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Rarely s-$(\tau_1, \tau_2)p$-continuous Multifunctions

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Abstract. This paper is concerned with the concepts of upper rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions and lower rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions. Furthermore, some characterizations and several properties concerning upper rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions and lower rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions are established.

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1. Introduction

Weaker and stronger forms of open sets such as semi-open sets [48], preopen sets [50], α-open sets [51], β-open sets [38], δ-open sets [67] and θ-open sets [67] play an important role in the research of generalizations of continuity in topological spaces. By using these sets, many authors introduced and studied various types of continuity for functions and multifunctions. Viriyapong and Boonpok [69] investigated some characterizations of (Λ, sp)-continuous functions by utilizing the notions of (Λ, sp)-open sets and (Λ, sp)-closed sets due to Boonpok and Khampakdee [12]. Dungthaisong et al. [35] introduced and studied the concept of $g_{(m,n)}$-continuous functions. Duangphui et al. [34] introduced and investigated the notion of $(\mu, \mu')^{(m,n)}$-continuous functions. Moreover, some characterizations of almost (Λ, p)-continuous functions, strongly $\theta(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $(\Lambda, p(\star))$-continuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions,

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pairwise almost M-continuous functions, (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions, weakly (τ_1, τ_2)-continuous functions, faintly (τ_1, τ_2)-continuous functions, almost quasi (τ_1, τ_2)-continuous functions and weakly quasi (τ_1, τ_2)-continuous functions were presented in [61], [64], [16], [56], [25], [11], [8], [10], [4], [1], [2], [26], [23], [18], [62], [46] and [33], respectively. Popa [54] introduced the concept of rare continuity as a generalization of weak continuity [47] which has been further investigated by Long and Herrington [49] and Jafari [39, 40]. Jafari [41] also generalized the concept of rare continuity to rare β-continuity by involving the notion of β-open sets. Caldas [30] introduced a new class of functions called rarely $\beta\theta$-continuous functions by utilizing the notion of β-θ-open sets and investigated some characterizations of rarely $\beta\theta$-continuous functions. Jafari [42] introduced and studied the concept of rare α-continuity as a generalization of rare continuity and weak α-continuity [52]. Caldas and Jafari [31] introduced and investigated a new class of functions called rarely g-continuous functions which is a generalization of both the class of rarely continuous functions and the class of weakly g-continuous functions. Quite recently, Thongmoon et al. [66] introduced and studied the concept of rarely (τ_1, τ_2)-continuous functions.

In 2005, Caldas et al. [32] introduced and studied the new notion of rarely g-continuous multifunctions is a generalization of weakly continuous multifunctions [53]. Viriyapong and Boonpok [70] introduced and studied the concept of weakly quasi (Λ, sp)-continuous multifunctions. Furthermore, several characterizations of $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly (τ_1, τ_2)-continuous multifunctions, $\star$-continuous multifunctions, $\beta(\star)$-continuous multifunctions, α-$\star$-continuous multifunctions, almost α-$\star$-continuous multifunctions, almost quasi $\star$-continuous multifunctions, weakly α-$\star$-continuous multifunctions, $s\beta(\star)$-continuous multifunctions, weakly $s\beta(\star)$-continuous multifunctions, $\theta(\star)$-quasi continuous multifunctions, almost $\imath^\star$-continuous multifunctions, weakly (Λ, sp)-continuous multifunctions, $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\alpha(\Lambda, sp)$-continuous multifunctions, weakly $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\beta(\Lambda, sp)$-continuous multifunctions, slightly (Λ, sp)-continuous multifunctions, (τ_1, τ_2)-continuous multifunctions, almost (τ_1, τ_2)-continuous multifunctions, weakly (τ_1, τ_2)-continuous multifunctions, weakly quasi (τ_1, τ_2)-continuous multifunctions, almost quasi (τ_1, τ_2)-continuous multifunctions, c-(τ_1, τ_2)-continuous multifunctions and c-quasi (τ_1, τ_2)-continuous multifunctions were established in [5], [28], [3], [7], [17], [24], [6], [21], [20], [15], [9], [19], [22], [43], [13], [27], [63], [14], [59], [45], [65], [60], [58], [44] and [57], respectively. Popa and Noiri [55] introduced and studied the notion of s-precontinuous multifunctions is a generalization of s-continuous multifunctions and precontinuous multifunctions. Ekici and Park [37] introduced and investigated the concept of weakly s-precontinuous multifunctions. The notion of weakly s-precontinuous multifunctions is a generalization of s-precontinuous multifunctions due to Popa and Noiri [55]. Ekici and Jafari [36] introduced and investigated the notion of rarely s-precontinuous multifunctions which is a generalization of weakly s-precontinuous multifunctions due to Ekici and Park [37]. In this paper, we introduce the notions of upper rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions and lower rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions. We also investigate several characterizations of upper rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions and lower rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$*-closed* [29] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$*-open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$*-closure* [29] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$*-interior* [29] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [29] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).

(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.

(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).

(5) $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.

A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-connected* [29] if X cannot be written as the union of two nonempty disjoint $\tau_1\tau_2$-open sets. A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)r$*-open* [68] (resp. $(\tau_1, \tau_2)s$*-open* [5], $(\tau_1, \tau_2)p$*-open* [5], $(\tau_1, \tau_2)\beta$*-open* [5], $\alpha(\tau_1, \tau_2)$*-open*) [71]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open, $\alpha(\tau_1, \tau_2)$-open) set is called $(\tau_1, \tau_2)r$*-closed* (resp. $(\tau_1, \tau_2)s$*-closed*, $(\tau_1, \tau_2)p$*-closed*, $(\tau_1, \tau_2)\beta$*-closed*, $\alpha(\tau_1, \tau_2)$*-closed*). A subset R of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-rare set* [66] if $\tau_1\tau_2$-Int$(R) = \emptyset$. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)p$-closed sets of X containing A is called the $(\tau_1, \tau_2)p$*-closure* of A and is denoted by (τ_1, τ_2)-pCl(A). The union of all $(\tau_1, \tau_2)p$-open sets of X contained in A is called the $(\tau_1, \tau_2)p$*-interior* of A and is denoted by (τ_1, τ_2)-pInt(A).

Lemma 2. [72] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) A is $(\tau_1, \tau_2)p$-closed if and only if (τ_1, τ_2)-pCl$(A) = A$;

(2) (τ_1, τ_2)-pCl$(A) = \tau_1\tau_2 Cl(\tau_1\tau_2 Int(A)) \cup A$;

(3) (τ_1, τ_2)-pCl$((\tau_1, \tau_2)$-pCl$(A)) = (\tau_1, \tau_2)$-pCl(A).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A} F(x)$.

3. Upper and lower rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions

In this section, we introduce the notions of upper rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions and lower rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions. Moreover, some characterizations of upper rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions and lower rarely s-$(\tau_1, \tau_2)p$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper rarely s-$(\tau_1, \tau_2)p$-continuous at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement such that $F(x) \subseteq V$, there exists a $\sigma_1\sigma_2$-rare set R_V with $\tau_1\tau_2$-$Cl(R_V) \cap V = \emptyset$ and a $(\tau_1, \tau_2)p$-open set U of X containing x such that $F(U) \subseteq V \cup R_V$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper rarely s-$(\tau_1, \tau_2)p$-continuous if F is upper rarely s-$(\tau_1, \tau_2)p$-continuous at each point x of X.*

Theorem 1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper rarely s-$(\tau_1, \tau_2)p$-continuous at $x \in X$;

(2) for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement with $F(x) \subseteq V$, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2$-$Cl(R_V) \cap V = \emptyset$ such that

$$x \in (\tau_1, \tau_2)\text{-}pInt(F^+(V \cup R_V));$$

(3) for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement with $F(x) \subseteq V$, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2$-$Cl(V) \cap R_V = \emptyset$ such that

$$x \in (\tau_1, \tau_2)\text{-}pInt(F^+(\sigma_1\sigma_2\text{-}Cl(V) \cup R_V));$$

(4) for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement with $F(x) \subseteq V$, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that

$$\sigma_1\sigma_2\text{-}Int(F(U) \cap (Y - V)) = \emptyset;$$

(5) for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement with $F(x) \subseteq V$, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that

$$\sigma_1\sigma_2\text{-}Int(F(U)) \subseteq \sigma_1\sigma_2\text{-}Cl(V);$$

(6) for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement with $F(x) \subseteq V$, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2$-Cl$(R_V) \cap V = \emptyset$ such that

$$x \in F^+(V \cup R_V) \cap \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(F^+(V \cup R_V))).$$

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement such that $F(x) \subseteq V$. Since F is upper rarely s-$(\tau_1, \tau_2)p$-continuous at $x \in X$, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2$-Cl$(R_V) \cap V = \emptyset$ and a $(\tau_1, \tau_2)p$-open set U of X containing x such that $F(U) \subseteq V \cup R_V$. Thus, $x \in U \subseteq F^+(V \cup R_V)$ and hence

$$x \in (\tau_1, \tau_2)\text{-pInt}(F^+(V \cup R_V)).$$

(2) ⇒ (3): Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement such that $F(x) \subseteq V$. By (2), there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2$-Cl$(R_V) \cap V = \emptyset$ such that $x \in (\tau_1, \tau_2)$-pInt$(F^+(V \cup R_V))$. Since $\sigma_1\sigma_2$-Cl$(R_V) \cap V = \emptyset$, we have $R_V \subseteq Y - V$ and $Y - V = [Y - \sigma_1\sigma_2\text{-Cl}(V)] \cup [\sigma_1\sigma_2\text{-Cl}(V) - V]$. Thus,

$$R_V \subseteq [R_V \cap (Y - \sigma_1\sigma_2\text{-Cl}(V))] \cup [\sigma_1\sigma_2\text{-Cl}(V) - V].$$

Put $W_V = R_V \cap (Y - \sigma_1\sigma_2\text{-Cl}(V))$. Then, W_V is a $\tau_1\tau_2$-rare set with $\sigma_1\sigma_2$-Cl$(V) \cap W_V = \emptyset$. Therefore, $x \in (\tau_1, \tau_2)$-pInt$(F^+(V \cup R_V)) \subseteq (\tau_1, \tau_2)$-pInt$(F^+(\sigma_1\sigma_2\text{-Cl}(V) \cup W_V))$.

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement such that $F(x) \subseteq V$. By (3), there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2$-Cl$(V) \cap R_V = \emptyset$ such that $x \in (\tau_1, \tau_2)$-pInt$(F^+(\sigma_1\sigma_2\text{-Cl}(V) \cup R_V))$. Let $U = (\tau_1, \tau_2)$-pInt$(F^+(\sigma_1\sigma_2\text{-Cl}(V) \cup R_V))$. Then, U is a $(\tau_1, \tau_2)p$-open set of X containing x and $F(U) \subseteq \sigma_1\sigma_2$-Cl$(V) \cup R_V$. Thus,

$$\begin{aligned}
&\sigma_1\sigma_2\text{-Int}(F(U) \cap (Y - V)) \\
&= \sigma_1\sigma_2\text{-Int}(F(U)) \cap \sigma_1\sigma_2\text{-Int}(Y - V) \\
&\subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V) \cup R_V) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V)) \\
&= \sigma_1\sigma_2\text{-Int}([\sigma_1\sigma_2\text{-Cl}(V) \cup R_V] \cap [Y - \sigma_1\sigma_2\text{-Cl}(V)]) \\
&= \sigma_1\sigma_2\text{-Int}([\sigma_1\sigma_2\text{-Cl}(V) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V))] \cup [R_V \cap (Y - \sigma_1\sigma_2\text{-Cl}(V))]) \\
&= \sigma_1\sigma_2\text{-Int}(R_V \cap (Y - \sigma_1\sigma_2\text{-Cl}(V))) \\
&= \sigma_1\sigma_2\text{-Int}(R_V) \cap \sigma_1\sigma_2\text{-Int}(Y - \sigma_1\sigma_2\text{-Cl}(V)) \\
&= \emptyset.
\end{aligned}$$

(4) ⇒ (5): Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement such that $F(x) \subseteq V$. By (4), there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $\sigma_1\sigma_2$-Int$(F(U) \cap (Y - V)) = \emptyset$. Since $\sigma_1\sigma_2$-Int$(F(U) \cap (Y - V)) = \emptyset$, we have $\sigma_1\sigma_2$-Int$(F(U)) \subseteq V \subseteq \sigma_1\sigma_2$-Cl$(V)$.

(5) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement with $F(x) \subseteq V$. By (5), there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $\sigma_1\sigma_2$-Int$(F(U)) \subseteq \sigma_1\sigma_2$-Cl$(V)$. Thus,

$$F(U) = (F(U) - \sigma_1\sigma_2\text{-Int}(F(U))) \cup \sigma_1\sigma_2\text{-Int}(F(U))$$

$$\begin{aligned}&\subseteq (F(U) - \sigma_1\sigma_2\text{-Int}(F(U))) \cup \sigma_1\sigma_2\text{-Cl}(V)\\&= (F(U) - \sigma_1\sigma_2\text{-Int}(F(U))) \cup V \cup (\sigma_1\sigma_2\text{-Cl}(V) - V)\\&= [(F(U) - \sigma_1\sigma_2\text{-Int}(F(U))) \cap (Y - V)] \cup V \cup (\sigma_1\sigma_2\text{-Cl}(V) - V).\end{aligned}$$

Let $W_V = (F(U) - \sigma_1\sigma_2\text{-Int}(F(U))) \cap (Y - V)$ and $W'_V = \sigma_1\sigma_2\text{-Cl}(V) - V$. Then, W_V and W'_V are $\sigma_1\sigma_2$-rare sets and $R_V = W_V \cup W'_V$ is a $\sigma_1\sigma_2$-rare set such that $\sigma_1\sigma_2\text{-Cl}(R_V) \cap V = \emptyset$ and $F(U) \subseteq V \cup R_V$. Thus, F is upper rarely s-$(\tau_1, \tau_2)p$-continuous at x.

(2) $\Leftrightarrow$ (6): It follows from the fact that

$$(\tau_1, \tau_2)\text{-pInt}(F^+(V \cup R_V)) = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^+(V \cup R_V))) \cap F^+(V \cup R_V).$$

Definition 2. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly s-$(\tau_1, \tau_2)p$-continuous at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement such that $x \in F^+(V)$, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $U \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly s-$(\tau_1, \tau_2)p$-continuous if F is upper weakly s-$(\tau_1, \tau_2)p$-continuous at each point x of X.*

Definition 3. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower weakly s-$(\tau_1, \tau_2)p$-continuous at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement such that $x \in F^-(V)$, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $U \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower weakly s-$(\tau_1, \tau_2)p$-continuous if F is lower weakly s-$(\tau_1, \tau_2)p$-continuous at each point x of X.*

Definition 4. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called strongly $(\tau_1, \tau_2)p$-open if $F(U)$ is $\sigma_1\sigma_2$-open in Y for every $(\tau_1, \tau_2)p$-open set U of X.*

Theorem 2. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper rarely s-$(\tau_1, \tau_2)p$-continuous and strongly $(\tau_1, \tau_2)p$-open, then F is upper weakly s-$(\tau_1, \tau_2)p$-continuous.*

Proof. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement with $F(x) \subseteq V$. Since F is upper rarely s-$(\tau_1, \tau_2)p$-continuous, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $\sigma_1\sigma_2\text{-Int}(F(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. It follows from that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(F(U))) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, F is upper weakly s-$(\tau_1, \tau_2)p$-continuous.

Definition 5. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower rarely s-$(\tau_1, \tau_2)p$-continuous at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement such that $F(x) \cap V \neq \emptyset$, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2\text{-}Cl(R_V) \cap V = \emptyset$ and a $(\tau_1, \tau_2)p$-open set U of X containing x such that $F(z) \cap (V \cup R_V) \neq \emptyset$ for each $z \in U$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower rarely s-$(\tau_1, \tau_2)p$-continuous if F is lower rarely s-$(\tau_1, \tau_2)p$-continuous at each point x of X.*

Theorem 3. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower rarely s-(τ_1, τ_2)p-continuous at $x \in X$;

(2) for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement with $F(x) \cap V \neq \emptyset$, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2$-Cl$(R_V) \cap V = \emptyset$ such that

$$x \in (\tau_1, \tau_2)\text{-}pInt(F^{-}(V \cup R_V));$$

(3) for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-connected complement with $F(x) \cap V \neq \emptyset$, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2$-Cl$(R_V) \cap V = \emptyset$ such that

$$x \in F^{-}(V \cup R_V) \cap \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(F^{-}(V \cup R_V))).$$

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement such that $F(x) \subseteq V$. Since F is lower rarely s-$(\tau_1, \tau_2)p$-continuous at $x \in X$, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2$-Cl$(R_V) \cap V = \emptyset$ and a $(\tau_1, \tau_2)p$-open set U of X containing x such that $F(z) \cap (V \cup R_V) \neq \emptyset$ for each $z \in U$. Thus, $x \in U \subseteq F^{-}(V \cup R_V)$ and hence $x \in (\tau_1, \tau_2)$-pInt$(F^{-}(V \cup R_V))$.

(2) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-connected complement with $F(x) \cap V \neq \emptyset$. By (2), there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2$-Cl$(R_V) \cap V = \emptyset$ such that $x \in (\tau_1, \tau_2)$-pInt$(F^{-}(V \cup R_V))$. Let $U = (\tau_1, \tau_2)$-pInt$(F^{-}(V \cup R_V))$. Then, U is a $(\tau_1, \tau_2)p$-open set U of X containing x. Furthermore, $F(z) \cap (V \cup R_V) \neq \emptyset$ for every $z \in U$. Thus, F is lower rarely s-$(\tau_1, \tau_2)p$-continuous at $x \in X$.

(2) $\Leftrightarrow$ (3): It follows from the fact that

$$(\tau_1, \tau_2)\text{-pInt}(F^{-}(V \cup R_V)) = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^{-}(V \cup R_V))) \cap F^{-}(V \cup R_V).$$

Definition 6. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called rarely s-(τ_1, τ_2)p-continuous at $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2$-Cl$(R_V) \cap V = \emptyset$ and a (τ_1, τ_2)p-open set U of X containing x such that $f(U) \subseteq V \cup R_V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called rarely s-(τ_1, τ_2)p-continuous if f is rarely s-(τ_1, τ_2)p-continuous at each point x of X.*

Corollary 1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is rarely s-(τ_1, τ_2)p-continuous at $x \in X$;

(2) for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2$-Cl$(R_V) \cap V = \emptyset$ such that

$$x \in (\tau_1, \tau_2)\text{-}pInt(f^{-1}(V \cup R_V));$$

(3) for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2\text{-}Cl(V) \cap R_V = \emptyset$ such that

$$x \in (\tau_1, \tau_2)\text{-}pInt(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V) \cup R_V));$$

(4) for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that

$$\sigma_1\sigma_2\text{-}Int(f(U) \cap (Y - V)) = \emptyset;$$

(5) for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that

$$\sigma_1\sigma_2\text{-}Int(f(U)) \subseteq \sigma_1\sigma_2\text{-}Cl(V);$$

(6) for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $\sigma_1\sigma_2$-rare set R_V with $\sigma_1\sigma_2\text{-}Cl(R_V) \cap V = \emptyset$ such that

$$x \in f^{-1}(V \cup R_V) \cap \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(f^{-1}(V \cup R_V))).$$

Theorem 4. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is rarely s-$(\tau_1, \tau_2)p$-continuous if and only if for every $\sigma_1\sigma_2$-open set V of Y, there exists a $\sigma_1\sigma_2$-rare set R_V with*

$$\sigma_1\sigma_2\text{-}Cl(R_V) \cap V = \emptyset$$

such that $f^{-1}(V) \subseteq (\tau_1, \tau_2)\text{-}pInt(f^{-1}(V \cup R_V))$.

Proof. It is an immediate consequence of the above corollary.

Definition 7. *A bitopologcal space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-rarely separate if for every pair of distinct points x and y in X, there exist $\tau_1\tau_2$-open sets V_x and V_y containing x and y, respectively, and $\tau_1\tau_2$-rare sets R_{V_x}, R_{V_y} with $\tau_1\tau_2\text{-}Cl(R_{V_x}) \cap V_x = \emptyset$ and*

$$\tau_1\tau_2\text{-}Cl(R_{V_y}) \cap V_y = \emptyset$$

such that $(V_x \cup R_{V_x}) \cap (V_y \cup R_{V_y}) = \emptyset$.

Definition 8. *A bitopologcal space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)p$-Hausdorff if for any distinct pair of points x and y in X, there exist $(\tau_1, \tau_2)p$-open sets U and V of X containing x and y, respectively, such that $U \cap V = \emptyset$.*

Theorem 5. *If (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-rarely separate and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a rarely s-$(\tau_1, \tau_2)p$-continuous injection, then (X, τ_1, τ_2) is $(\tau_1, \tau_2)p$-Hausdorff.*

Proof. Let x and y be any distinct points in X. Then, $f(x) \neq f(y)$. Since (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-rarely separate, there exist $\sigma_1\sigma_2$-open sets V and W of Y containing $f(x)$ and $f(y)$, respectively, and $\sigma_1\sigma_2$-rare sets R_V and R_W with $\sigma_1\sigma_2\text{-Cl}(R_V) \cap V = \emptyset$ and

$$\sigma_1\sigma_2\text{-Cl}(R_W) \cap W = \emptyset$$

such that $(V \cup R_V) \cap (W \cup R_W) = \emptyset$. Thus,

$$(\tau_1, \tau_2)\text{-pInt}(f^{-1}(V \cup R_V)) \cap (\tau_1, \tau_2)\text{-pInt}(f^{-1}(W \cup R_W)) = \emptyset.$$

By Theorem 4, we have $x \in f^{-1}(V) \subseteq (\tau_1, \tau_2)\text{-pInt}(f^{-1}(V \cup R_V))$ and

$$y \in f^{-1}(W) \subseteq (\tau_1, \tau_2)\text{-pInt}(f^{-1}(W \cup R_W)).$$

Since $(\tau_1, \tau_2)\text{-pInt}(f^{-1}(V \cup R_V))$ and $(\tau_1, \tau_2)\text{-pInt}(f^{-1}(W \cup R_W))$ are $(\tau_1, \tau_2)p$-open sets, (X, τ_1, τ_2) is a $(\tau_1, \tau_2)p$-Hausdorff space.

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