{Z^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right) \quad \Big| \because \Psi_q(\bar{u})=\bar{u}^{q-1}\\
&= \left(\frac{Z^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right)^{q-1}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}g(\tau,w(\tau))d\tau\right)\right)d\sigma\right|\\
&\le \left(\frac{Z^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right)^{q-1}\frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\rho-1}d\sigma\\
&\le \left(\frac{Z^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right)^{q-1}\cdot\frac{Z^{\rho}}{\mathscr{B}(\rho)\Gamma(\rho)}.
\end{aligned}
$$

Thus, we have

$$
\max_{\Upsilon\in\Omega}|B_{\overline{w}}| \le {}^{\mathcal{AB}}I_{0_+}^{\rho}|\Theta| + \frac{1}{\mathscr{B}(\rho)}\left(\frac{Z^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right)^{q-1}\left\{1-\rho+\frac{Z^{\rho}}{\Gamma(\rho)}\right\}.
$$

Finally

$$
\begin{aligned}
\max_{\Upsilon\in\Omega}|A_w - A_{\overline{w}}| &= \max_{\Upsilon\in\Omega}\left|\left(Q(\Upsilon)+\frac{1}{\Gamma(\gamma)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\gamma-1}f(\sigma,w(\sigma))d\sigma\right)\right.\\
&\qquad\left.-\left(Q(\Upsilon)+\frac{1}{\Gamma(\gamma)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\gamma-1}f(\sigma,\overline{w}(\sigma))d\sigma\right)\right|\\
&\le \frac{1}{\Gamma(\gamma)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\gamma-1}\max_{\Upsilon\in\Omega}|f(\sigma,w(\sigma))-f(s,\overline{w}(s))|d\sigma\\
&\le \frac{L_f Z^{\gamma}}{\Gamma(\gamma+1)}\|w-\overline{w}\|.
\end{aligned}
$$

Then (19) becomes

$$
\begin{aligned}
&\|Aw - A\overline{w}\|\\
&\le \left[\left(L_Q Z + |Q(0)| + \frac{Z^{\gamma}\|F\|_{L^1}}{\Gamma(\gamma+1)}\right)\left(\frac{(q-1)\mathcal{M}^{q-2}Z^{\beta}}{\mathscr{B}(\rho)\Gamma(\beta+1)}\left\{1-\rho+\frac{Z^{\rho}}{\Gamma(\rho)}\right\}L_g\right)\right.\\
&\quad\left.+\left({}^{\mathcal{AB}}I_{0_+}^{\rho}|\Theta|+\frac{1}{\mathscr{B}(\rho)}\left(\frac{Z^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right)^{q-1}\left\{1-\rho+\frac{Z^{\rho}}{\Gamma(\rho)}\right\}\right)\left(\frac{L_f Z^{\gamma}}{\Gamma(\gamma+1)}\right)\right]\|w-\overline{w}\|.
\end{aligned}
$$

Consequently (18) becomes

$$
\|\Phi w - \Phi\overline{w}\|
$$

$$\leq \Bigg[\left(L_Q Z + |Q(0)| + \frac{Z^\gamma \|F\|_{L^1}}{\Gamma(\gamma+1)} \right) \left(\frac{(q-1)\mathcal{M}^{q-2} Z^\beta}{\mathscr{B}(\rho)\Gamma(\beta+1)} \left\{ 1 - \rho + \frac{Z^\rho}{\Gamma(\rho)} \right\} L_g \right)$$
$$+ \left({}^{\mathcal{AB}}I^\rho_{0_+} |\Theta| + \frac{1}{\mathscr{B}(\rho)} \left(\frac{Z^\beta \|G\|_{L^1}}{\Gamma(\beta+1)} \right)^{q-1} \left\{ 1 - \rho + \frac{Z^\rho}{\Gamma(\rho)} \right\} \right) \left(\frac{L_f Z^\gamma}{\Gamma(\gamma+1)} \right) + L_h \Bigg] \|w - \overline{w}\|$$
$$\leq \Delta \|w - \overline{w}\|.$$

From equation (17), we have the condition $\Delta < 1$, which guarantees that the operator Φ is a contraction. By applying Banach's FPT, it follows that the hybrid system of $m\mathcal{ABC}$-HFIDEs described in (1) possesses a unique solution, which corresponds to the fixed points of the operator Φ.

Subsequently, utilizing Krasnoselskii FPT [14, 16], we establish the existence of solutions for the system (1).

Theorem 2. *Under the conditions set by hypotheses* $(A1)-(A3)$, *the* $m\mathcal{ABC}$*-HFIDEs* (1) *is guaranteed to have at least one solution if*

$$L_h < 1. \tag{20}$$

Proof. Fix $\mathcal{B} = AC(\Omega, \mathbb{R})$ and define a subset $\mathbb{S}$ of $\mathcal{B}$ by

$$\mathbb{S} = \{ w \in \mathcal{B} : \|w\| \leq \Lambda \},$$

where

$$\Lambda = \left(L_Q Z + |Q(0)| + \frac{Z^\gamma \|F\|_{L^1}}{\Gamma(\gamma+1)} \right) \left({}^{\mathcal{AB}}I^\rho_{0_+} |\Theta| + \frac{1}{\mathscr{B}(\rho)} \left(\frac{Z^\beta \|G\|_{L^1}}{\Gamma(\beta+1)} \right)^{q-1} \left\{ 1 - \rho + \frac{Z^\rho}{\Gamma(\rho)} \right\} \right)$$
$$+ L_h \|\overline{w}\| + H_0$$

with $H_0 = \max_{\Upsilon \in \Omega} |h(\Upsilon, 0)|$. It is evident that $\mathbb{S}$ is a closed, convex, and bounded subset of the Banach space $\mathcal{B}$.

We will now examine two operators A and B that map from $\mathbb{S}$ to $\mathcal{B}$, which are defined as specified in equations (12) and (13), respectively.

At this point, the expression in (9) can be rewritten as the operator equation

$$w(\Upsilon) = (Aw)(\Upsilon) + (Bw)(\Upsilon), \quad \Upsilon \in \Omega.$$

Step 1: Let $w, \overline{w} \in \mathbb{S}$. Then from (13) and (A1), we have

$$\begin{aligned} \|Bw - B\overline{w}\| &= \max_{\Upsilon \in \Omega} |h(\Upsilon, w(\Upsilon)) - h(\Upsilon, \overline{w}(\Upsilon))| \\ &\leq L_h \|w - \overline{w}\|. \end{aligned}$$

Therefore, according to (20), the operator B acts as a contraction on $\mathbb{S}$ with a constant $L_h < 1$.

Step 2: We will demonstrate that A is a compact operator mapping from $\mathbb{S}$ to $\mathcal{B}$. It suffices to show that $A(\mathbb{S})$ forms a uniformly bounded and equi-continuous subset within $\mathcal{B}$. First, consider an arbitrary element $w \in \mathbb{S}$. Then, from (12) and (A2), we have

$$\begin{aligned}\|Aw\| \leq& \max_{\Upsilon\in\Omega}\left(|Q(\Upsilon)| + \frac{1}{\Gamma(\gamma)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\gamma-1}|f(\sigma, w(\sigma))|d\sigma\right)\\ &(\times)\Bigg[{}^{\mathcal{AB}}I^{\rho}_{0_+}|\Theta| + \frac{1-\rho}{\mathscr{B}(\rho)}\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\beta-1}|g(\sigma, w(\sigma))|d\sigma\right)\\ &+\frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}|g(\tau, w(\tau))|d\tau\right)\right)d\sigma\Bigg]\\ \leq& \left(L_Q Z + |Q(0)| + \frac{Z^{\gamma}\|F\|_{L^1}}{\Gamma(\gamma+1)}\right)\left({}^{\mathcal{AB}}I^{\rho}_{0_+}|\Theta| + \frac{1}{\mathscr{B}(\rho)}\left(\frac{Z^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right)^{q-1}\left\{1-\rho+\frac{Z^{\rho}}{\Gamma(\rho)}\right\}\right).\end{aligned}$$

This indicates that $A(\mathbb{S})$ is uniformly bounded within $\mathcal{B}$. Conversely, let $\Upsilon_1, \Upsilon_2 \in \Omega$ be chosen arbitrarily such that $\Upsilon_1 < \Upsilon_2$. Then, for any $w \in \mathbb{S}$, we obtain

$$\begin{aligned}&|(Aw)(\Upsilon_2) - (Aw)(\Upsilon_1)|\\ &\leq \left(|Q(\Upsilon_2)| + \frac{Z^{\gamma}\|F\|_{L^1}}{\Gamma(\gamma+1)}\right)\left(\frac{(q-1)\mathcal{M}^{q-2}Z^{\beta}}{\mathscr{B}(\rho)\Gamma(\beta+1)}\left\{1-\rho+\frac{Z^{\rho}}{\Gamma(\rho)}\right\}\left|\int_{\Upsilon_1}^{\Upsilon_2}G(\sigma)d\sigma\right|\right)\\ &\quad+\left({}^{\mathcal{AB}}I^{\rho}_{0_+}|\Theta| + \frac{1}{\mathscr{B}(\rho)}\left(\frac{Z^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right)^{q-1}\left\{1-\rho+\frac{Z^{\rho}}{\Gamma(\rho)}\right\}\right)\Bigg(L_Q|\Upsilon_2-\Upsilon_1|\\ &\quad+\frac{Z^{\gamma}}{\Gamma(\gamma+1)}\left|\int_{\Upsilon_1}^{\Upsilon_2}F(\sigma)d\sigma\right|\Bigg)\\ &= \left(|Q(\Upsilon_2)| + \frac{Z^{\gamma}\|F\|_{L^1}}{\Gamma(\gamma+1)}\right)\left(\frac{(q-1)\mathcal{M}^{q-2}Z^{\beta}}{\mathscr{B}(\rho)\Gamma(\beta+1)}\left\{1-\rho+\frac{Z^{\rho}}{\Gamma(\rho)}\right\}|\xi(\Upsilon_2)-\xi(\Upsilon_1)|\right)\\ &\quad+\left({}^{\mathcal{AB}}I^{\rho}_{0_+}|\Theta| + \frac{1}{\mathscr{B}(\rho)}\left(\frac{Z^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right)^{q-1}\left\{1-\rho+\frac{Z^{\rho}}{\Gamma(\rho)}\right\}\right)\Bigg(L_Q|\Upsilon_2-\Upsilon_1|\\ &\quad+\frac{Z^{\gamma}}{\Gamma(\gamma+1)}|\zeta(\Upsilon_2)-\zeta(\Upsilon_1)|\Bigg),\end{aligned}$$

where $\xi(\Upsilon) = \int_0^{\Upsilon}G(\sigma)d\sigma$ and $\zeta(\Upsilon) = \int_0^{\Upsilon}F(\sigma)d\sigma$. Given that the functions ξ and ζ are continuous on the compact interval Ω, they are also uniformly continuous. Therefore, for any $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that for all $\Upsilon_1, \Upsilon_2 \in \Omega$ and $w \in \mathbb{S}$, the following holds:

$$|\Upsilon_2 - \Upsilon_1| < \delta \implies |(Aw)(\Upsilon_2) - (Aw)(\Upsilon_1)| < \varepsilon.$$

This establishes that $A(\mathbb{S})$ is an equi-continuous subset of $\mathcal{B}$. Since $A(\mathbb{S})$ is both uniformly bounded and equi-continuous in $\mathcal{B}$, it follows from the Arzelà-Ascoli theorem that $A(\mathbb{S})$ is relatively compact. Consequently, we conclude that A is a compact operator on $\mathbb{S}$.

Step 3: To demonstrate that A is a continuous operator from $\mathbb{S}$ to $\mathcal{B}$, consider a sequence $\{w_n\}$ in $\mathbb{S}$ that converges to a point $w \in \mathbb{S}$. By applying the Lebesgue dominated convergence theorem, we can derive the following result:

$$\begin{aligned}
&\lim_{n\to\infty}(Aw_n)(\curlyvee)\\
&=\lim_{n\to\infty}\Bigg\{\left(Q(\curlyvee)+\frac{1}{\Gamma(\gamma)}\int_0^{\curlyvee}(\curlyvee-\sigma)^{\gamma-1}f(\sigma,w_n(\sigma))d\sigma\right)\\
&\quad(\times)\Bigg[{}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta)+\frac{1-\rho}{\mathscr{B}(\rho)}\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\curlyvee}(\curlyvee-\sigma)^{\beta-1}g(\sigma,w_n(\sigma))d\sigma\right)\\
&\quad+\frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\curlyvee}(\curlyvee-\sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}g(\tau,w_n(\tau))d\tau\right)\right)d\sigma\Bigg]\Bigg\}\\
&=\left(Q(\curlyvee)+\frac{1}{\Gamma(\gamma)}\int_0^{\curlyvee}(\curlyvee-\sigma)^{\gamma-1}\lim_{n\to\infty}f(\sigma,w_n(\sigma))d\sigma\right)\\
&\quad(\times)\Bigg[{}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta)+\frac{1-\rho}{\mathscr{B}(\rho)}\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\curlyvee}(\curlyvee-\sigma)^{\beta-1}\lim_{n\to\infty}g(\sigma,w_n(\sigma))d\sigma\right)\\
&\quad+\frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\curlyvee}(\curlyvee-\sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}\lim_{n\to\infty}g(\tau,w_n(\tau))d\tau\right)\right)d\sigma\Bigg]\\
&=\left(Q(\curlyvee)+\frac{1}{\Gamma(\gamma)}\int_0^{\curlyvee}(\curlyvee-\sigma)^{\gamma-1}f(\sigma,w(\sigma))d\sigma\right)\\
&\quad(\times)\Bigg[{}^{\mathcal{AB}}I^{\rho}_{0_+}(\Theta)+\frac{1-\rho}{\mathscr{B}(\rho)}\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\curlyvee}(\curlyvee-\sigma)^{\beta-1}g(\sigma,w(\sigma))d\sigma\right)\\
&\quad+\frac{\rho}{\mathscr{B}(\rho)\Gamma(\rho)}\int_0^{\curlyvee}(\curlyvee-\sigma)^{\rho-1}\left(\Psi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}g(\tau,w(\tau))d\tau\right)\right)d\sigma\Bigg]\\
&=(Aw)(\curlyvee)
\end{aligned}$$

for all $\curlyvee \in \Omega$. This establishes that the sequence $\{Aw_n\}$ converges point-wise to Aw on the interval Ω. Furthermore, by employing a similar argument as in Step 2, we can demonstrate that the sequence $\{Aw_n\}$ is equi-continuous. Consequently, it follows that $\{Aw_n\}$ converges uniformly to Aw, thereby confirming that A is a continuous operator on $\mathbb{S}$.

Step 4: We demonstrate that $Aw + B\overline{w} \in \mathbb{S}$ for all $w, \overline{w} \in \mathbb{S}$. For any $w, \overline{w} \in \mathbb{S}$ and $\curlyvee \in \Omega$, it follows that

$$\begin{aligned}
&|(Aw)(\curlyvee)+(B\overline{w})(\curlyvee)|\\
&\le\left(L_QZ+|Q(0)|+\frac{Z^{\gamma}\|F\|_{L^1}}{\Gamma(\gamma+1)}\right)\left({}^{\mathcal{AB}}I^{\rho}_{0_+}|\Theta|+\frac{1}{\mathscr{B}(\rho)}\left(\frac{Z^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right)^{q-1}\left\{1-\rho+\frac{Z^{\rho}}{\Gamma(\rho)}\right\}\right)\\
&\quad+L_f\|\overline{w}\|+F_0
\end{aligned}$$

$\leq \Lambda,$

it follows that $Aw + B\overline{w} \in \mathbb{S}$ for all $w, \overline{w} \in \mathbb{S}$. This confirms that all the conditions specified in [17, Theorem 2.6] are satisfied, leading to the conclusion that the operator equation $Aw + B\overline{w} = w$ has a solution in the set $\mathbb{S}$. Consequently, the $m\mathcal{ABC}$-HFIDEs (1) has a solution that is defined on the interval Ω.

4. Stability Analysis

This section is dedicated to the study of U-H-stability of the $m\mathcal{ABC}$-HFIDEs (1). To proceed, we will first recall the following definition:

Definition 7. *The integral equation* (11) *is Ulam-Hyers stable, if for some* $\lambda_1 > 0$, *we have* $\vartheta > 0$ *with* w *satisfying*

$$\|w - \Phi w\| < \vartheta \tag{21}$$

with $\overline{w}(\Upsilon)$ *of* (11) *with*

$$\overline{w}(\Upsilon) = \Phi\overline{w}(\Upsilon) \tag{22}$$

and $\|w - \overline{w}\| < \vartheta\lambda_1$.

Theorem 3. *Under the conditions of Theorem 1, the system* (11) *demonstrates U-H stability, which implies the U-H stability of the hybrid system of* $m\mathcal{ABC}$*-FDEs* (1).

Proof. For any $w, \overline{w}^* \in AC(\Omega, \mathbb{R})$, we have

$$\begin{aligned}\|\Phi w - \Phi\overline{w}^*\| &= \max_{\Upsilon\in\Omega} |((A+B)w)(\Upsilon) - ((A+B)\overline{w})(\Upsilon)| \\ &\leq \max_{\Upsilon\in\Omega} |(Aw)(\Upsilon) - (A\overline{w})(\Upsilon)| + \max_{\Upsilon\in\Omega} |(Bw)(\Upsilon) - (B\overline{w})(\Upsilon)|.\end{aligned}$$

In view of Theorem 1, we have

$$\begin{aligned}&\|\Phi w - \Phi\overline{w}^*\| \\ &\leq \Bigg[\left(L_Q Z + |Q(0)| + \frac{Z^\gamma\|F\|_{L^1}}{\Gamma(\gamma+1)}\right)\left(\frac{(q-1)\mathcal{M}^{q-2}Z^\beta}{\mathscr{B}(\rho)\Gamma(\beta+1)}\left\{1-\rho+\frac{Z^\rho}{\Gamma(\rho)}\right\}L_g\right) \\ &\quad + \left({}^{\mathcal{AB}}I^\rho_{0_+}|\Theta| + \frac{1}{\mathscr{B}(\rho)}\left(\frac{Z^\beta\|G\|_{L^1}}{\Gamma(\beta+1)}\right)^{q-1}\left\{1-\rho+\frac{Z^\rho}{\Gamma(\rho)}\right\}\right)\left(\frac{L_f Z^\gamma}{\Gamma(\gamma+1)}\right) + L_h\Bigg]\|w - \overline{w}^*\| \\ &= \Delta\|w - \overline{w}^*\|.\end{aligned} \tag{23}$$

For $\Delta < 1$, by (21)-(23), consider the following norm

$$\|w - \overline{w}^*\| = \|w - \Phi w + \Phi w - \overline{w}^*\|$$

$$\leq \|w - \Phi w\| + \|\Phi w - \Phi \overline{w}^*\|$$
$$\leq \vartheta + \Delta \|w - \overline{w}^*\|.$$

Hence

$$\|w - \overline{w}^*\| \leq \frac{\vartheta}{1 - \Delta}$$

with $\lambda = \dfrac{1}{1-\Delta}$. Hence (11) is stable. This implies the stability of the addressing system represented by (1).

5. Example

In this section, we will provide a justification for our findings by presenting an illustrative example.

Consider the given $m\mathcal{ABC}$-HFIDEs

$$\begin{cases} {}^{C}\mathcal{D}^{\beta}\left(\Psi_p\left({}^{m\mathcal{ABC}}\mathcal{D}^{\rho}_{0+}\left(\dfrac{w(\Upsilon) - \frac{1}{16}\sin w(\Upsilon)}{\pi + \sin\Upsilon + \frac{1}{\Gamma\left(\frac{1}{3}\right)}\int_0^{\Upsilon}(\Upsilon-\sigma)^{-\frac{2}{3}}\frac{1}{\sigma+25}\sin w(\sigma)d\sigma}\right)\right)\right) \\ = \dfrac{1}{\Upsilon+36}\cos w(\Upsilon), \quad \Upsilon \in [0,1], \\ w(0) = \dfrac{1}{16}\sin w(0) + \pi. \end{cases} \tag{24}$$

Set $\beta = \frac{1}{4}, \rho = \frac{1}{2}, \gamma = \frac{1}{3}, Z = 1, L_Q = 1, \Theta = 0.735, {}^{\mathcal{AB}}I^{\rho}_{0+}\Theta = 1, p = q = 2, \mathcal{M} = 1, \mathscr{B}(\rho) = 1 - \rho + \frac{\rho}{\Gamma(\rho)}, h(\Upsilon, w(\Upsilon)) = \frac{1}{16}\sin w(\Upsilon), Q(\Upsilon) = \pi + \sin\Upsilon, Q(0) = \pi, f(\Upsilon, w(\Upsilon)) = \frac{1}{\Upsilon+25}\sin w(\Upsilon), g(\Upsilon, w(\Upsilon)) = \frac{1}{\Upsilon+36}\cos w(\Upsilon)$.

Let $w, \overline{w} \in AC([0,1])$. Then, we have

$$|f(\Upsilon, w(\Upsilon)) - f(\Upsilon, \overline{w}(\Upsilon))| \leq \frac{1}{26}\left|w(\Upsilon) - \overline{w}(\Upsilon)\right|,$$

$$|g(\Upsilon, w(\Upsilon)) - g(\Upsilon, \overline{w}(\Upsilon))| \leq \frac{1}{37}\left|w(\Upsilon) - \overline{w}(\Upsilon)\right|,$$

and

$$|h(\Upsilon, w(\Upsilon)) - h(\Upsilon, \overline{w}(\Upsilon))| \leq \frac{1}{16}\left|w(\Upsilon) - \overline{w}(\Upsilon)\right|.$$

Then the assumptions (A1)-(A3) holds with $L_f = \frac{1}{26}, L_g = \frac{1}{37}, L_h = \frac{1}{16}, L_Q = 1, \|F\|_{L^1} = \ln\left(\frac{26}{25}\right) = 0.03922, \|G\|_{L^1} = \ln\left(\frac{37}{36}\right) = 0.02731$.

At this point, we will examine the conditions outlined in the theorems to ensure they are satisfied. By carefully analyzing these conditions, we can derive the necessary conclusions and results that follow from them. This thorough verification process will allow us to confirm the validity of our findings.

Now

$$\begin{aligned}\Lambda = \Bigg[&\left(L_Q Z + |Q(0)| + \frac{Z^{\gamma}\|F\|_{L^1}}{\Gamma(\gamma+1)}\right)\left(\frac{(q-1)\mathcal{M}^{q-2}Z^{\beta}}{\mathscr{B}(\rho)\Gamma(\beta+1)}\left\{1-\rho+\frac{Z^{\rho}}{\Gamma(\rho)}\right\}L_g\right) \\ &+ \left({}^{\mathcal{AB}}I^{\rho}_{0_+}|\Theta| + \frac{1}{\mathscr{B}(\rho)}\left(\frac{Z^{\beta}\|G\|_{L^1}}{\Gamma(\beta+1)}\right)^{q-1}\left\{1-\rho+\frac{Z^{\rho}}{\Gamma(\rho)}\right\}\right)\left(\frac{L_f Z^{\gamma}}{\Gamma(\gamma+1)}\right) + L_h\Bigg] \\ = 0.2782 < 1.&\end{aligned}$$

Consequently, we have established that the conditions specified in Theorem 1 are indeed met. As a result of this verification, we can confidently conclude that problem (24) possesses a unique solution.

Next,

$$L_h = 0.0625 < 1.$$

Therefore, we can confirm that the criteria outlined in Theorem 2 are also fulfilled. This affirmation leads us to conclude that the problem presented in equation (24) has at least one solution. Also $1 - 0.2782 = 0.7218 \neq 0$. Thus (24) is U-H stable.

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