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Characterization of Three-dimensional Jacobi-Poisson Manifolds

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Abstract. After a reminder of essential notions concerning Jacobi and Poisson manifolds, we study the relationships between their structures. We also show that for any three-dimensional Jacobi manifold, we can construct a Poisson structure on the same manifold. The paper concludes with some examples of such manifolds.

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1. Introduction

Jacobi and Poisson manifolds are nowadays a research subject in expansion in differential geometry. The Jacobi manifolds are introduced by A. Lichnerowicz [1] as a generalisation of both symplectic, Poisson and contact manifold. A Jacobi bracket is just a Lie bracket on the algebra of smooth functions given by bilinear first order differential operator.

The Jacobi and Poisson manifolds formalize the Hamitonnian geometry and are used to quantify physical systems. This motivates severals studies of such manifolds. In this context M. Boucetta studies the compatibility between Poisson and pseudo-Riemannian structures [2–4]. This allows better characterization of symplectic leaves [5, 6]. In the same way, Y. A. Amrane and A. Zeglaoui study the compatibility between Riemannian structures and Jacobi structures [7].

To better understand the geometry behind the Jacobi structure, some researchers try to look at this structure as a generalization of a Poisson structure. So they rewrite notions

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on Jacobi manifolds so that they generalize the same notions on Poisson manifolds. Thus, one can ask about the possibility of having a Jacobi structure and a Poisson structure on the same manifold, and the relationship that can exist between the two structures. The first answer is given in [8, 9]. They show that from any n-dimensional Jacobi manifold one can construct a natural Poisson structure on the cone $M \times \mathbb{R}_+^*$.

In this paper we try to give an answer to the above question. We study a special case of a three-dimensional Jacobi manifold. We show that for any three-dimensional Jacobi manifold we can construct a Poisson structure on the same manifold. We give some examples of such manifolds.

The organisation of the paper is as follows: in section 2 we begin with a brief review of the Jacobi structure and its correspondence with Poisson structures. Section 3 is devoted to the compatibility between Riemannian and Jacobi structures on a manifold as a generalisation of the compatibility between Riemannian and Poisson structures. In section 4 we give the connection between Jacobi and Poisson manifolds in $\mathbb{R}^3$. In addition, we give the partial compatibility with the Riemannian metric g and the Jacobi structure (π, E); we also prove that if (M, π, E) is a Jacobi manifold, then (M, P) is a Poisson manifold under certain conditions. And some examples of 3-dimensional Jacobi-Poisson manifolds are given.

2. Jacobi manifolds and Poisson manifolds

Let (M, g, π) be an n-dimensional manifold with a pseudo-Riemannian metric g and a bivector field π. Let E be a vector field on M. The pair (π, E) defines a Jacobi structure on M if we have the following relations

$$[\pi, \pi] = 2E \wedge \pi \qquad \text{and} \qquad [E, \pi] := \mathcal{L}_E \pi = 0, \tag{1}$$

where $[.,.]$ is the Schouten-Nijenhuis bracket, see [7].

In local coordinates $(x_1, \ldots, x_n)$ the tensor π is determined by the matrix $\pi_{ij}(x) = \{x_i, x_j\}$. The rank of this matrix is called the rank of π at x.

Note that a be vector π is a Poisson tensor ((M, π) is a Poisson manifold) if $[\pi, \pi] = 0$. A Poisson structure is called regular if the rank of π is constant on M. If this matrix is invertible a each x, then π is called non-degenerate or symplectic. We call (M, π, E) a Jacobi manifold.

If $E = 0$, then $[\pi, \pi] = 0$, which correspond for Poisson structure (M, π). Thus the Jacobi manifold generalised at once the Poisson manifolds, the contact manifolds and the locally conformal symplectic manifolds. Furthermore, if g is a Hermitian metric on M, then (M, g, π, E, λ) is called a pseudo-Riemannian Jacobi manifold, with λ a contact form [7].

A correspondence between Jacobi and Poisson structures is given by the following fact:

Fact . (π, E) is a Jacobi structure on M if and only if $P = e^{-t}(\pi + \frac{\partial}{\partial t} \wedge E)$ is a Poisson structure on the cone $\overline{M} = M \times \mathbb{R}_+^{*}$ † (more generally on $M \times \mathbb{R}$, and the Poisson structure

† The non-degeneracy of the canonical metric motivates the choice of the cone $M \times \mathbb{R}_+^*$.

is homogeneous, see [10, 11]).

Let $\pi = \sum_{1\le i<j\le 4} \pi^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$ be a bivector field on M. Then by definition of the Schouten-Nijenhuis bracket, one has locally

$$\begin{aligned}&\big[\pi^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j},\pi^{sk}\frac{\partial}{\partial x^s}\wedge\frac{\partial}{\partial x^k}\big] =\\ &\quad \pi^{ij}\frac{\partial \pi^{sk}}{\partial x^i}\frac{\partial}{\partial x^s}\wedge\frac{\partial}{\partial x^k}\wedge\frac{\partial}{\partial x^j} - \pi^{sk}\frac{\partial \pi^{ij}}{\partial x^s}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^k}\wedge\frac{\partial}{\partial x^j}\\ &\qquad +\pi^{sk}\frac{\partial \pi^{ij}}{\partial x^k}\frac{\partial}{\partial x^s}\wedge\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j} - \pi^{ij}\frac{\partial \pi^{sk}}{\partial x^j}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^s}\wedge\frac{\partial}{\partial x^k}. \qquad (2)\end{aligned}$$

Theorem 1. *Let (M,π,E) be a Jacobi manifold. If the pair (π,E) is a Jacobi structure on M then $P = e^{-t}(\pi+\partial_t\wedge E)$ is a Poisson structure on the cone $\overline{M} = M\times\mathbb{R}^*_+$.*

Proof. It's clear that (π,E) satisfies (1). Now it suffices to prove that $[P,P]=0$, where $P = e^{-t}\pi + e^{-t}\partial_t\wedge E$.
By (2), we get

$$\begin{aligned}[P,P] &= [e^{-t}\pi + e^{-t}\partial_t\wedge E, e^{-t}\pi + e^{-t}\partial_t\wedge E]\\ &= [e^{-t}\pi, e^{-t}\pi] + [e^{-t}\pi, e^{-t}\partial_t\wedge E] + [e^{-t}\partial_t\wedge E, e^{-t}\pi] + [e^{-t}\partial_t\wedge E, e^{-t}\partial_t\wedge E]\\ &= e^{-2t}[\pi,\pi] + 2[e^{-t}\partial_t\wedge E, e^{-t}\pi]\\ &= e^{-2t}[\pi,\pi] + 2e^{-t}\partial_t[E, e^{-t}\pi] + 2[e^{-t}\partial_t, e^{-t}\pi]\wedge E.\end{aligned}$$

Since

$$[e^{-t}\pi, e^{-t}\partial_t\wedge E] = [e^{-t}\partial_t\wedge E, e^{-t}\pi] \ \text{ and } \ [e^{-t}\partial_t\wedge E, e^{-t}\partial_t\wedge E] = 0,$$

one has

$$\begin{aligned}[P,P] &= e^{-2t}[\pi,\pi] + 2e^{-2t}\partial_t\wedge[E,\pi] - 2e^{-2t}\pi\wedge E\\ &= e^{-2t}\Big([\pi,\pi] - 2\pi\wedge E\Big) + 2e^{-2t}\partial_t[E,\pi].\end{aligned}$$

So by equation (1);

$$[P,P]=0 \iff [\pi,\pi]-2\pi\wedge E = 0 \ \text{ and } \ [E,\pi]=0.$$

Let's denote by $\sharp_\pi : T^*M \to TM$ the anchor map given by $\beta(\sharp_\pi(\alpha)) = \pi(\alpha,\beta)$, and by $[,]_\pi$ the Koszul bracket given by

$$[\alpha,\beta]_\pi = \mathcal{L}_{\sharp_\pi(\alpha)}\beta - \mathcal{L}_{\sharp_\pi(\beta)}\alpha - d(\pi(\alpha,\beta)), \qquad \alpha,\beta\in\Omega^1(M),$$

and its inverse $\flat_g : TM \to T^*M$ taking $X \mapsto X^{\flat_g} := g(X,\cdot)$, that for $X, Y \in TM$ and $\alpha,\beta\in\Omega^1(M)$ satisfies

$$g(X^{\flat_g}, Y^{\flat_g}) := g(X,Y) = X^{\flat_g}(Y) \text{ and } g^*(\alpha,\beta) := g(\sharp_\pi(\alpha),\sharp_\pi(\beta)) = \alpha(\sharp_\pi(\beta)),$$

where g^* has an associated dual metric g.
Let's consider the map bundle (see e.g. [7]) $\sharp_{\pi,E} : T^*M \to TM$, by

$$\sharp_{\pi,E}(\alpha) = \sharp_\pi(\alpha) + \alpha(E)E,$$

and, for $\lambda \in \Omega^1(M)$ a 1-form, the map $[\cdot,\cdot]^\lambda_{\pi,E} : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$ defined by

$$[\alpha,\beta]^\lambda_{\pi,E} := [\alpha,\beta]_\pi + \alpha(E)\Big(\mathcal{L}_E\beta - \beta\Big) - \beta(E)\Big(\mathcal{L}_E\alpha - \alpha\Big) - \pi(\alpha,\beta)\lambda,$$

where $\Omega^1(M)$ denotes the space of differential 1-forms on M.
Recall that for the differential forms $\alpha, \beta, \gamma \in \Omega^1(M)$ we have

$$\gamma\Big(\sharp_\pi([\alpha,\beta]_\pi) - [\sharp_\pi(\alpha), \sharp_\pi(\beta)]\Big) = \frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma),$$

and for any functions $f_1, f_2, f_3 \in \mathcal{C}^\infty(M)$ one has

$$\begin{aligned}&[df_1,[df_2,df_3]_\pi]_\pi + [df_2,[df_3,df_1]_\pi]_\pi + [df_3,[df_1,df_2]_\pi]_\pi \\ &= -\frac{1}{2}d\Big([\pi,\pi])(df_1,df_2,df_3)\Big).\end{aligned}$$

Proposition 1. *Let* $\alpha \in \Omega^1(M)$*. Then*

$$\mathcal{L}_E(\sharp_{\pi,E}\alpha) = \sharp_{\pi,E}(\mathcal{L}_E\alpha).$$

Proof. By definition of $\sharp_{\pi,E}$, we have

$$\mathcal{L}_E(\sharp_{\pi,E}\alpha) = \mathcal{L}_E(\sharp_\pi\alpha + \alpha(E)E) = \mathcal{L}_E(\sharp_\pi\alpha) + \mathcal{L}_E(\alpha(E)E).$$

Likewise

$$\sharp_{\pi,E}(\mathcal{L}_E\alpha) = \sharp_\pi(\mathcal{L}_E\alpha) + (\mathcal{L}_E\alpha)(E)E.$$

Then a straightforward calculation yields, for any $\beta \in \Omega^1(M)$,

$$\begin{aligned}
&\beta\Big(\mathcal{L}_E(\sharp_{\pi,E}\alpha) - \sharp_{\pi,E}(\mathcal{L}_E\alpha)\Big) \\
=\ &\beta\Big(\mathcal{L}_E(\sharp_\pi\alpha) + \mathcal{L}_E(\alpha(E)E) - \sharp_\pi(\mathcal{L}_E\alpha) - (\mathcal{L}_E\alpha(E))E\Big) \\
=\ &\beta\Big(\mathcal{L}_E(\sharp_\pi\alpha) - \sharp_\pi(\mathcal{L}_E\alpha) + [\mathcal{L}_E(\alpha(E)E) - (\mathcal{L}_E\alpha)(E)E]\Big) \\
=\ &\Big(\mathcal{L}_E(\beta(\sharp_\pi\alpha)) - (\mathcal{L}_E\beta)(\sharp_\pi\alpha)\Big) - \pi(\mathcal{L}_E\alpha,\beta) \\
=\ &\mathcal{L}_E\pi(\alpha,\beta) - (\mathcal{L}_E\beta)\Big(\sharp_\pi(\alpha)\Big) - \pi(\mathcal{L}_E\alpha,\beta) \\
=\ &-(\mathcal{L}_E\beta)\Big(\sharp_\pi(\alpha)\Big) - \pi(\mathcal{L}_E\alpha,\beta) \\
=\ &-\pi(\alpha,\mathcal{L}_E\beta) - \pi(\mathcal{L}_E\alpha,\beta) \\
=\ &(\mathcal{L}_E\pi)(\alpha,\beta) \\
=\ &0.
\end{aligned}$$

3. Riemann-Poisson and Riemann-Jacobi manifolds

Let D be the Levi-Civita contravariant connection associated with (P, g^*). (M, g^*, P) is said to be a pseudo-Riemannian-Poisson manifold, if

$$D_\alpha P(\beta,\gamma) := \sharp_P(\alpha)\cdot P(\beta,\gamma) - P(D_\alpha\beta,\gamma) - P(\beta, D_\alpha\gamma) = 0,$$

where $\alpha,\beta,\gamma \in \Omega^1(M)$. D is characterized by

$$\begin{aligned} 2g^*(D_\alpha\beta,\gamma) &= \sharp_P(\alpha)g^*(\beta,\gamma) + \sharp_P(\beta)g^*(\alpha,\gamma) - \sharp_P(\gamma)g^*(\alpha,\beta) \\ &\quad + g^*([\alpha,\beta]_P,\gamma) + g^*([\gamma,\alpha],\beta) + g^*([\gamma,\beta]_P,\alpha). \end{aligned} \tag{3}$$

Moreover, once we introduce the bundle map $J^* : T^*M \to T^*M$, which behaves like a contravariant (almost) complex structure [12], such that $D_\alpha J^*\beta = J^* D_\alpha\beta$, for all $\alpha,\beta,\gamma \in \Omega^1(M)$ the space of 1-forms on M. We denote by $\flat_g : TM \to T^*M$, bundle maps called musical isomorphisms such that $\flat_g(X)(Y) = g(X,Y)$ and $\sharp_g$ its inverse, by g^* the dual of g, such that $g^*(\alpha,\beta) = g(\sharp_g(\alpha),\sharp_g(\beta))$ and such that

$$g(J\sharp_g(\alpha),\sharp_g(\beta)) = \pi(\alpha,\beta), \text{ and } g^*(\alpha, J^*\beta) = \pi(\alpha,\beta), \tag{4}$$

so $J^* = \flat_g \circ J \circ \sharp_g$.

Definition 1. *A triple (π, g, J) of structures on a tangent space TM is called compatible if for each $\alpha,\beta \in \Omega^1(M)$ the relation $\pi(\alpha,\beta) = g^*(\alpha, J^*\beta)$ is satisfied.*

To the triple (π, E, g) we associate the differential 1-form λ defined by $\lambda = g(E,E)\flat_g(E) - \flat_g(JE)$.

We call the contravariant Levi-Civita derivative associated with the triplet (π, E, g) the unique derivative D, symmetric with respect to the bracket $[\cdot,\cdot]_{\pi,E}$ and compatible with the metric g^*. It is characterised by the formula

$$\begin{aligned} 2g^*(D_\alpha\beta,\gamma) &= \sharp_{\pi,E}(\alpha)g^*(\beta,\gamma) + \sharp_{\pi,E}(\beta)g^*(\alpha,\gamma) - \sharp_{\pi,E}(\gamma)g^*(\alpha,\beta) \\ &\quad + g^*([\alpha,\beta]_{\pi,E},\gamma) + g^*([\gamma,\alpha]_{\pi,E},\beta) + g^*([\gamma,\beta]_{\pi,E},\alpha) \end{aligned}$$

Proposition 2. *If (π, E) is a Jacobi structure on M, then*

$$\sharp_{\pi,E}(D_\alpha J^*\beta) = J\sharp_{\pi,E}(D_\alpha\beta)$$

for all $\alpha,\beta \in \Omega^1(M)$.

Proof. By [7, Proposition 3.1 and 2.6], we have $\sharp_{\pi,E}(D_\alpha\beta) = \nabla_{\sharp_{\pi,E}(\alpha)}\sharp_{\pi,E}(\beta)$, where ∇ is the Levi-Civita connection (covariant) associated with g. Since any Hermitian manifold is a Kähler manifold if $\nabla J = 0$ which is true (see [13]). Then (M, g, π) is a Kähler-Poisson manifold and consequently, we have $\nabla_X JY = J\nabla_X Y$, for all $X, Y \in TM$. If we set $X = \sharp_{\pi,E}(\alpha)$ and $Y = \sharp_{\pi,E}(\beta)$, then

$$\sharp_{\pi,E}(D_\alpha J^*\beta) = \nabla_{\sharp_{\pi,E}(\alpha)} J\sharp_{\pi,E}(\beta) = J\nabla_{\sharp_{\pi,E}(\alpha)}\sharp_{\pi,E}(\beta) = J\sharp_{\pi,E}(D_\alpha\beta).$$

Let $\omega \in \Omega^2(M)$ be a non-degenerate 2-form and $\theta \in \Omega^1(M)$. Suppose that the pair (π, E) is associated with the pair (ω, θ).

Definition 2. *We say that the Riemannian metric g is associated with the pair (ω, θ) if $\sharp_{\omega,\theta} := \sharp_{\pi,E}$ is an isometry, i.e. if*

$$g\Big(\sharp_{\omega,\theta}(\alpha), \sharp_{\omega,\theta}(\beta)\Big) = g^*(\alpha, \beta),$$

for all $\alpha, \beta \in \Omega^1(M)$.

If J and J^* are the endomorphism fields defined by the formulae (4), then

$$\begin{aligned} g\Big(\sharp_{\omega,\theta}(\alpha), \sharp_{\omega,\theta}(\beta)\Big) &= g\Big(\sharp_{\omega}(\alpha), \sharp_{\omega}(\beta)\Big) \\ &= g^*\Big(\flat_g(\sharp_\omega(\alpha)), \flat_g(\sharp_\omega(\beta)\Big) \\ &= g^*(J^*\alpha, J^*\beta), \end{aligned}$$

for all $\alpha, \beta \in \Omega^1(M)$.

Theorem 2. *An almost Jacobi manifold (M, π, g^*, J^*) with non-degenerate Poisson bivector field π is a Jacobi manifold if and only if (M, ω, g, J) is a symplectic manifold, where $\omega := \pi^{-1}$.*

Proof. We know that if g is positive definite, the pair (ω, g) is an almost Hermitian structure on M and that J is the associated almost complex structure, i.e. we have

$$g(JX, JY) = g(X, Y) \text{ and } \omega(X, Y) = g(X, JY),$$

for all $X, Y \in TM$.
From the equivalence $J^* = \flat_g \circ J \circ \sharp_g \iff \sharp_g \circ J^* \circ \flat_g = J$, we get $J^* \circ \flat_g = \flat_g \circ J$ and $\omega^\flat \circ \pi^\sharp = \mathrm{Id}$, and therefore (π, g^*, J^*) is Jacobian if and only if (ω, g, J) is symplectic. The conclusion follows from [7, Corollary 2.2], to see that

$$D\pi(\alpha, \beta, \gamma) = \nabla\omega(\sharp_{\omega,\theta}(\alpha), \sharp_{\omega,\theta}(\beta), \sharp_{\omega,\theta}(\gamma)).$$

.

Theorem 3. *Assume that (π, E) is a Jacobi structure and $\alpha, \beta, \gamma \in \Omega^1(M)$. We have the following property*

$$(\mathcal{L}_{\sharp_{\pi,E}(\alpha)}\pi)(\beta, \gamma) = \mathcal{L}_{\sharp_\pi(\alpha)}\pi(\beta, \gamma) - \Big\{\gamma(E)\pi\Big(\beta, d\alpha(E)\Big) - \beta(E)\pi\Big(\gamma, d\alpha(E)\Big)\Big\}.$$

Proof. Since $\sharp_{\pi,E}(\alpha) = \sharp_\pi(\alpha) + \alpha(E)E$, we have

$$\begin{aligned} (\mathcal{L}_{\sharp_{\pi,E}(\alpha)}\pi)(\beta, \gamma) &= \sharp_\pi(\alpha)\pi(\beta, \gamma) + \alpha(E)E \cdot \pi(\beta, \gamma) \\ &\quad -\pi(\beta, \mathcal{L}_{\sharp_\pi(\alpha)+\alpha(E)E}\gamma) - \pi(\mathcal{L}_{\sharp_\pi(\alpha)+\alpha(E)E}\beta, \gamma). \end{aligned} \tag{5}$$

Moreover, we know that, for all vector fields X and Y on TM, $\alpha \in \Omega^1(M)$ and f a smooth function on M, we have

$$\mathcal{L}_{X+fY}\alpha = \mathcal{L}_X\alpha + f\mathcal{L}_Y\alpha + \alpha(Y)df.$$

So by setting $X = \sharp_\pi(\alpha)$, $Y = E$ and $f = \alpha(E)$, we have

$$\begin{aligned}
\pi(\beta, \mathcal{L}_{\sharp_\pi(\alpha)+\alpha(E)E}\gamma) &= \pi(\beta, \mathcal{L}_{\sharp_\pi(\alpha)}\gamma) + \pi(\beta, \alpha(E)\mathcal{L}_E\gamma + \gamma(E)d(\alpha(E))) \\
&= \pi(\beta, \mathcal{L}_{\sharp_\pi(\alpha)}\gamma) + \pi(\beta, \alpha(E)\mathcal{L}_E\gamma) + \pi(\beta, \gamma(E)d(\alpha(E))) \\
\pi(\mathcal{L}_{\sharp_\pi(\alpha)+\alpha(E)E}\beta, \gamma) &= \pi(\mathcal{L}_{\sharp_\pi(\alpha)}\beta, \gamma) + \pi(\alpha(E)\mathcal{L}_E\beta + \beta(E)d(\alpha(E)), \gamma) \\
&= \pi(\mathcal{L}_{\sharp_\pi(\alpha)}\beta, \gamma) + \pi(\alpha(E)\mathcal{L}_E\beta, \gamma) + \pi(\beta(E)d(\alpha(E)), \gamma).
\end{aligned}$$

Then (5), become

$$\begin{aligned}
(\mathcal{L}_{\sharp_{\pi,E}(\alpha)}\pi)(\beta,\gamma) &= \sharp_\pi(\alpha)\pi(\beta,\gamma) + \alpha(E)E\cdot\pi(\beta,\gamma) \\
&\quad -\Big(\pi(\beta, \mathcal{L}_{\sharp_\pi(\alpha)}\gamma) + \pi(\beta, \alpha(E)\mathcal{L}_E\gamma) + \pi(\beta, \gamma(E)d(\alpha(E)))\Big) \\
&\quad -\Big(\pi(\mathcal{L}_{\sharp_\pi(\alpha)}\beta, \gamma) + \pi(\alpha(E)\mathcal{L}_E\beta, \gamma) + \pi(\beta(E)d(\alpha(E)), \gamma)\Big) \\
&= \sharp_\pi(\alpha)\pi(\beta,\gamma) - \pi(\beta, \mathcal{L}_{\sharp_\pi(\alpha)}\gamma) - \pi(\mathcal{L}_{\sharp_\pi(\alpha)}\beta, \gamma) \\
&\quad +\alpha(E)\Big(E\cdot\pi(\beta,\gamma) - \pi(\beta, \mathcal{L}_E\gamma) - \pi(\mathcal{L}_E\beta,\gamma)\Big) \\
&\quad -\Big(\pi(\beta, \gamma(E)d\alpha(E)) + \pi(\beta(E)d\alpha(E), \gamma)\Big) \\
&= (\mathcal{L}_{\sharp_\pi(\alpha)}\pi)(\beta,\gamma) + \alpha(E)\mathcal{L}_E\pi(\beta,\gamma) \\
&\qquad -\Big(\pi(\beta, \gamma(E)d\alpha(E)) + \pi(\beta(E)d\alpha(E), \gamma)\Big) \\
&= (\mathcal{L}_{\sharp_\pi(\alpha)}\pi)(\beta,\gamma) - \Big\{\gamma(E)\pi\Big(\beta, d\alpha(E)\Big) - \beta(E)\pi\Big(\gamma, d\alpha(E)\Big)\Big\}.
\end{aligned}$$

We can conclude by [13, section 3, definition and properties, iii)], since $\mathcal{L}_E\pi = 0$.Gives us a necessary condition so that $\left(\mathbb{R}^3, g^*, P\right)$ is a Riemann-Poisson manifold using divergence, where g^* is the canonical metric.

Remark 1. *If $\sharp_\pi(\alpha) = E$, then we have a Hermitian-Poisson manifold, since $\mathcal{L}_E\pi = 0$.*

The characterization of finite dimensional Riemann-Poisson manifolds started with [4]. The following theorem gives another method for the characterization obtained in [4].

Theorem 4. *If the bivector field P is compatible with g^* then there exists a differential function μ on $\mathbb{R}^3$ such that*

$$P = -\left(\frac{\partial\mu}{\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + \left(\frac{\partial\mu}{\partial y}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} - \left(\frac{\partial\mu}{\partial x}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$$

Proof. According to the work of M. Boucetta [4], the compatibility between P and g^* implies that the application being $C^\infty\left(\mathbb{R}^3\right)$-bilinear

$$(\alpha,\beta) \longmapsto \mathcal{L}_{H_h}g^*(\alpha,\beta) = g^*\left(\mathcal{D}^P_\alpha dh, \beta\right) + g^*\left(\alpha, \mathcal{D}^P_\beta dh\right)$$

which is anti-symmetric for any differentiable function h on $\mathbb{R}^3$, where H_h is the Hamiltonian field of h, consequently the vanishing of modular vector field given by

$$\phi(h) := g^*\left(\mathcal{D}^P_{dx}dh, dx\right) + g^*\left(\mathcal{D}^P_{dy}dh, dy\right) + g^*\left(\mathcal{D}^P_{dz}dh, dz\right) \quad \forall \;\; h \in C^\infty\left(\mathbb{R}^3\right).$$

We calculate $\phi(h)$, as

$$\begin{aligned} g^*\left(\mathcal{D}^P_{dx}dh, dx\right) &= \frac{\partial h}{\partial x} g^*\left(\mathcal{D}^P_{dx}dx, dx\right) + \frac{\partial h}{\partial y} g^*\left(\mathcal{D}^P_{dx}dy, dx\right) + \frac{\partial h}{\partial z} g^*\left(\mathcal{D}^P_{dx}dz, dx\right) + \sharp_P(dx) \cdot \left(\frac{\partial h}{\partial x}\right) \\ &= \frac{\partial h}{\partial x} \cdot \Gamma^{11}_1 + \frac{\partial h}{\partial y} \cdot \Gamma^{12}_1 + \frac{\partial h}{\partial z} \cdot \Gamma^{13}_1 + \left(\frac{\partial P_{12}}{\partial y} + \frac{\partial P_{13}}{\partial z}\right)\left(\frac{\partial h}{\partial x}\right) \\ &= \frac{\partial h}{\partial y} \cdot \frac{\partial P_{12}}{\partial x} + \frac{\partial h}{\partial z} \cdot \frac{\partial P_{13}}{\partial x} + P_{12}\frac{\partial^2 h}{\partial y \partial x} + P_{13}\frac{\partial^2 h}{\partial z \partial x}, \end{aligned}$$

So

$$\begin{aligned} g^*\left(\mathcal{D}^P_{dy}dh, dy\right) &= \frac{\partial h}{\partial x} \cdot \Gamma^{21}_2 + \frac{\partial h}{\partial y} \cdot \Gamma^{22}_2 + \frac{\partial h}{\partial z} \cdot \Gamma^{23}_2 + \left(-\frac{\partial P_{12}}{\partial x} + \frac{\partial P_{23}}{\partial z}\right)\left(\frac{\partial h}{\partial y}\right) \\ &= -\frac{\partial h}{\partial x} \cdot \frac{\partial P_{12}}{\partial y} + \frac{\partial h}{\partial z} \cdot \frac{\partial P_{23}}{\partial y} - P_{12}\frac{\partial^2 h}{\partial x \partial y} + P_{23}\frac{\partial^2 h}{\partial z \partial y} \end{aligned}$$

and

$$\begin{aligned} g^*\left(\mathcal{D}^P_{dz}dh, dz\right) &= \frac{\partial h}{\partial x} \cdot \Gamma^{31}_3 + \frac{\partial h}{\partial y} \cdot \Gamma^{32}_3 + \frac{\partial h}{\partial z} \cdot \Gamma^{33}_3 + \left(-\frac{\partial P_{13}}{\partial x} - \frac{\partial P_{23}}{\partial y}\right)\left(\frac{\partial h}{\partial z}\right) \\ &= -\frac{\partial h}{\partial x} \cdot \frac{\partial P_{13}}{\partial z} - \frac{\partial h}{\partial y} \cdot \frac{\partial P_{23}}{\partial z} - P_{13}\frac{\partial^2 h}{\partial x \partial z} - P_{23}\frac{\partial^2 h}{\partial y \partial z}. \end{aligned}$$

Now, let us compute the contravariant connection D. We will use the Christoffel symbols Γ^{ij}_k. For example, $D_{dx}dx = \Gamma^{11}_1 dx + \Gamma^{11}_2 dy + \Gamma^{11}_3 dz$. From (3), one has

$$\begin{aligned} \Gamma^{11}_1 &= 0, \qquad \Gamma^{11}_2 = -\frac{\partial^2 \mu}{\partial x \partial z}, \qquad \Gamma^{11}_3 = \frac{\partial^2 \mu}{\partial x \partial y}, \\ \Gamma^{12}_1 &= \frac{\partial^2 \mu}{\partial x \partial z}, \qquad \Gamma^{12}_2 = 0, \qquad \Gamma^{12}_3 = \frac{1}{2}\Big(-\frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} + \frac{\partial^2 \mu}{\partial z^2}\Big) \\ \Gamma^{21}_1 &= 0, \qquad \Gamma^{21}_2 = -\frac{\partial^2 \mu}{\partial y \partial z}, \qquad \Gamma^{21}_3 = \frac{1}{2}\Big(-\frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} - \frac{\partial^2 \mu}{\partial z^2}\Big) \\ \Gamma^{13}_1 &= -\frac{\partial^2 \mu}{\partial x \partial y}, \qquad \Gamma^{13}_2 = \frac{1}{2}\Big(\frac{\partial^2 \mu}{\partial x^2} - \frac{\partial^2 \mu}{\partial y^2} - \frac{\partial^2 \mu}{\partial z^2}\Big), \qquad \Gamma^{13}_3 = 0 \\ \Gamma^{31}_1 &= 0, \qquad \Gamma^{31}_2 = \frac{1}{2}\Big(\frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} - \frac{\partial^2 \mu}{\partial z^2}\Big), \qquad \Gamma^{31}_3 = \frac{\partial^2 \mu}{\partial y \partial z}, \\ \Gamma^{22}_1 &= \frac{\partial^2 \mu}{\partial y \partial z}, \qquad \Gamma^{22}_2 = 0, \qquad \Gamma^{22}_3 = -\frac{\partial^2 \mu}{\partial x \partial y}, \\ \Gamma^{23}_1 &= \frac{1}{2}\Big(\frac{\partial^2 \mu}{\partial x^2} - \frac{\partial^2 \mu}{\partial y^2} + \frac{\partial^2 \mu}{\partial z^2}\Big), \qquad \Gamma^{23}_2 = \frac{\partial^2 \mu}{\partial x \partial y}, \qquad \Gamma^{23}_3 = 0, \end{aligned}$$

$$\begin{aligned}\Gamma_1^{32} &= \frac{1}{2}\Big(-\frac{\partial^2\mu}{\partial x^2}-\frac{\partial^2\mu}{\partial y^2}+\frac{\partial^2\mu}{\partial z^2}\Big), \qquad \Gamma_2^{32}=0, \qquad \Gamma_3^{32}=-\frac{\partial^2\mu}{\partial x\partial z},\\ \Gamma_1^{33} &= -\frac{\partial^2\mu}{\partial y\partial z}, \qquad \Gamma_2^{33}=\frac{\partial^2\mu}{\partial x\partial z}, \qquad \Gamma_3^{33}=0.\end{aligned}$$

Therefore

$$\begin{aligned}\phi(h) &= \frac{\partial h}{\partial y}\cdot\frac{\partial P_{12}}{\partial x}+\frac{\partial h}{\partial z}\cdot\frac{\partial P_{13}}{\partial x}-\frac{\partial h}{\partial x}\cdot\frac{\partial P_{12}}{\partial y}+\frac{\partial h}{\partial z}\cdot\frac{\partial P_{23}}{\partial y}-\frac{\partial h}{\partial x}\cdot\frac{\partial P_{13}}{\partial z}\\ &= \left(-\frac{\partial P_{13}}{\partial z}-\frac{\partial P_{12}}{\partial y}\right)\frac{\partial h}{\partial x}+\left(\frac{\partial P_{12}}{\partial x}-\frac{\partial P_{23}}{\partial z}\right)\frac{\partial h}{\partial y}+\left(\frac{\partial P_{23}}{\partial y}+\frac{\partial P_{13}}{\partial x}\right)\frac{\partial h}{\partial z}\\ &= \left\{\left(-\frac{\partial P_{13}}{\partial z}-\frac{\partial P_{12}}{\partial y}\right)\frac{\partial}{\partial x}+\left(\frac{\partial P_{12}}{\partial x}-\frac{\partial P_{23}}{\partial z}\right)\frac{\partial}{\partial y}+\left(\frac{\partial P_{23}}{\partial y}+\frac{\partial P_{13}}{\partial x}\right)\frac{\partial}{\partial z}\right\}\cdot h;\end{aligned}$$

since $\left\{\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\}$ is a basis of $C^\infty\left(\mathbb{R}^3\right)$-free module of $\mathcal{A}\left(\mathbb{R}^3\right)$ then, $\phi(h)=0\ \forall\ h\in C^\infty\left(\mathbb{R}^3\right)$ is equivalent to the system of equations

$$\begin{cases} -\frac{\partial P_{13}}{\partial z}-\frac{\partial P_{12}}{\partial y} = 0\\ \frac{\partial P_{12}}{\partial x}-\frac{\partial P_{23}}{\partial z} = 0\\ \frac{\partial P_{23}}{\partial y}+\frac{\partial P_{13}}{\partial x} = 0,\end{cases}$$

which is equivalent to:

$$\frac{\partial}{\partial x}\left(P_{12}+P_{13}\right)+\frac{\partial}{\partial y}\left(-P_{12}+P_{23}\right)+\frac{\partial}{\partial z}\left(-P_{13}-P_{23}\right)=0.$$

By setting

$$a=-P_{13}-P_{23}, \quad b=-P_{12}+P_{23}, \quad c=P_{12}+P_{13},$$

one has

$$\frac{\partial c}{\partial x}+\frac{\partial b}{\partial y}+\frac{\partial a}{\partial z}=0 \quad \Leftrightarrow div\cdot v=0 \ \text{ where } \ v=(c,b,a).$$

We can say that the 2-form $\alpha=adx\wedge dy+bdx\wedge dz+cdy\wedge dz$ on $\mathbb{R}^3$ is closed. Since $\mathbb{R}^3$ is simply connected, then α is exact, there are three differentiable functions η, μ, ν and an 1-form $\beta=\eta dx+\mu dy+\nu dz$ on $\mathbb{R}^3$ such that $\alpha=d\beta$

$$d\beta=\left(\frac{\partial\mu}{\partial x}-\frac{\partial\eta}{\partial y}\right)dx\wedge dy+\left(\frac{\partial\nu}{\partial x}-\frac{\partial\eta}{\partial z}\right)dx\wedge dz+\left(\frac{\partial\eta}{\partial y}-\frac{\partial\mu}{\partial z}\right)dy\wedge dz\Rightarrow$$

$$\begin{cases} a &= -P_{13} - P_{23} = \dfrac{\partial \mu}{\partial x} - \dfrac{\partial \eta}{\partial y} \\ b &= -P_{12} + P_{23} = \dfrac{\partial \eta}{\partial z} - \dfrac{\partial \nu}{\partial x} \\ c &= P_{12} + P_{13} = \dfrac{\partial \nu}{\partial y} - \dfrac{\partial \mu}{\partial z} \end{cases} \Rightarrow \text{a+b+c=0} \tag{6}$$

which is equivalent to

$$\frac{\partial}{\partial x}(\mu - \nu) + \frac{\partial}{\partial y}(\nu - \eta) + \frac{\partial}{\partial z}(\eta - \mu) = 0$$

which is also equivalent to

$$\begin{cases} \eta &= \nu \\ \eta &= \mu \\ \mu &= \nu \end{cases} \Leftrightarrow \quad \eta = \mu = \nu.$$

Therefore the system (6) becomes

$$\begin{cases} a = -P_{13} - P_{23} &= \dfrac{\partial \mu}{\partial x} - \dfrac{\partial \mu}{\partial y} \\ b = -P_{12} + P_{23} &= \dfrac{\partial \mu}{\partial z} - \dfrac{\partial \mu}{\partial x} \\ c = P_{12} + P_{13} &= \dfrac{\partial \mu}{\partial y} - \dfrac{\partial \mu}{\partial z}. \end{cases}$$

By identification we obtain

$$P_{12} = -\frac{\partial \mu}{\partial z}, \quad P_{13} = \frac{\partial \mu}{\partial y}, \quad P_{23} = -\frac{\partial \mu}{\partial x}.$$

Which gives the desired result.

Remark 2. *By setting $f = -\mu$ we find the results of M. Boucetta [4].*

4. Link between Jacobi structures and Poisson structures

Contrary to Theorem 1, here we show the existence of a Poisson structure from a Jacobi structure on a manifold of dimension 3.
Let (M, π, E) and (M, P) respectively a Jacobi manifold and Poisson manifold, where

$$\pi = \pi_{12}\partial_x \wedge \partial_y + \pi_{13}\partial_x \wedge \partial_z + \pi_{23}\partial_y \wedge \partial_z, \; E = E_1\partial_x + E_2\partial_y + E_3\partial_z.$$

Theorem 5. *Let (M, π, E) be a Jacobi manifold. Then there exists a vector field Z on TM different from E such that the bivector P defined by $P = \pi + E \wedge Z$, is a Poisson* *tensor on M. There is a natural one-to-one correspondence between Jacobi manifold and Poisson manifold ($[P,P]=0$), which is, if and only if*

$$-2E_1\pi_{13}+2E_2\pi_{13}-2E_3\pi_{12} = \Big(2\pi_{12}-A\Big)\frac{\partial B}{\partial x}+\Big(2\pi_{13}-C\Big)\frac{\partial A}{\partial y}+A\Big(\frac{2\partial\pi_{13}}{\partial x}-\frac{\partial C}{\partial y}\Big) +B\Big(2\frac{\partial\pi_{12}}{\partial x}-\frac{\partial C}{\partial z}\Big)+C\Big(2\frac{\partial\pi_{12}}{\partial y}-\frac{\partial B}{\partial z}\Big)+2\pi_{13}\frac{\partial A}{\partial x}, \quad (7)$$

wehere

$$\begin{aligned} \pi &= \pi_{12}\partial x\wedge\partial y+\pi_{13}\partial x_\wedge\partial z+\pi_{23}\partial y\wedge\partial_z \\ E &= = E_1\partial_x+E_2\partial_y+E_3\partial_z \\ Z &= Z_1\partial_x+Z_2\partial_y+Z_3\partial_z. \\ A &= E_1Z_2-E_2Z_1,\ B=E_1Z_3-E_3Z_1 \ \ \textit{and}\ \ C=E_2Z_3-E_3Z_2. \end{aligned}$$

Moreover

$$P=(\pi_{12}+E_1Z_2-E_2Z_1)\partial_x\wedge\partial_y+(\pi_{13}+E_1Z_3-E_3Z_1)\partial_x\wedge\partial_z+(\pi_{23}+E_2Z_3-E_3Z_2)\partial_y\wedge\partial_z.$$

Proof. To obtain the result it is enough to show that, by using equation (2),

$$\begin{aligned} [P,P] &= [\pi+E\wedge Z,\pi+E\wedge Z] \\ &= [\pi,\pi]_\pi+[\pi,E\wedge Z]+[E\wedge Z,\pi]+[E\wedge Z,E\wedge Z] \\ &= 0 \end{aligned}$$

We can assume that $E\neq Z$. Then one has

$$\begin{aligned} E\wedge Z &= \Big(E_1\partial_x+E_2\partial_y+E_3\partial_z\Big)\wedge\Big(Z_1\partial_x+Z_2\partial_y+Z_3\partial_z\Big) \\ &= \Big(E_1Z_2-E_2Z_1\Big)\partial_x\wedge\partial_y+\Big(E_1Z_3-E_3Z_1\Big)\partial_x\wedge\partial_z+\Big(E_2Z_3-E_3Z_2\Big)\partial_y\wedge\partial_z \\ &= A\partial_x\wedge\partial_y+B\partial_x\wedge\partial_z+C\partial_y\wedge\partial_z. \end{aligned}$$

A straightforward computation show that

$$[E\wedge Z,E\wedge Z]=\Big(-A\frac{\partial B}{\partial x}-C\frac{\partial A}{\partial y}-A\frac{\partial C}{\partial y}-C\frac{\partial B}{\partial z}-B\frac{\partial C}{\partial z}\Big)\partial_x\wedge\partial_y\wedge\partial_z. \quad (8)$$

$$\begin{aligned} [\pi,\pi] &= 2E\wedge\pi \\ &= 2\Big(E_1\pi_{23}-E_2\pi_{13}+E_3\pi_{12}\Big)\partial_x\wedge\partial_y\wedge\partial_z \end{aligned} \quad (9)$$

$$\begin{aligned} [\pi,E\wedge Z]+[E\wedge Z,\pi] &= 2\Big(B\frac{\partial\pi_{12}}{\partial x}-\pi_{12}\frac{\partial B}{\partial x}+C\frac{\partial\pi_{12}}{\partial_y}+A\frac{\partial\pi_{13}}{\partial x} \\ &\quad +\pi_{13}\frac{\partial A}{\partial x}+\pi_{23}\frac{\partial A}{\partial y}\Big)\partial_x\wedge\partial_y\wedge\partial_z \end{aligned} \quad (10)$$

So equation (8) by (9)-(10), becomes

$$\begin{aligned}[P,P] &= \Big(2E_1\pi_{23} - 2E_2\pi_{13} + 2E_3\pi_{12} + (2\pi_{12} - A)\frac{\partial B}{\partial x} + (2\pi_{23} - C)\frac{\partial A}{\partial y} \\ &\quad + A(\frac{2\partial\pi_{13}}{\partial x} - \frac{\partial C}{\partial y}) + B(2\frac{\partial\pi_{12}}{\partial x} - \frac{\partial C}{\partial z}) + C(2\frac{\partial\pi_{12}}{\partial y} - \frac{\partial B}{\partial z}) + 2\pi_{13}\frac{\partial A}{\partial x}\Big)\partial_x \wedge \partial_y \wedge \partial_z.\end{aligned}$$

We deduce the result (7).

Example 1. *Let us equip $\mathbb{R}^3$ with the bivector field π defined by*

$$\pi = (2y + 3y^2)\partial_x \wedge \partial_y + (2 + 3y)\partial_y \wedge \partial_z$$

and the vector field E defined by $E = -2\partial_x - 3\partial_z$.
A direct calculation shows that

$$[\pi,\pi] = 2E \wedge \pi = 2(-9y^2 - 4 - 12y)\partial_x \wedge \partial_y \wedge \partial_z \quad \text{and} \quad [E,\pi] = 0.$$

By choosing $Z_1 = f(x) = -x^2 - 1,\ Z_2 = g(z),\ Z_3 = h(y) = y^2 + 1$, in this example, we have $A = -2Z_2,\ B = -2Z_3 + 3Z_1,\ C = 3Z_2$ and by (7)

$$[P,P] = \Big(-8 - 24y - 18y^2 + (-2h(y) + 3f(x))(-\frac{\partial g}{\partial z}) + (4 + 12y)(3g(z))\Big)\partial_x \wedge \partial_y \wedge \partial_z.$$

$$\begin{aligned}[P,P] = 0 &\iff \Big(2h(y) - 3f(x)\Big)g'(x) + (12 + 36y)g(z) = 18y^2 + 24y + 8 \\ &\iff \alpha g'(z) + \beta g(z) = \gamma\end{aligned}$$

We get

$$g(z) = Ke^{-\frac{\beta}{\alpha}z} + \frac{\alpha\gamma}{\beta},\ \forall K \in \mathbb{R},$$

where $\alpha = 2y^2 + 3x^2 + 5,\ \beta = 12 + 36y,\ \gamma = 18y^2 + 24y + 8$ with $y \neq -\frac{1}{3}$.
Thus we obtain

$$Z = -(x^2 + 1)\partial_x + Z_2\partial_y + (y^2 + 1)\partial_z$$

with

$$Z_2 = K\exp\Big(\frac{-12 - 36y)}{3x^2 + 2y^2 + 5}z\Big) + \frac{(9x^2 + 6y^2 + 15)(y + \frac{2}{3})^2}{6y + 2},\ \forall K \in \mathbb{R}.$$

Corollary 1. *Let us assume that $E = f\partial_z$ where $f \not\equiv 0$. Let (π, E) a Jacobi structure on M. $P = \pi + E \wedge Z$, $(Z = \partial_x + \partial_y + \partial_z)$ is a Poisson structure on M if and ally if*

$$\frac{\partial\pi_{12}}{\partial_z} = 0, \tag{11}$$

$$\pi_{12}\frac{\partial f}{\partial y} + \pi_{13}\frac{\partial f}{\partial z} + f\frac{\partial\pi_{13}}{\partial z} = 0, \tag{12}$$

$$\pi_{12}\frac{\partial f}{\partial x} + \pi_{23}\frac{\partial f}{\partial z} + f\frac{\partial\pi_{23}}{\partial z} = 0, \tag{13}$$

$$2f\Big(\frac{\partial\pi_{12}}{\partial x} + \frac{\partial\pi_{12}}{\partial y} + \frac{1}{2}\frac{\partial\pi_{13}}{\partial y} + \frac{1}{2}\frac{\partial\pi_{13}}{\partial z}\Big) = \pi_{13}\Big(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\Big) + 2\pi_{12}\Big(f + \frac{\partial f}{\partial x}\Big). \tag{14}$$

Proof. Since

$$E \wedge Z = -f\partial_x \wedge \partial_z - f\partial_y \wedge \partial_z,$$

one has:

$$\begin{aligned}
[P,P] &= [\pi + E\wedge Z, \pi + E \wedge Z] \\
&= [\pi,\pi] + [\pi, E\wedge Z] + [E\wedge Z, \pi] + [E\wedge Z, E\wedge Z] \\
&= 2E\wedge\pi + \pi_{13}\Big(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\Big)\partial_x\wedge\partial_y\wedge\partial_z + 2\pi_{12}\frac{\partial f}{\partial x}\partial_x\wedge\partial_y\wedge\partial_z \\
&\qquad -f\Big(2\frac{\partial \pi_{12}}{\partial x} + 2\frac{\partial \pi_{12}}{\partial y} + \frac{\partial \pi_{13}}{\partial y} + \frac{\partial \pi_{13}}{\partial z}\Big)\partial_x\wedge\partial_y\wedge\partial_z \\
&= -2f\Big(\frac{\partial \pi_{12}}{\partial x} + \frac{\partial \pi_{12}}{\partial y} + \frac{1}{2}\frac{\partial \pi_{13}}{\partial y} + \frac{1}{2}\frac{\partial \pi_{13}}{\partial z}\Big)\partial_x\wedge\partial_y\wedge\partial_z \\
&\qquad \pi_{13}\Big(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\Big)\partial_x\wedge\partial_y\wedge\partial_z + 2\pi_{12}\Big(f + \frac{\partial f}{\partial x}\Big)\partial_x\wedge\partial_y\wedge\partial_z.
\end{aligned}$$

So $[P,P]=0$ if and only if

$$2f\Big(\frac{\partial \pi_{12}}{\partial x} + \frac{\partial \pi_{12}}{\partial y} + \frac{1}{2}\frac{\partial \pi_{13}}{\partial y} + \frac{1}{2}\frac{\partial \pi_{13}}{\partial z}\Big) = \pi_{13}\Big(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\Big) + 2\pi_{12}\Big(f + \frac{\partial f}{\partial x}\Big)$$

which give (14).

$$\begin{aligned}
[E,\pi] &= f\frac{\partial \pi_{12}}{\partial_z}\partial_x\wedge\partial_y + \Big(\pi_{12}\frac{\partial f}{\partial_x} + f\frac{\partial \pi_{23}}{\partial_z} + \pi_{23}\frac{\partial f}{\partial_z}\Big))\partial_y\wedge\partial_z \\
&\qquad +\Big(\pi_{12}\frac{\partial f}{\partial_y} + f\frac{\partial \pi_{13}}{\partial_z} + \pi_{13}\frac{\partial f}{\partial_z}\Big)\partial_x\wedge\partial_z \\
&= 0.
\end{aligned}$$

Since $f\neq 0$, one has

$$\begin{cases} \frac{\partial \pi_{12}}{\partial_z} &= 0 \\ \pi_{12}\frac{\partial f}{\partial_x} + f\frac{\partial \pi_{23}}{\partial_z} + \pi_{23}\frac{\partial f}{\partial_z} &= 0 \\ \pi_{12}\frac{\partial f}{\partial_y} + f\frac{\partial \pi_{13}}{\partial_z} + \pi_{13}\frac{\partial f}{\partial_z} &= 0. \end{cases}$$

Thus we obtain (11),(12) and (13).

Example 2. *Let*

$$\pi = (x^4+y^4)\partial_x\wedge\partial_y - x\partial_x\wedge\partial_z + y\partial_y\wedge\partial_z$$

and $E = f\partial_z$ *with* f *is smooth function.*

Then a straightforward calculation yields

$$\begin{cases} (x^4+y^4)\frac{\partial f}{\partial_x} + y\frac{\partial f}{\partial z} &= 0, \\ (x^4+y^4)\frac{\partial f}{\partial_y} - x\frac{\partial f}{\partial z} &= 0, \\ 2f(x^4+y^4+4x^3+4y^3) + 2(x^4+y^4)\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial_y} + x\frac{\partial f}{\partial z} &= 0. \end{cases}$$

On can obtain

$$f(x,y,z) = \exp\Big(zg(x,y) + k(x,y)\Big).$$

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