EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5693
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Refinements of Reverse Young Inequality for Scalars and Matrices

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Abstract. In this article, we introduce some refinements of the reverse Young inequality for scalars. As applications of our results, we establish corresponding inequalities for matrices. The obtained inequalities in this article can be viewed as refinements of the derived inequalities by Burqan and Khandaqji [4].

2020 Mathematics Subject Classifications: 47A63, 15A60.
Key Words and Phrases: Young inequality, positive matrices, unitarily invariant norms, singular values.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. For $T = [t_{ij}] \in M_n$, the Hilbert-Schmidt norm, the trace norm, and the spectral norm of T are defined by $\|T\|_2 = (\sum_{j=1}^{n} s_j^2(T))^{\frac{1}{2}}$,$\|T\|_1 = \sum_{j=1}^{n} s_j(T)$ and $\|T\| = s_1(T)$, respectively, where $s_1(T) \geq ... \geq s_n(T)$ are the singular values of T, that is, the eigenvalues of the positive semidefinite matrix $|T| = (T^*T)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity.

The classical Young inequality says that, if $a, b \geq 0$ and $0 \leq \mu \leq 1$, then

$$a^{\mu} b^{1-\mu} \leq \mu a + (1-\mu) b, \tag{1.1}$$

with equality if and only if $a = b$.
Kittaneh and Manasrah [5] obtained a refinement of inequality (1.1) as follows

$$a^{\mu} b^{1-\mu} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \mu a + (1-\mu) b, \tag{1.2}$$

where $r_0 = min\{\mu,\ 1-\mu\}$.
Kittaneh and Manasrah [6] gave a reverse of inequality (1.2) as follows

$$\mu a + (1-\mu) b \leq a^{\mu} b^{1-\mu} + R_0(\sqrt{a} - \sqrt{b})^2, \tag{1.3}$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5693

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where $R_0 = max\{\mu,\ 1-\mu\}$.
Also, Kai [9] gave a refinement of inequality (1.1) as follows
If $0 \le \mu \le \frac{1}{2}$, then

$$\left[(\mu a)^{\mu} b^{1-\mu}\right]^2 + \mu^2(a-b)^2 \le \mu^2 a^2 + (1-\mu)^2 b^2. \tag{1.4}$$

If $\frac{1}{2} \le \mu \le 1$, then

$$\left[a^{\mu}((1-\mu)b)^{1-\mu}\right]^2 + (1-\mu)^2(a-b)^2 \le \mu^2 a^2 + (1-\mu)^2 b^2. \tag{1.5}$$

Reverses of inequalities (1.4), (1.5) were established by Burqan and Khandaqji [4] as follows
If $0 \le \mu \le \frac{1}{2}$, then

$$\mu^2 a^2 + (1-\mu)^2 b^2 \le (1-\mu)^2(a-b)^2 + \left[a^{\mu}((1-\mu)b)^{1-\mu}\right]^2. \tag{1.6}$$

If $\frac{1}{2} \le \mu \le 1$, then

$$\mu^2 a^2 + (1-\mu)^2 b^2 \le \mu^2(a-b)^2 + \left[(\mu a)^{\mu} b^{1-\mu}\right]^2. \tag{1.7}$$

Moreover, Nasiri, Shokoori, and Liao [8] obtaind refinements of Kai results as follows
If $0 \le \mu \le \frac{1}{2}$, then

$$(\mu a)^{2\mu} b^{2-2\mu} + \mu^2(a-b)^2 + r_0 b\left(\sqrt{\mu a} - \sqrt{b}\right)^2 \le \mu^2 a^2 + (1-\mu)^2 b^2, \tag{1.8}$$

where $r_0 = min\{2\mu,\ 1-2\mu\}$.
If $\frac{1}{2} \le \mu \le 1$, then

$$\begin{aligned} a^{2\mu}[(1-\mu)b]^{2-2\mu} + (1-\mu)^2(a-b)^2 + r_0 a\left(\sqrt{a} - \sqrt{(1-\mu)b}\right)^2 \\ \le \mu^2 a^2 + (1-\mu)^2 b^2, \end{aligned} \tag{1.9}$$

where $r_0 = min\{2\mu - 1,\ 2-2\mu\}$.
A matrix version of (1.1) proved in [1] says that if $A, B \in M_n$ are positive semidefinit, then

$$\|A^{\mu} B^{1-\mu}\| \le \|\mu A + (1-\mu)B\|, \text{ for } 0 \le \mu \le 1. \tag{1.10}$$

Kosaki [7], Bhatia and Parthasarathy [3] proved that if $A, B, X \in M_n$ such that A and B are positive semidefinite, then

$$\|A^{\mu} X B^{1-\mu}\|_2^2 \le \|\mu AX + (1-\mu)XB\|_2^2, \text{ for } 0 \le \mu \le 1. \tag{1.11}$$

Based on the refined Young inequalities (1.4) and (1.5), Kai [9] have showed that if $A, B, X \in M_n$ such that A and B are positive semidefinite, then

$$\begin{aligned} \mu^2\|AX - XB\|_2^2 + \mu^{2\mu}\|A^{\mu} X B^{1-\mu}\|_2^2 + 2\mu(1-\mu)\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\|_2^2 \\ \le \|\mu AX + (1-\mu)XB\|_2^2, \end{aligned} \tag{1.12}$$

for $0 \leq \mu \leq \frac{1}{2}$.

$$(1-\mu)^2\|AX-XB\|_2^2+(1-\mu)^{2-2\mu}\|A^\mu XB^{1-\mu}\|_2^2+2\mu(1-\mu)\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2 \leq \|\mu AX+(1-\mu)XB\|_2^2, \tag{1.13}$$

for $\frac{1}{2} \leq \mu \leq 1$.

Burqan and Khandaqji [4] gave matrix versions of the inequalities (1.6) and (1.7) as follows Let $A, B, X \in M_n$ such that A and B are positive semidefinite. If $0 \leq \mu \leq \frac{1}{2}$, then

$$\|\mu AX+(1-\mu)XB\|_2^2 \leq (1-\mu)^2\|AX-XB\|_2^2+2\mu(1-\mu)\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2 + (1-\mu)^{2(1-\mu)}\|A^\mu XB^{1-\mu}\|_2^2. \tag{1.14}$$

If $\frac{1}{2} \leq \mu \leq 1$, then

$$\|\mu AX+(1-\mu)XB\|_2^2 \leq \mu^2\|AX-XB\|_2^2+2\mu(1-\mu)\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2 + \mu^{2\mu}\|A^\mu XB^{1-\mu}\|_2^2. \tag{1.15}$$

In this paper, we introduce reverses of the inequalities (1.8) and (1.9) which are refinements of the inequalities (1.6) and (1.7). As applications of our results, we obtain corresponding inequalities for matrices.

2. Main Results

We will divide our main results into two categories, the first is about scalars and the other is about matrices.

2.1. Inequalities for Scalars

We will start this section with the following results for scalars

Theorem 1. *Let $a, b \geq 0$. If $0 \leq \mu \leq \frac{1}{2}$, then*

$$\mu^2a^2+(1-\mu)^2b^2 \leq (\mu a)^{2\mu}b^{2-2\mu}+\mu^2(a-b)^2+R_0b\Big(\sqrt{\mu a}-\sqrt{b}\Big)^2, \tag{2.1}$$

where $R_0 = max\{2\mu,\ 1-2\mu\}$.
If $\frac{1}{2} \leq \mu \leq 1$, then

$$\mu^2a^2+(1-\mu)^2b^2 \leq a^{2\mu}[(1-\mu)b]^{2-2\mu}+(1-\mu)^2(a-b)^2 + R_0a\Big(\sqrt{a}-\sqrt{(1-\mu)b}\Big)^2, \tag{2.2}$$

where $R_0 = max\{2\mu-1,\ 2-2\mu\}$.

Proof. If $0 \leq \mu \leq \frac{1}{2}$, then by inequality (1.3), we have

$$\begin{aligned} \mu^2 a^2 + (1-\mu)^2 b^2 - \mu^2 (a-b)^2 &= b^2 - 2\mu b^2 + 2ab\mu^2 \\ &= b[(1-2\mu)b + 2\mu(\mu a)] \\ &\leq b\Big[b^{1-2\mu}(\mu a)^{2\mu} + R_0\Big(\sqrt{\mu a} - \sqrt{b}\Big)^2\Big] \\ &= (\mu a)^{2\mu} b^{2-2\mu} + R_0 b\Big(\sqrt{\mu a} - \sqrt{b}\Big)^2, \end{aligned}$$

and so

$$\mu^2 a^2 + (1-\mu)^2 b^2 \leq (\mu a)^{2\mu} b^{2-2\mu} + \mu^2 (a-b)^2 + R_0 b\Big(\sqrt{\mu a} - \sqrt{b}\Big)^2.$$

On the other hand, if $\frac{1}{2} \leq \mu \leq 1$, then by inequality (1.3), we have

$$\begin{aligned} \mu^2 a^2 + (1-\mu)^2 b^2 - (1-\mu)^2 (a-b)^2 &= -a^2 + 2ab + 2\mu^2 ab + 2\mu a^2 - 4\mu ab \\ &= a[(2\mu-1)a + 2(1-\mu)(1-\mu)b] \\ &\leq a\Big[a^{2\mu-1}((1-\mu)b)^{2-2\mu} + R_0\Big(\sqrt{a} - \sqrt{(1-\mu)b}\Big)^2\Big] \\ &= a^{2\mu}[(1-\mu)b]^{2-2\mu} + R_0 a\Big(\sqrt{a} - \sqrt{(1-\mu)b}\Big)^2, \end{aligned}$$

and so

$$\mu^2 a^2 + (1-\mu)^2 b^2 \leq a^{2\mu}[(1-\mu)b]^{2-2\mu} + (1-\mu)^2 (a-b)^2 + R_0 a\Big(\sqrt{a} - \sqrt{(1-\mu)b}\Big)^2.$$

This completes the proof.

2.2. Inequalities for Matrices

In the following theorem we introduce matrix versions of the inequalities (2.1) and (2.2), using the spectral theorem for positive semidefinite matrices.

Theorem 2. *Let $A, B, X \in M_n$ such that A and B are positive semidefinite.*
If $0 \leq \mu \leq \frac{1}{2}$, then

$$\begin{aligned} \|\mu AX + (1-\mu)XB\|_2^2 \leq{}& \mu^{2\mu}\|A^\mu X B^{1-\mu}\|_2^2 + \mu^2\|AX - XB\|_2^2 \\ &+ R_0\Big[\mu\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\|_2^2 + \|XB\|_2^2 - 2\sqrt{\mu}\|A^{\frac{1}{4}} X B^{\frac{3}{4}}\|_2^2\Big] \\ &+ 2\mu(1-\mu)\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\|_2^2, \end{aligned} \tag{2.3}$$

where $R_0 = max\{2\mu,\ 1-2\mu\}$.
If $\frac{1}{2} \leq \mu \leq 1$, then

$$\begin{aligned} \|\mu AX + (1-\mu)XB\|_2^2 \leq{}& (1-\mu)^{2(1-\mu)}\|A^\mu X B^{1-\mu}\|_2^2 + (1-\mu)^2\|AX - XB\|_2^2 \\ &+ R_0\Big[(1-\mu)\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\|_2^2 + \|AX\|_2^2 \\ &- 2\sqrt{(1-\mu)}\|A^{\frac{3}{4}} X B^{\frac{1}{4}}\|_2^2\Big] + 2\mu(1-\mu)\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\|_2^2, \end{aligned} \tag{2.4}$$

where $R_0 = max\{2\mu - 1,\ 2 - 2\mu\}$.

Proof. Since every positive semidefinite matrix is unitarily diagonalizable, hence it follows that there are unitary matrices $U, V \in M_n$ such that $A = UCU^*$and $B = VDV^*$, where $C = diag(\alpha_1, ..., \alpha_n)$, $D = diag(\beta_1, ..., \beta_n)$, and $\alpha_i, \beta_i \geq 0,\ i = 1, ..., n$.
Let $Y = U^*XV = \left[y_{ij}\right]$. Then we have

$$\mu AX + (1-\mu)XB = U\left[(\mu\alpha_i + (1-\mu)\beta_j)y_{ij}\right]V^*,$$

$$AX - XB = U\left[(\alpha_i - \beta_j)y_{ij}\right]V^*,$$

$$A^{\frac{1}{2}}XB^{\frac{1}{2}} = U\left[(\alpha_i^{\frac{1}{2}}\beta_j^{\frac{1}{2}})y_{ij}\right]V^*$$

and

$$A^{\mu}XB^{1-\mu} = U\left[(\alpha_i^{\mu}\beta_j^{1-\mu})y_{ij}\right]V^*.$$

It is known that the Hilbert-Schmidt norm is unitarily invariant, so
if $0 \leq \mu \leq \frac{1}{2}$, inequality (2.1) yields that

$$\begin{aligned}
\|\mu AX + (1-\mu)XB\|_2^2 &= \sum_{i,j=1}^{n} \Big(\mu\alpha_i + (1-\mu)\beta_j\Big)^2 |y_{ij}|^2 \\
&\leq \mu^2 \sum_{i,j=1}^{n} \Big(\alpha_i - \beta_j\Big)^2 |y_{ij}|^2 + \mu^{2\mu} \sum_{i,j=1}^{n} \Big(\alpha_i^{\mu}\beta_j^{1-\mu}\Big)^2 |y_{ij}|^2 \\
&+ 2\mu(1-\mu) \sum_{i,j=1}^{n} \Big(\alpha_i^{\frac{1}{2}}\beta_j^{\frac{1}{2}}\Big)^2 |y_{ij}|^2 \\
&+ R_0\Big[\mu \sum_{i,j=1}^{n} \Big(\alpha_i^{\frac{1}{2}}\beta_j^{\frac{1}{2}}\Big)^2 |y_{ij}|^2 + \sum_{i,j=1}^{n} \beta_j^2 |y_{ij}|^2 \\
&- 2\sqrt{\mu} \sum_{i,j=1}^{n} \Big(\alpha_i^{\frac{1}{4}}\beta_j^{\frac{3}{4}}\Big)^2 |y_{ij}|^2\Big] \\
&= \mu^2\|AX - XB\|_2^2 + \mu^{2\mu}\|A^{\mu}XB^{1-\mu}\|_2^2 + 2\mu(1-\mu)\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2 \\
&+ R_0\Big[\mu\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2 + \|XB\|_2^2 - 2\sqrt{\mu}\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\|_2^2\Big]
\end{aligned}$$

and so

$$\begin{aligned}
\|\mu AX + (1-\mu)XB\|_2^2 \leq \mu^{2\mu}\|A^{\mu}XB^{1-\mu}\|_2^2 &+ \mu^2\|AX - XB\|_2^2 \\
&+ R_0\Big[\mu\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2 + \|XB\|_2^2 - 2\sqrt{\mu}\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\|_2^2\Big] \\
&+ 2\mu(1-\mu)\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2,
\end{aligned}$$

Thus, we get (2.3).
If $\frac{1}{2} \leq \mu \leq 1$, then by the inequality (2.2) and the same method above, we have the

inequality (2.4).
This completes the proof.

Finally, we obtain refinements of the trace versions of Young type inequalities. To achieve this, we need the following lemmas that can be found in [2].

Lemma 1. *Let $A, B \in M_n$. Then*

$$\sum_{j=1}^{n} s_j(AB) \leq \sum_{j=1}^{n} s_j(A)s_j(B).$$

Lemma 2. *(Cauchy-Schwarz Inequality). Let $a_i \geq 0,\ b_i \geq 0$ for $i = 1, ..., n$. Then*

$$\sum_{i=1}^{n} a_i b_i \leq \Big(\sum_{i=1}^{n} a_i^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{n} b_i^2\Big)^{\frac{1}{2}}.$$

Theorem 3. *Let $A, B, X \in M_n$ such that A and B are positive semidefinite.*
If $0 \leq \mu \leq \frac{1}{2}$,then

$$\begin{aligned} tr(\mu^2 A^2 + (1-\mu)^2 B^2) \leq \mu^{2\mu} \|A^\mu\|_2 \|B^{1-\mu}\|_2 + \mu^2 \Big[\|A\|_2^2 + \|B\|_2^2 - 2\|AB\|_1\Big] \\ + R_0 \Big[\mu \|A\|_2 \|B\|_2 + \|B\|_2^2 - 2\sqrt{\mu} \|A^{\frac{1}{2}} B^{\frac{3}{2}}\|_1\Big]. \end{aligned} \tag{2.5}$$

where $R_0 = max\{2\mu,\ 1 - 2\mu\}$.
If $\frac{1}{2} \leq \mu \leq 1$, then

$$\begin{aligned} tr(\mu^2 A^2 + (1-\mu)^2 B^2) \leq (1-\mu)^{2\mu} \|A^\mu\|_2 \|B^{1-\mu}\|_2 \\ + (1-\mu)^2 \Big[\|A\|_2^2 + \|B\|_2^2 - 2\|AB\|_1\Big] \\ + R_0 \Big[(1-\mu)\|A\|_2 \|B\|_2 + \|A\|_2^2 - 2\sqrt{1-\mu} \|A^{\frac{3}{2}} B^{\frac{1}{2}}\|_1\Big]. \end{aligned} \tag{2.6}$$

where $R_0 = max\{2\mu - 1,\ 2 - 2\mu\}$.

Proof. If $0 \leq \mu \leq \frac{1}{2}$, then

$$\begin{aligned} tr(\mu^2 A^2 + (1-\mu)^2 B^2) &= \mu^2 tr A^2 + (1-\mu)^2 tr B^2 \\ &= \sum_{j=1}^{n} \Big(\mu^2 s_j^2(A) + (1-\mu)^2 s_j^2(B)\Big). \end{aligned}$$

Inequality (2.1) yields that

$$\begin{aligned} tr(\mu^2 A^2 + (1-\mu)^2 B^2) \leq \mu^{2\mu} \sum_{j=1}^{n} s_j(A^{2\mu}) s_j(B^{2(1-\mu)}) \\ + \mu^2 \Big[\sum_{j=1}^{n} s_j^2(A) + \sum_{j=1}^{n} s_j^2(B) - 2\sum_{j=1}^{n} s_j(A)s_j(B)\Big] \\ + R_0 \Big[\mu \sum_{j=1}^{n} s_j(A)s_j(B) + \sum_{j=1}^{n} s_j^2(B) - 2\sqrt{\mu} \sum_{j=1}^{n} s_j^{\frac{1}{2}}(A) s_j^{\frac{3}{2}}(B)\Big]. \end{aligned}$$

Thus, using Lemma 1 and Lemma 2, we have

$$\begin{aligned}tr(\mu^2A^2+(1-\mu)^2B^2) \leq\ & \mu^{2\mu}\Big(\sum_{j=1}^{n} s_j^2(A^{2\mu})\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{n} s_j^2(B^{2(1-\mu)})\Big)^{\frac{1}{2}}\\ &+\mu^2\Big[\sum_{j=1}^{n} s_j^2(A)+\sum_{j=1}^{n} s_j^2(B)-2\sum_{j=1}^{n} s_j(AB)\Big]\\ &+R_0\Big[\mu\Big(\sum_{j=1}^{n} s_j^2(A)\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{n} s_j^2(B)\Big)^{\frac{1}{2}}+\sum_{j=1}^{n} s_j^2(B)\\ &-2\sqrt{\mu}\sum_{j=1}^{n} s_j(A^{\frac{1}{2}}B^{\frac{3}{2}})\Big].\end{aligned}$$

and so,

$$\begin{aligned}tr(\mu^2A^2+(1-\mu)^2B^2) \leq\ & \mu^{2\mu}\|A^{\mu}\|_2\|B^{1-\mu}\|_2+\mu^2\Big[\|A\|_2^2+\|B\|_2^2-2\|AB\|_1\Big]\\ &+R_0\Big[\mu\|A\|_2\|B\|_2+\|B\|_2^2-2\sqrt{\mu}\|A^{\frac{1}{2}}B^{\frac{3}{2}}\|_1\Big].\end{aligned}$$

Thus, we get (2.5).
If $\frac{1}{2} \leq \mu \leq 1$, then by the inequality (2.2) and the same method above, we get the inequality (2.6).
This completes the proof.

Acknowledgements

The authors are grateful to the referees for their valuable comments and suggestions. This research is funded by Zarqa University- Jordan.

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