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Generalized Extension of Watson's theorem for the Series ${}_3F_2(1)$

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Abstract. The ${}_3F_2$ hypergeometric function holds a pivotal position in the realm of hypergeometric and generalized hypergeometric series. Its significance extends beyond mathematics, impacting various fields such as physics and statistics.
This research paper aspires to uncover the explicit expression of the ${}_3F_2$ Watson's classical summation theorem, an endeavor that promises to deepen our understanding and expand the applications of this remarkable function:

$$
{}_3F_2\left[\begin{array}{ccc} a, & b, & c \\ & & ;\quad 1 \\ \frac{1}{2}(a+b+i+1), & 2c+j & \end{array}\right]
$$

For any arbitrary i and j, setting $i=j=0$ leads directly to Watson's theorem for the series ${}_3F_2(1)$. This highlights the theorem's critical relevance.

2020 Mathematics Subject Classifications: 33C05, 33C20, 33C70

Key Words and Phrases: Hypergeometric Summation Theorems, Watson's Theorem

1. Introduction

The generalized hypergeometric function, represented by ${}_rF_s$, is an excellent mathematical construct that demonstrates the remarkable depth and beauty of special functions. The foundation of our research lies in understanding and expanding the realm of these functions, especially in the context of long-established summation theorems and their innovative extensions.

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The generalized hypergeometric function characterized by p numerator and q denominator parameters, is formally defined in [1]

$$
{}_pF_q\left[\begin{array}{ccc} \nu_1, & ..., & \nu_p \\ & & ; \quad z \\ \xi_1, & ..., & \xi_q \end{array}\right] = \sum_{m=0}^{\infty} \frac{(\nu_1)_m \dots (\nu_p)_m}{(\xi_1)_m \dots (\xi_q)_m} \frac{z^m}{m!}, \tag{1}
$$

where $(\nu)_m$ denotes the shifted factorial defined for any complex number μ, by

$$
(\mu)_n = \begin{cases} \mu(\mu+1)...(\mu+n-1); & n = 1, 2, 3, ... \\ 1; & n = 0 \end{cases}.
$$

Using the main property $\Gamma(\mu+1) = \mu\Gamma(\mu)$, $(\mu)_n$ can be written as

$$
(\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(\mu)}.
$$

It is essential to recognize that the representation of hypergeometric and generalized hypergeometric functions in terms of the Gamma function has profound theoretical and practical implications. A limited number of summation theorems exist in the literature, specifically known as classical summation theorems, which include Gauss, Gauss's second theorem, Kummer, and Bailey for the ${}_2F_1$ series, along with Watson, Dixon, and Whipple for the ${}_3F_2$ series.

The ${}_3F_2$ hypergeometric function is of paramount importance in the theory of hypergeometric and generalized hypergeometric series. Furthermore, this function has an extensive range of applications in mathematics, as detailed in references [2–10], and it significantly contributes to the fields of physics and statistics, as outlined in references [11–17].

Now, we begin by introducing the classical Watson's summation theorem ${}_3F_2$ of unit argument [18], which takes the form:

$$
\begin{aligned}
&{}_3F_2\left[\begin{array}{ccc} \nu, & \xi, & \eta \\ & & ; \quad 1 \\ \frac{1}{2}(\nu+\xi+1), & 2\eta & \end{array}\right] \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(\eta+\frac{1}{2})\Gamma(\frac{\nu}{2}+\frac{\xi}{2}+\frac{1}{2})\Gamma(\eta-\frac{\nu}{2}-\frac{\xi}{2}+\frac{1}{2})}{\Gamma(\frac{\nu}{2}+\frac{1}{2})\Gamma(\frac{\xi}{2}+\frac{1}{2})\Gamma(\eta-\frac{\nu}{2}+\frac{1}{2})\Gamma(\eta-\frac{\xi}{2}+\frac{1}{2})}
\end{aligned} \tag{2}
$$

where $Re(2\eta-\nu-\xi) > -1$.

In [19], Watson demonstrated the formula given in (2) for cases where one of the parameters, a or b, is a negative integer. This result was later established more generally in the non-terminating case by Whipple in [20].

The standard proof of (2), presented in [18, p.149] and [21, p.54], relies on a transformation developed by Thomae [22]. Additionally, MacRobert [23] provided an alternative and more interesting proof by utilizing the quadratic transformation for Gauss's hypergeometric function, as stated in [1, Theorem 25, p. 67].

Recently Rathie and Paris [24] gave a basic confirmation of (2) that just depends on the Gauss summation theorem for the ${}_2F_1$ hypergeometric function, namely, [25] while Rakha in [26] gave an extremely straightforward proof of(2) by using the Gauss's second summation theorem.

In 1987, Lavoie [27] made a significant contribution by establishing two powerful summation formulas that have important implications in the field:

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} \nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi+1), & 2\eta+1 & \end{matrix} ; 1\right] \\ &= \frac{2^{\nu+\xi-2}\,\Gamma(\eta+\frac{1}{2})\Gamma(\frac{\nu}{2}+\frac{\xi}{2}+\frac{1}{2})\Gamma(\eta-\frac{\nu}{2}-\frac{\xi}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\nu)\,\Gamma(\xi)} \\ &\times\left\{\frac{\Gamma(\frac{\nu}{2})\Gamma(\frac{\xi}{2})}{\Gamma(\eta-\frac{\nu}{2}+\frac{1}{2})\,\Gamma(\eta-\frac{\xi}{2}+\frac{1}{2})} - \frac{\Gamma(\frac{\nu}{2}+\frac{1}{2})\,\Gamma(\frac{\xi}{2}+\frac{1}{2})}{\Gamma(\eta-\frac{\nu}{2}+1)\,\Gamma(\eta-\frac{\xi}{2}+1)}\right\} \end{aligned} \tag{3}$$

provided $Re(2\eta-\nu-\xi) > -3$, and

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} \nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi+1), & 2\eta-1 & \end{matrix} ; 1\right] \\ &= \frac{2^{\nu+\xi-2}\,\Gamma(\eta-\frac{1}{2})\Gamma(\frac{\nu}{2}+\frac{\xi}{2}+\frac{1}{2})\Gamma(\eta-\frac{\nu}{2}-\frac{\xi}{2}-\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\nu)\,\Gamma(\xi)} \\ &\times\left\{\frac{\Gamma(\frac{\nu}{2})\Gamma(\frac{\xi}{2})}{\Gamma(\eta-\frac{\nu}{2}-\frac{1}{2})\,\Gamma(\eta-\frac{\xi}{2}-\frac{1}{2})} - \frac{\Gamma(\frac{\nu}{2}+\frac{1}{2})\,\Gamma(\frac{\xi}{2}+\frac{1}{2})}{\Gamma(\eta-\frac{\nu}{2})\,\Gamma(\eta-\frac{\xi}{2})}\right\} \end{aligned} \tag{4}$$

provided $Re(2\eta-\nu-\xi) > 1$.

In 1992, Lavoie et al. in [28], took out explicit expression of the series

$${}_3F_2\left[\begin{matrix} \nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi+i+1), & 2\eta+j & \end{matrix} ; 1\right] \tag{5}$$

for $i, j = 0, \pm1, \pm2$, where at $i = j = 0$ we obtain (2).

Other remarkable results of such computations, are:

1. In 1997, Stainislaw [29] presented an analytical formula for (5), establishing a framework with a fixed value of j and allowing for arbitrary values of i.

2. Kim et al. [30] subsequently derived the aforementioned result (5) specifically for the case where $j = 0$ and i takes values of $0, \pm1, \pm2, \ldots, \pm5$.

3. In 2012, Chu [31] conducted an investigation into the generalized Watson's series, incorporating two additional integer parameters by integrating the linearization method with Dougall's summation approach for well-poised ${}_5F_4$-series.

4. Rakha et al., in their 2013 study [32], established the results pertaining to (5) for $i = 0, \pm 1, \pm 2, \ldots, \pm 5$ and $j = 0, \pm 1, \pm 2$.

The primary objective of this paper is to identify explicit extensions of the classical Watson's summation theorem for arbitrary values of i and j. This endeavor aims to yield additional summation theorems as well as further contiguous relations concerning the hypergeometric series denoted as ${}_3F_2(1)$.

2. Main Results

Our main results in this paper can be formed in the following theorem.

Theorem 1. *For $i, j \in \mathbb{Z}$,*

$$\begin{aligned}&(2\eta + j)\ f_{i,j+1}(\nu, \xi, \eta)\\ &= (2\eta + j)\ f_{i,j}(\nu, \xi, \eta) - \frac{2\nu\xi\eta}{(\nu + \xi + i + 1)(2\eta + j + 1)} f_{i,j}(\nu + 1, \xi + 1, \eta + 1).\end{aligned} \tag{6}$$

Proof. Let us consider that

$$\begin{aligned} f_{i,j}(\nu, \xi, \eta) &= {}_3F_2 \left[\begin{matrix} \nu & \xi & \eta \\ \frac{\nu+\xi+i+1}{2} & 2\eta + j & \end{matrix} ; 1 \right] \\ &= \sum_{n=0}^{\infty} \frac{(\nu)_n (\xi)_n (\eta)_n}{\left(\frac{\nu+\xi+i+1}{2}\right)_n (2\eta + j)_n} \frac{1}{n!}. \end{aligned} \tag{7}$$

It is clear that

$$\begin{aligned}&(2\eta + j)\ f_{i,j+1}(\nu, \xi, \eta)\\ &= \sum_{n=0}^{\infty} \frac{(\nu)_n (\xi)_n (\eta)_n (2\eta + j)}{\left(\frac{\nu+\xi+i+1}{2}\right)_n (2\eta + j + 1)_n} \frac{1}{n!}\\ &= \sum_{n=0}^{\infty} \frac{(\nu)_n (\xi)_n (\eta)_n (2\eta + j + n)}{\left(\frac{\nu+\xi+i+1}{2}\right)_n (2\eta + j + 1)_n} \frac{1}{n!} - \sum_{n=0}^{\infty} \frac{(\nu)_n (\xi)_n (\eta)_n\, n}{\left(\frac{\nu+\xi+i+1}{2}\right)_n (2\eta + j + 1)_n} \frac{1}{n!}\\ &= (2\eta + j) \sum_{n=0}^{\infty} \frac{(\nu)_n (\xi)_n (\eta)_n}{\left(\frac{\nu+\xi+i+1}{2}\right)_n (2\eta + j)_n} \frac{1}{n!} - \sum_{n=0}^{\infty} \frac{(\nu)_{n+1} (\xi)_{n+1} (\eta)_{n+1}}{\left(\frac{\nu+\xi+i+1}{2}\right)_{n+1} (2\eta + j + 1)_{n+1}} \frac{1}{n!}\\ &= (2\eta + j)\ f_{i,j}(\nu, \xi, \eta) - \frac{2abc}{(\nu + \xi + i + 1)(2\eta + j + 1)} \sum_{n=0}^{\infty} \frac{(\nu + 1)_{n+1} (\xi + 1)_{n+1} (\eta + 1)_{n+1}}{\left(\frac{\nu+\xi+i+3}{2}\right)_{n+1} (2\eta + j + 2)_{n+1}} \frac{1}{n!}\end{aligned}$$

$$= (2\eta + j)\ f_{i,j}(\nu, \xi, \eta) - \frac{2\nu\xi\eta}{(\nu + \xi + i + 1)(2\eta + j + 1)} f_{i,j}(\nu + 1, \xi + 1, \eta + 1),$$

Thus, we have successfully reached the outcome specified in equation (6). This accomplishment not only confirms the validity of our findings but also effectively concludes the derivation associated with (6).

Remark 1. *If we know* $f_{i,0}(\nu, \xi, \eta) = \left[\begin{matrix} \nu & \xi & \eta \\ \frac{\nu+\xi+i+1}{2} & 2\eta & \end{matrix} ; 1\right]$*, we can generate* $f_{i,j}(\nu, \xi, \eta)$ *for any values of* i *and* j*.*

3. Special Cases

3.1. Special Cases

(i) When $i = j = 0$ in (6), we obtain

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} \nu, & \xi, & \eta \\ \frac{1}{2}(\nu + \xi + 1), & 2\eta + 1 & \end{matrix} ; 1\right] \\
&= {}_3F_2\left[\begin{matrix} \nu, & \xi, & \eta \\ \frac{1}{2}(\nu + \xi + 1), & 2\eta & \end{matrix} ; 1\right] \\
&- \frac{\nu\xi}{(2\eta + 1)(\nu + \xi + 1)}\, {}_3F_2\left[\begin{matrix} \nu + 1, & \xi + 1, & \eta + 1 \\ \frac{1}{2}(\nu + \xi + 3), & 2\eta + 2 & \end{matrix} ; 1\right] \\
&= \frac{2^{\nu+\xi-2}\ \Gamma(\eta + \frac{1}{2})\Gamma(\frac{\nu}{2} + \frac{\xi}{2} + \frac{1}{2})\Gamma(\eta - \frac{\nu}{2} - \frac{\xi}{2} + \frac{1}{2})}{\Gamma(\frac{1}{2})\ \Gamma(\nu)\ \Gamma(\xi)} \\
&\times \left\{ \frac{\Gamma(\frac{\nu}{2})\Gamma(\frac{\xi}{2})}{\Gamma(\eta - \frac{\nu}{2} + \frac{1}{2})\ \Gamma(\eta - \frac{\xi}{2} + \frac{1}{2})} - \frac{\Gamma(\frac{\nu}{2} + \frac{1}{2})\ \Gamma(\frac{\xi}{2} + \frac{1}{2})}{\Gamma(\eta - \frac{\nu}{2} + 1)\ \Gamma(\eta - \frac{\xi}{2} + 1)} \right\}
\end{aligned}$$

which appeared in [32, Eq.(3.18), pp. 229], [27, Result (2), pp.269],[28, Result (1),pp.24], [33, Eq.(4.4),pp.12] and [34, Theorem 4, p.147].

(a) In such a case, the result when $\nu = 1$, $\xi = 1$ and $\eta = 1$, appeared in [35, Result 209, p. 459].

(b) In such a case the result when $\nu = \frac{2}{3}$, $\xi = \frac{4}{3}$ and $\eta = 1$, appeared in [36, Eq. 26].

(ii) When $i = 1$ and $j = 0$ in (6), we obtain

$$ {}_3F_2\left[\begin{matrix} \nu, & \xi, & \eta \\ \frac{1}{2}(\nu + \xi + 2), & 2\eta + 1 & \end{matrix} ; 1\right]$$

$$= {}_3F_2\left[\begin{array}{ccc} \nu, & \xi, & \eta \\ & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+2), & 2\eta & \end{array}\right]$$

$$- \frac{\nu\xi}{(2\eta+1)(\nu+\xi+2)}\,{}_3F_2\left[\begin{array}{ccc} \nu+1, & \xi+1, & \eta+1 \\ & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+4), & 2\eta+2 & \end{array}\right]$$

$$= \frac{2^{\nu+\xi-1}\Gamma\left(\frac{\nu+\xi+2}{2}\right)\Gamma\left(\eta+\frac{1}{2}\right)\Gamma\left(\eta-\frac{\nu}{2}-\frac{\xi}{2}\right)}{2(\nu-\xi)\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)\Gamma(\xi)}\left\{\frac{(2\eta-\nu+\xi)\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\xi}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+1\right)\Gamma\left(\eta-\frac{\xi}{2}+\frac{1}{2}\right)}\right.$$

$$\left. - \frac{(2\eta+\nu-\xi)\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\xi+1}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}+1\right)}\right\}$$

which appeared in [32, Eq.(3.18), pp.229], [34, Theorem 5, pp.148], [28, Result (1), pp.24], and [34, Theorem 2, pp. 144].

In such a case, the results when $\nu = \frac{1}{2}$, $\xi = 1$, $\eta = \frac{5}{4}$; $\nu = \frac{1}{2}$, $\xi = \frac{3}{2}$, $\eta = 1$ and $\nu = \xi = \eta = 1$; appeared in [35, Results 187, 188 & 211, pages 458, 458 & 459], respectively.

(iii) When $i = 2$ and $j = 0$ in (6), we obtain

$${}_3F_2\left[\begin{array}{ccc} \nu, & \xi, & \eta \\ & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+3), & 2\eta+1 & \end{array}\right]$$

$$= {}_3F_2\left[\begin{array}{ccc} \nu, & \xi, & \eta \\ & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+3), & 2\eta & \end{array}\right]$$

$$- \frac{\nu\xi}{(2\eta+1)(\nu+\xi+3)}\,{}_3F_2\left[\begin{array}{ccc} \nu+1, & \xi+1, & \eta+1 \\ & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+5), & 2\eta+2 & \end{array}\right]$$

$$= \frac{2^{\nu+\xi}\Gamma\left(\frac{\nu+\xi+3}{2}\right)\Gamma\left(\eta+\frac{1}{2}\right)\Gamma\left(\eta-\frac{\nu}{2}-\frac{\xi}{2}-\frac{1}{2}\right)}{4(\nu-\xi-1)(\nu-\xi+1)\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)\Gamma(\xi)}\left\{\frac{(2\eta(\nu+\xi-1)-(\nu-\xi)^2+1)\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\xi}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}+\frac{1}{2}\right)}\right.$$

$$\left. - \frac{(8\eta^2-2\eta(\nu+\xi-1)-(\nu-\xi)^2+1)\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\xi+1}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+1\right)\Gamma\left(\eta-\frac{\xi}{2}+1\right)}\right\}$$

which appeared in [32, Eq.3.18, pp.229],[33, Eq.(4.5),pp.12] and [28, Result(1),pp.24].

In such a case, the results when $\nu = 1$, $\xi = \frac{3}{2}$, $\eta = \frac{3}{4}$; $\nu = 1$, $\xi = 2$, $\eta = 1$ and $\nu = \frac{3}{2}$, $\xi = \frac{3}{2}$, $\eta = 1$; appeared in [35, Results 204, 234 & 242, pages 459 & 460], respectively.

(iv) When $i=-1$ and $j=0$ in (6), we obtain

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}\nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi), & 2\eta+1 & \end{matrix}\;;\;1\right]\\
&= {}_3F_2\left[\begin{matrix}\nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi), & 2\eta & \end{matrix}\;;\;1\right]\\
&- \frac{\nu\xi}{(2\eta+1)(\nu+\xi)}\,{}_3F_2\left[\begin{matrix}\nu+1, & \xi+1, & \eta+1 \\ \frac{1}{2}(\nu+\xi+2), & 2\eta+2 & \end{matrix}\;;\;1\right]\\
&= \frac{2^{\nu+\xi-2}\Gamma\left(\frac{\nu+\xi}{2}\right)\Gamma\left(\eta+\frac{1}{2}\right)\Gamma\left(\eta-\frac{\nu}{2}-\frac{\xi}{2}+1\right)}{2(\nu-\xi)\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)\Gamma(\xi)}\left\{\frac{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\xi}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+1\right)\Gamma\left(\eta-\frac{\xi}{2}+\frac{1}{2}\right)}\right.\\
&\left.+\frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\xi+1}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}+1\right)}\right\}
\end{aligned}
$$

which appeared in [28, Result (1), pp.24], [32, Eq.(3.18), pp.229], [31, Example (11), pp.9] and [34, Theorem (1), pp.143].

(v) When $i=-2$ and $j=0$ in (6), we obtain

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}\nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi-1), & 2\eta+1 & \end{matrix}\;;\;1\right]\\
&= {}_3F_2\left[\begin{matrix}\nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi-1), & 2\eta & \end{matrix}\;;\;1\right]\\
&- \frac{\nu\xi}{(2\eta+1)(\nu+\xi-1)}\,{}_3F_2\left[\begin{matrix}\nu+1, & \xi+1, & \eta+1 \\ \frac{1}{2}(\nu+\xi+1), & 2\eta+2 & \end{matrix}\;;\;1\right]\\
&= \frac{2^{\nu+\xi-3}\Gamma\left(\frac{\nu+\xi-1}{2}\right)\Gamma\left(\eta+\frac{1}{2}\right)\Gamma\left(\eta-\frac{\nu}{2}-\frac{\xi}{2}+\frac{1}{2}\right)}{2(\nu-\xi)\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)\Gamma(\xi)}\left\{\frac{(\nu+\xi-1)\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\xi}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}+\frac{1}{2}\right)}\right.\\
&\left.+\frac{(4\eta-\nu-\xi+1)\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\xi+1}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+1\right)\Gamma\left(\eta-\frac{\xi}{2}+1\right)}\right\}
\end{aligned}
$$

which appeared in [28, Result(1), pp.24] and [32, Eq. (3.18), pp.229].

(vi) When $i = 0$ and $j = 1$ in (6), we obtain

$$
\begin{aligned}
&{}_3F_2\left[\begin{array}{ccc} \nu, & \xi, & \eta \\ & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+1), & 2\eta+2 & \end{array}\right] \\
&= {}_3F_2\left[\begin{array}{ccc} \nu, & \xi, & \eta \\ & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+1), & 2\eta+1 & \end{array}\right] \\
&- 2\frac{\nu\xi\eta}{(2\eta+1)(2\eta+2)(\nu+\xi+1)}\,{}_3F_2\left[\begin{array}{ccc} \nu+1, & \xi+1, & \eta+1 \\ & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+3), & 2\eta+3 & \end{array}\right] \\
&= \frac{2^{\nu+\xi-2}\Gamma\left(\frac{\nu+\xi+1}{2}\right)\Gamma\left(\eta+\frac{3}{2}\right)\Gamma\left(\eta-\frac{\nu}{2}-\frac{\xi}{2}+\frac{1}{2}\right)}{2(\eta+1)\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)\Gamma(\xi)} \\
&\times\left\{\frac{[(\eta-\nu+1)(\eta-\xi+1)+\eta(\eta+1)]\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\xi}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+\frac{3}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}+\frac{3}{2}\right)} - \frac{4\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\xi+1}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+1\right)\Gamma\left(\eta-\frac{\xi}{2}+1\right)}\right\}
\end{aligned}
$$

which appeared in [32, Eq.(3.20), pp. 230] and [28, Result (1), pp.24].

In such a case, the results when $\nu = \frac{1}{4}$, $\xi = 1$, $\eta = \frac{1}{8}$;$\nu = \frac{1}{3}$, $\xi = 1$, $\eta = \frac{1}{6}$; $\nu = \frac{1}{2}$, $\xi = 1$ $\eta = \frac{1}{4}$; $\nu = \frac{1}{2}$, $\xi = 1$ $\eta = \frac{3}{8}$; $\nu = \frac{3}{4}$, $\xi = 1$ $\eta = \frac{3}{8}$ and $\nu = \frac{3}{2}$, $\xi = 1$ $\eta = \frac{3}{4}$ appeared in [35, Results 123, 130, 136, 137, 160 & 203, pages 456, 457 & 459].

(vii) When $i = 1$ and $j = 1$ in (6) we obtain

$$
\begin{aligned}
&{}_3F_2\left[\begin{array}{ccc} \nu, & \xi, & \eta \\ & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+2), & 2\eta+2 & \end{array}\right] \\
&= {}_3F_2\left[\begin{array}{ccc} \nu, & \xi, & \eta \\ & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+2), & 2\eta+1 & \end{array}\right] \\
&- 2\frac{\nu\xi\eta}{(2\eta+1)(2\eta+2)(\nu+\xi+2)}\,{}_3F_2\left[\begin{array}{ccc} \nu+1, & \xi+1, & \eta+1 \\ & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+4), & 2\eta+3 & \end{array}\right] \\
&= \frac{2^{\nu+\xi-1}\Gamma\left(\frac{\nu+\xi+2}{2}\right)\Gamma\left(\eta+\frac{3}{2}\right)\Gamma\left(\eta-\frac{\nu}{2}-\frac{\xi}{2}\right)}{2(\eta+1)(\nu-\xi)\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)\Gamma(\xi)}\left\{\frac{[2\eta(\eta+1)-(\nu-\xi)(\eta-\xi+1)]\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\xi}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+1\right)\Gamma\left(\eta-\frac{\xi}{2}+\frac{3}{2}\right)}\right. \\
&\left.-\frac{[2\eta(\eta+1)+(\nu-\xi)(\eta-\nu+1)]\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\xi+1}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+\frac{3}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}+1\right)}\right\}
\end{aligned}
$$

which appeared in [28, Result (1), pp. 24], [34, Theorem (5), pp.148] and [31, Example (9), pp.8].

In such a case, the result when $\nu = \frac{3}{2}$, $\xi = 1$ and $\eta = \frac{1}{4}$, appeared in [35, Result 148,p.457].

(viii) When $i = 2$ and $j = 1$ in (6), we obtain

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} \nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi+3), & 2\eta+2 & \end{matrix} ; 1\right] \\
&= {}_3F_2\left[\begin{matrix} \nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi+3), & 2\eta+1 & \end{matrix} ; 1\right] \\
&- 2\frac{\nu\xi\eta}{(2\eta+1)(2\eta+2)(\nu+\xi+3)}\,{}_3F_2\left[\begin{matrix} \nu+1, & \xi+1, & \eta+1 \\ \frac{1}{2}(\nu+\xi+5), & 2\eta+3 & \end{matrix} ; 1\right] \\
&= \frac{2^{\nu+\xi}\Gamma\left(\frac{\nu+\xi+3}{2}\right)\Gamma\left(\eta+\frac{3}{2}\right)\Gamma\left(\eta-\frac{\nu}{2}-\frac{\xi}{2}-\frac{1}{2}\right)}{8(\eta+1)(\nu-\xi-1)(\nu-\xi+1)\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)\Gamma(\xi)}\left\{\frac{k\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\xi}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+\frac{3}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}+\frac{3}{2}\right)}\right. \\
&\left.-\frac{[4(2\eta+\nu-\xi+1)(2\eta-\nu+\xi+1)]\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\xi+1}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}+1\right)\Gamma\left(\eta-\frac{\xi}{2}+1\right)}\right\}
\end{aligned}
$$

where

$$
\begin{aligned}
k = {} & 2\eta(\eta+1)\left[(2\eta+1)(\nu+\xi-1)-\nu(\nu-1)-\xi(\xi-1)\right] \\
& -(\nu-\xi-1)(\nu-\xi+1)\left[(\eta+1)(2\eta-\nu-\xi+1)+\nu\xi\right]
\end{aligned}
$$

which appeared in [28, Result(1), pp.24].

In such a case, the result when $\nu = \frac{1}{2}$, $\xi = 1$ and $\eta = \frac{1}{4}$, appeared in [35, Result 139, p. 456].

(ix) When $i = 0$ and $j = -1$ in (6), we obtain

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} \nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi+1), & 2\eta-1 & \end{matrix} ; 1\right] \\
&= {}_3F_2\left[\begin{matrix} \nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi+1), & 2\eta & \end{matrix} ; 1\right] \\
&+ \frac{\nu\xi}{(2\eta-1)(\nu+\xi+1)}\,{}_3F_2\left[\begin{matrix} \nu+1, & \xi+1, & \eta+1 \\ \frac{1}{2}(\nu+\xi+3), & 2\eta+1 & \end{matrix} ; 1\right] \\
&= \frac{2^{\nu+\xi-2}\,\Gamma(\eta-\frac{1}{2})\Gamma(\frac{\nu}{2}+\frac{\xi}{2}+\frac{1}{2})\Gamma(\eta-\frac{\nu}{2}-\frac{\xi}{2}-\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\nu)\,\Gamma(\xi)}
\end{aligned}
$$

$$\times\left\{\frac{\Gamma(\frac{\nu}{2})\Gamma(\frac{\xi}{2})}{\Gamma(\eta-\frac{\nu}{2}-\frac{1}{2})\,\Gamma(\eta-\frac{\xi}{2}-\frac{1}{2})}-\frac{\Gamma(\frac{\nu}{2}+\frac{1}{2})\,\Gamma(\frac{\xi}{2}+\frac{1}{2})}{\Gamma(\eta-\frac{\nu}{2})\,\Gamma(\eta-\frac{\xi}{2})}\right\}$$

which appeared in [28, Result(1),pp.24], [32, Eq.(3.19), pp. 230], [27, Result(1), pp. 269] and [34, Theorem (7), pp. 152].

In such a case, the result when $\nu=\frac{1}{2}$, $\xi=\frac{1}{2}$ and $\eta=2$, appeared in [35, Result 172, p.458] & [37, Result(2.9),pp.5].

(x) When $i=1$ and $j=-1$ in (6), we obtain

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}\nu, & \xi, & \eta \\ & & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+2), & 2\eta-1 & \end{matrix}\right] \\
&= {}_3F_2\left[\begin{matrix}\nu, & \xi, & \eta \\ & & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+2), & 2\eta & \end{matrix}\right] \\
&+\frac{\nu\xi}{(2\eta-1)(\nu+\xi+2)}\,{}_3F_2\left[\begin{matrix}\nu+1, & \xi+1, & \eta+1 \\ & & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+4), & 2\eta+1 & \end{matrix}\right] \\
&=\frac{2^{\nu+\xi-1}\Gamma\left(\frac{\nu+\xi+2}{2}\right)\Gamma\left(\eta-\frac{1}{2}\right)\Gamma\left(\eta-\frac{\nu}{2}-\frac{\xi}{2}\right)}{(\nu-\xi)\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)\Gamma(\xi)}\left\{\frac{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\xi}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}-\frac{1}{2}\right)}\right. \\
&\left.-\frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\xi+1}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}-\frac{1}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}\right)}\right\}
\end{aligned}$$

which appeared in [28, Result(1), pp. 24],[32, Eq.(3.20), pp. 230], [31, Example(10), pp. 8] and [34, Theorem (8), pp.153].

(xi) When $i=2$ and $j=-1$ in (6), we obtain

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}\nu, & \xi, & \eta \\ & & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+3), & 2\eta-1 & \end{matrix}\right] \\
&= {}_3F_2\left[\begin{matrix}\nu, & \xi, & \eta \\ & & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+3), & 2\eta & \end{matrix}\right] \\
&+\frac{\nu\xi}{(2\eta-1)(\nu+\xi+3)}\,{}_3F_2\left[\begin{matrix}\nu+1, & \xi+1, & \eta+1 \\ & & & ;\ 1 \\ \frac{1}{2}(\nu+\xi+4), & 2\eta+1 & \end{matrix}\right] \\
&=\frac{2^{\nu+\xi}\Gamma\left(\frac{\nu+\xi+3}{2}\right)\Gamma\left(\eta-\frac{1}{2}\right)\Gamma\left(\eta-\frac{\nu}{2}-\frac{\xi}{2}-\frac{1}{2}\right)}{2(\nu-\xi-1)(\nu-\xi+1)\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)\Gamma(\xi)}\left\{\frac{(\nu+\xi-1)\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\xi}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}-\frac{1}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}-\frac{1}{2}\right)}\right.
\end{aligned}$$

$$-\frac{(4\eta-\nu-\xi-3)\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\xi+1}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}\right)}\Bigg\}$$

which appeared in [28, Result(1), pp.24].

In such a case, the results when $\nu = 1$, $\xi = 2$ and $\eta = 2$ appeared in [35, Results 243, page 460].

(xii) When $i = 1$ and $j = -2$ in (6), we obtain

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}\nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi+2), & 2\eta-2 & \end{matrix};1\right] \\
&= {}_3F_2\left[\begin{matrix}\nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi+2), & 2\eta-1 & \end{matrix};1\right] \\
&+\frac{\nu\xi\eta}{(2\eta-1)(\eta-1)(\nu+\xi+2)}\,{}_3F_2\left[\begin{matrix}\nu+1, & \xi+1, & \eta+1 \\ \frac{1}{2}(\nu+\xi+4), & 2\eta & \end{matrix};1\right] \\
&=\frac{2^{\nu+\xi-1}\Gamma\left(\frac{\nu+\xi+2}{2}\right)\Gamma\left(\eta-\frac{1}{2}\right)\Gamma\left(\eta-\frac{\nu}{2}-\frac{\xi}{2}-1\right)}{(\eta-1)(\nu-\xi)\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)\Gamma(\xi)}\Bigg\{\frac{(\eta-\xi-1)\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\xi}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}-1\right)\Gamma\left(\eta-\frac{\xi}{2}-\frac{1}{2}\right)} \\
&-\frac{(\eta-\nu-1)\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\xi+1}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}-\frac{1}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}-1\right)}\Bigg\}
\end{aligned}
$$

which appeared in [28, Result(1), pp.24], [31, Example(10), pp. 8] and [34, Theorem(8), pp. 153].

(xiii) When $i = -1$ and $j = -2$ in (6), we obtain

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}\nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi), & 2\eta-1 & \end{matrix};1\right] \\
&= {}_3F_2\left[\begin{matrix}\nu, & \xi, & \eta \\ \frac{1}{2}(\nu+\xi), & 2\eta-2 & \end{matrix};1\right] \\
&-\frac{\nu\xi\eta}{(2\eta-1)(\eta-1)(\nu+\xi)}\,{}_3F_2\left[\begin{matrix}\nu+1, & \xi+1, & \eta+1 \\ \frac{1}{2}(\nu+\xi+2), & 2\eta & \end{matrix};1\right] \\
&=\frac{2^{\nu+\xi-3}\Gamma\left(\frac{\nu+\xi}{2}\right)\Gamma\left(\eta-\frac{1}{2}\right)\Gamma\left(\eta-\frac{\nu}{2}-\frac{\xi}{2}-1\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)\Gamma(\xi)}\Bigg\{\frac{(2\eta-\nu+\xi-2)\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\xi}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}-\frac{1}{2}\right)}
\end{aligned}
$$

$$-\frac{(2\eta+\nu-\xi-2)\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\xi+1}{2}\right)}{\Gamma\left(\eta-\frac{\nu}{2}-\frac{1}{2}\right)\Gamma\left(\eta-\frac{\xi}{2}\right)}\Bigg\}$$

which appeared in [28, Result(1), pp.24], [31, Example(12), pp. 9] and [34, Theorem(6), pp. 150].

Also, other special cases can be obtained as

- When $i = 3$ and $j = 0$ in (6), which appeared in [32, Eq.(3.18), pp.229] and [29, Result(2.23), pp.380]. In such a case, the result when $\nu = 1$, $\xi = 3$ and $\eta = 1$, appeared in [35, Result (237), p.460].
- When $i = 4$ and $j = 0$ in (6), which appeared in [32, Eq. (3.18), pp.229]. In such a case, the result when $\nu = 4$, $\xi = \eta = 1$ appeared in [35, Results 240 & 239, page 460].
- When $i = 5$ and $j = 0$ in (6), which appeared in [32, Eq.(3.18), pp.229] and [29, Result (2.22), pp.380]. In such a case, the results when $\nu = \eta = 1$, $\xi = 3$ and $\nu = \eta = 1$, $\eta = 5$ appeared in [35, Results 238 & 241, p. 460], respectively.
- When $i = -3$ and $j = 0$ in (6), which appeared in [32, Eq.(3.18), pp.229].
- When $i = -4$ and $j = 0$ in (6), which appeared in [32, Eq.(3.18), pp.229].
- When $i = -5$ and $j = 0$ in (6), which appeared in [32, Eq.(3.18), pp.229].

Remark 2. *We have already established a recursive relation (6), that generalized the extension of Watson summation theorem* ${}_3F_2(1)$*. Another explicit expression of (5) that generalize our result (6), can be presented in the next theorem.*

Theorem 2. *For* $i, j \in \mathbb{Z}$,

$$\begin{aligned} f_{i,j}(\nu,\xi,\eta) &= (2\eta+j)\ f_{i+1,j}(\nu-1,\xi,\eta) \\ &\quad - \frac{2\nu\xi}{(\nu+\xi+i+1)(2\eta+j)} f_{i+1,j-1}(\nu,\xi+1,\eta+1) \end{aligned}$$

where $f_{i,j}(\nu,\xi,\eta)$ *is defined as in (7).*
Proof. The proof left for the readers.

4. Concluding Remarks

1. Various other special cases of our result can be obtained.
2. Many new identities and relations which obtained from our result are under examinations and will be published later.

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Conflicts of interest

The authors declare no conflict of interest.

References

[1] E.D. Rainville. *Special Functions Macmillan.* 1960.

[2] P. Berglund, P. Candelas, X. De La Ossa, A. Font, T. Hübsch, D. Jančić, and F. Quevedo. Periods for calabi-yau and landau-ginzburg vacua. *Nuclear Physics B*, 419(2):352–403, 1994.

[3] L. De Branges. A proof of the bieberbach conjecture. *Acta Mathematica*, 154(1):137–152, 1985.

[4] M. Petkovsek, H.S. Wilf, and D. Zeilberger. $A = B$. CRC Press, 1996.

[5] R. Askey and G. Gasper. Positive jacobi polynomial sums, ii. *American Journal of Mathematics*, pages 709–737, 1976.

[6] N. D. Kazarinoff. Special functions and the bieberbach conjecture. *The American Mathematical Monthly*, 95(8):689–696, 1988.

[7] S. G. Samko and R. P. Cardoso. Integral equations of the first kind of sonine type. *International Journal of Mathematics and Mathematical Sciences*, 2003(57):3609–3632, 2003.

[8] S. G. Samko. Fractional integrals and derivatives. *Theory and applications*, 1993.

[9] N. Khan, N. Khan, M. Aman, T. Usman, M. Aman, and T. Usman. Extended beta, hypergeometric and confluent hypergeometric functions via multi-index mittag-leffler function. In *Proceedings of the Jangjeon Mathematical Society*, volume 25, pages 43–58, 2022.

[10] A. Belafhal, F. Chib, S.and Khannous, and T. Usman. Evaluation of integral transforms using special functions with applications to biological tissues. *Computational and Applied Mathematics*, 40(4):156, 2021.

[11] M. N. Barber and B. W. Ninham. *Random and restricted walks: Theory and applications*, volume 10. CRC Press, 1970.

[12] L. G. Cabral-Rosetti and M. A. Sanchis-Lozano. Generalized hypergeometric functions and the evaluation of scalar one-loop integrals in feynman diagrams. *Journal of computational and applied mathematics*, 115(1-2):93–99, 2000.

[13] S. Moch, P. Uwer, and S. Weinzierl. Nested sums, expansion of transcendental functions, and multiscale multiloop integrals. *Journal of Mathematical Physics*, 43(6):3363–3386, 2002.

[14] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii. *Quantum theory of angular momentum.* World Scientific, 1988.

[15] Daniel W. Lozier Frank W. J. Olver and et. *Nist handbook of mathematical functions.* chapter 34,pp.758-766, (2010).

[16] A. Belafhal, N. Nossir, and T. Usman. Integral transforms involving orthogonal polynomials and its application in diffraction of cylindrical waves. *Computational and Applied Mathematics*, 41(3):100, 2022.

[17] A. Belafhal, H. Benzehoua, and T. Usman. Certain integral transforms and their application to generate new laser waves: Exton-gaussian beams. *Adv. math. models appl*, 6:206–217, 2021.

[18] W. N. Bailey. Generalized hypergeometric series. 1935.

[19] G.N. Watson. A note on generalized hypergeometric series. *Proc. London Math. Soc*, 2(23):13–15, 1925.

[20] F.J.W Whipple. A group of generalized hypergeometric series: relations between 120 allied series of the type $f(a, b, c, e, f)$. *Proceedings of the London Mathematical Society*, 2(1):104–114, 1925.

[21] L. J. Slater. *Generalized Hypergeometric Functions.* Cambridge University Press, 1966.

[22] J. Thomae. Ueber die functionen, welche durch reihen von der form dargestellt werden. 1879.

[23] T. M. MacRobert. *Functions of a Complex Variable (5th Edition).* Macmillan, 1962.

[24] A. K. Rathie and R. B. Paris. A new proof of watson's theorem for the series ${}_3f_2(1)$. *Applied Mathematical Sciences 3 (4)*, 2009.

[25] L.J. Slater, M. Abramowitz, and I.A. Stegun. Handbook of mathematical functions. *Abramowitz and IA Stegun, Eds.(US Govt. Printing Office, Washington, DC, 1968) Appl. Math. Ser*, 55, 1965.

[26] M. A. Rakha. A new proof of the classical watson's summation theorem. *Appl. Math. E-Notes*, 11:278–282, 2011.

[27] J.L. Lavoie. Some summation formulas for the series ${}_3f_2(1)$. *Mathematics of computation*, pages 269–274, 1987.

[28] J.L. Lavoie, F. Grondin, and A.K. Rathie. Generalizations of watson's theorem on the sum of a ${}_3f_2$. *Indian J. Math*, 34(2):23–32, 1992.

[29] S. Lewanowicz. Generalized watson's summation formula for ${}_3f_2(1)$. *Journal of computational and applied mathematics*, 86(2):375–386, 1997.

[30] Y. S. Kim and A. K. Rathie. Applications of a generalized form of gauss's second theorem to the series ${}_3f_2$. *Mathematical Communications*, 16(2):481–489, 2011.

[31] W. Chu. Analytical formulae for extended ${}_3f_2$-series of watson-whipple-dixon with two extra integer parameters. *Mathematics of Computation*, 81(277):467–479, 2012.

[32] M. A. Rakha, A. k. Rathie, and U. Pandey. On a generalization of contiguous watson's theorem for the series ${}_3f_2(1)$. 11 2013.

[33] Y. S. Kim, M. A. Rakha, and A. K. Rathie. Extensions of certain classical summation theorems for the series ${}_2f_1$, ${}_3f_2$, and ${}_4f_3$ with applications in ramanujan's summations. *International Journal of Mathematics and Mathematical Sciences*, 2010(1):309503, 2010.

[34] W. Chu and R. R. Zhou. Watson–like formulae for terminating ${}_3f_2$-series. In *Advances*

in Combinatorics: Waterloo Workshop in Computer Algebra, W80, May 26-29, 2011, pages 139–159. Springer, 2013.

[35] A.P. Prudnikov, I.U.A. Brychkov, and O.I. Marichev. *Integrals and Series: Special functions.* Integrals and Series. Gordon and Breach Science Publishers, 1986.

[36] Michael Milgram. On hypergeometrics ${}_3f_2(1)$- a review. 11 2010.

[37] M. M. Awad, A. O. Mohammed, M. A. Rakha, and A. K. Rathie. New series identities for $\frac{1}{\pi}$. *Communications of the Korean Mathematical Society*, 32(4):865–874, 2017.