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Some Types of Tri-Locally Compactness Spaces

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Abstract. Three topologies, or tri-locally compact spaces, will be examined in this study in order to examine the locally compactness spaces attribute. Furthermore, these spaces' characteristics will be analyzed in light of locally limited spaces. Several well-known theorems about locally compact spaces have been expanded to apply to three topologies, and many theoretical results have been proposed and verified. The results are supported by illustrative instances.

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1. Introduction

The study of the connections between different classes of topological spaces located between countably paracompact spaces is one of the main areas of set theoretic topology [1–4]. Since it naturally lies between these classes, the class of locally compact spaces is important in this context. According to Dugundji (1966), a locally compact space is a topological space (X, ϑ) in which each point $a \in X$ has a neighborhood that is also contained within a compact space. Similarly, a tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is called a tri-locally metacompact space if every point $a \in X$ has a neighborhood that is contained within a tri-compact area.

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One important feature of such spaces is that several important separation axioms, such as normality and collection-wise Hausdorff, agree for them. There are significant theoretical and practical implications to problems that come from other areas of mathematics or from a strictly topological standpoint. If there is a fundamental system of nearly open neighborhoods for every point in a set X, then X is a tri-topological space. Keep in mind that the first people to examine almost open sets in a topological group were Ghosh and Lahiri [5]. The concept of a tri-topological group, or the tri-topologized form of a topological group, has previously been discussed in earlier research. Since all of the spaces examined in this study are assumed to be nonempty and T_0 spaces, any two open neighborhoods of a meet to generate another open neighborhood of a for any point a in the space.

The concept of a locally compact space in topological space (X, ϑ) was first proposed by Levine [6]. These subjects were examined in greater detail in more recent research [7–10].... The notions of tri-locally compact and tri-locally metacompact in tri-topological spaces, as well as related findings, are examined in this work. In tri-topological spaces, we present the notion of tri-locally compactness, analyze its properties, and apply it to different spaces. We go over common definitions that will be used in the parts that follow.

Typically, ϑ_u, ϑ_{dis}, ϑ_{cof}, and ϑ_{coc} represent discrete, co-finite, and co-countable topologies, respectively. $X = (X, \vartheta_1, \vartheta_2)$ is a representation of the concept of bitopological spaces, where ϑ_1, ϑ_2 are two topologies on X. This is related to earlier research on bitopological spaces, where a topology is a collection of points that satisfy a set of axioms. Kim's paper [5] described pairwise Hausdorff, pairwise regular, and pairwise normal spaces using a set of standard results known as the Tietze extension. Bitopological space study was further explored in [11, 12]. According to [7, 8], bitopological space can expand, nearly expand, and feebly expand. The primary goal of this paper is to introduce and investigate a new kind of tripartite compact space: the tripartite locally compact space. Tri-topological spaces are sets containing three topologies, where ϑ_1, ϑ_2, and ϑ_3 are topologies on X. They are represented as $X = (X, \vartheta_1, \vartheta_2, \vartheta_3)$. Tri-topological space variations match well-known topological space features.

2. Preliminaries

We will illustrate some of the fundamental concepts of tri-topological space in this part, including paracompactness, dense sets, and compact space.

Definition 1. *[1] $\vartheta \subset \mathcal{P}(X) = \{A : A \subseteq X\}$ is a collection of subsets of X, where X is a non-empty set. If ϑ satisfies the following requirements, it is considered a topology on X:*

(i) $\emptyset, X \in \vartheta$

(ii) *We have $A \cap B \in \vartheta$ for every $A, B \in \vartheta$.*

(iii) $\bigcup_{\alpha\in\lambda} A_\alpha \in \vartheta$ *if $E = \{A_\alpha : \alpha \in \lambda, A_\alpha \in \vartheta\}$ is any collection of sets in ϑ*

Definition 2. *[10]. Let X be a non-empty set, and for $i = 1, 2, 3$, $\vartheta_i \subset \mathcal{P}(X)$. $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-topological space if ϑ_i is a topology on X for all $i = 1, 2, 3$.*

Example 1. *Assume $X = \{a, b, c\}$ so that they*

(i) $\vartheta_1 = \{\emptyset, X, \{a\}\} \subset \mathcal{P}(X)$

(ii) $\vartheta_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \subset \mathcal{P}(X)$

(iii) $\vartheta_3 = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\} \subset \mathcal{P}(X)$

For $i = 1, 2, 3$, ϑ_i satisfies the requirements of a topological space. A tri-topological space is thus $(X, \vartheta_1, \vartheta_2, \vartheta_3)$.

For example, $\{a\} \cup \{b\} = \{a, b\} \notin \vartheta$, therefore the space $\vartheta = \{\emptyset, X, \{a\}, \{b\}\}$ is not a topological space.

Definition 3. *[11] Presuming that $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-topological space and that $A \subset X$*

(i) If $A \in \vartheta_i$ for some $i = 1, 2, 3$, then A is a ϑ_i-open set.

(ii) If $A^c \in \vartheta_i$ for some $i = 1, 2, 3$, then A is a ϑ_i-closed set.

(iii) If A and A^c are both in ϑ_i for some $i = 1, 2, 3$, then A is referred to as a ϑ_i-clopen set.

Definition 4. *[7] Given a tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$, $X \neq \emptyset$, and A as a subset of X, $a \in X$ is a tri-limit point of A if, for any ϑ_i-open set u_a containing a, $u_a \cap (A - \{a\}) \neq \emptyset$..*

$A' = \{a : a$ is a tri-limit point of $A\}$ is the representation of the tri-derived set, which is the set of all tri-limit points.

Theorem 1. *[8] If $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ and $A, B \subset X$ are tri-topological spaces, then:*

(i) $\emptyset' = \emptyset$

(ii) $(A \cup B)' = A' \cup B'$

(iii) $(A \cap B)' \subset A' \cap B'$

(iv) If $A \subset B$, then $A' \subset B'$

Definition 5. *[9] $\overline{A} = A \cup A'$ is the representation of the tri-closure set if $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-topological space, $X \neq \emptyset$, and A is a subset of X.*

Theorem 2. *[8] Let $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ be a tri-topological space and $A, B \subset X$. Then:*

(i) $\overline{X} = X$ *and* $\overline{\emptyset} = \emptyset$

(ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(iii) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

(iv) We have $U_a \cap A \neq \emptyset$ for every point $a \in \overline{A}$ and every ϑ_i-open set U_a containing a.

(v) A is a ϑ_i-closed set for each $i \in \{1, 2, 3\}$ if and only if $\overline{A} = A$.

Definition 6. *[4] allow $X \neq \emptyset$, A be a subset of X, and allow $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ be a tri-topological space. If there is at least one ϑ_i-neighborhood $N(a, \vartheta_i)$ of a for some $i \in \{1, 2, 3\}$ such that $N(a, \vartheta_i) \subseteq A$, then a point $a \in A$ is called a tri-interior point of A.*

A° or $INT(A)$ indicate the tri-interior of A, which is the set of all tri-interior points of A. $A^\circ = INT(A) = (\overline{A^c})^c$ is another way to express this, in which $\overline{A^c}$ is the tri-closure of the complement of A.

Theorem 3. *[7] The following characteristics are true given a tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ and $A, B \subset X$:*

(i) $X^\circ = X$ *and* $\emptyset^\circ = \emptyset$.

(ii) $(A \cap B)^\circ = A^\circ \cap B^\circ$ *and* $A^\circ \cup B^\circ \subset (A \cup B)^\circ$.

(iii) *A° is a ϑ_i-open set for each $i \in \{1, 2, 3\}$ if and only if there exists a ϑ_i-open set U_n such that $n \in U_n \subset A$ for each $i \in \{1, 2, 3\}$.*

Definition 7. *[8] Since $X \neq \phi$ and A is a subset of X, let $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ be a tri-topological space. If there is at least one neighborhood of a such that $N(a, \varepsilon) \cap A = \phi$, then a is a tri-exterior point of A. The tri-exterior set, which is the set of all tri-exterior points, is represented by the equation $EX(A) = Int(A^c) = \overline{A}^C$.*

Theorem 4. *If we define $EX(A) = Int(A^c)$ (the interior of the complement of A) given a tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ and $A, B \subset X$, then:*

(i) *$EX(\emptyset) = X$ and $EX(X) = \emptyset$. If and only if there is a ϑ_i-open set U_e such that $e \in U_e \subset A^c$, then for all $e \in X$, $e \in EX(A)$ $EX(B) \subset EX(A)$ if $A \subset B$.*

Proof. (i) $EX(\emptyset) = \text{Int}(\emptyset^c) = \text{Int}(X) = X$ and $EX(X) = \text{Int}(X^c) = \text{Int}(\emptyset) = \emptyset$.

(ii) If and only if there is a ϑ_i-open set U_e such that $e \in U_e \subset A^c$, then by definition of interior, $e \in \text{Int}(A^c) = EX(A)$.

Assume that $A \subset B$. $B^c \subset A^c$, then. Set inclusion is maintained by taking the interior, so $\text{Int}(B^c) \subset \text{Int}(A^c)$. $EX(B) \subset EX(A)$, so.

Definition 8. *[9] Let $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ be a Tri-topological space, and let $X \neq \emptyset$, A be a subset of X. If every neighborhood $N(a, \vartheta_i)$ of a (with respect to each topology ϑ_i, $i = 1, 2, 3$) fulfills both $N(a, \vartheta_i) \cap A \neq \emptyset$ and $N(a, \vartheta_i) \cap A^c \neq \emptyset$, then the point $a \in X$ is a tri-boundary point of A.*

The collection of all tri-boundary points of A is its tri-boundary, represented by $Bd(A)$. It may be written as follows: $Bd(A) = \overline{A} \cap \overline{A^c} = \overline{A} - A^\circ$, where $\overline{A}$ is the tri-closure of A and A° is the tri-interior of A.

Theorem 5. *Given $A, B \subset X$ and a tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$,*

(i) $Bd(\emptyset) = Bd(X) = \emptyset$

(ii) *$Bd(A)$ is a ϑ_i-closed set*

(iii) $b \in Bd(A)$ *if and only if for all* ϑ_i*-open sets* u_b *containing* b*, we have* $u_b \cap A \neq \emptyset$ *and* $u_b \cap A^c \neq \emptyset$

Proof. We only prove (iii) here. If $b \in Bd(A)$ *and* u_b *is a* ϑ_i*-open set containing* b*, then:*

$b \in Bd(A) = \overline{A} \cap \overline{A^c}$ *if and only if* $b \in \overline{A} \wedge b \in \overline{A^c}$
if and only if $b \in (A \cup A') \wedge b \in A^c \cup (A^c)'$
if and only if $(b \in A \vee b \in A') \wedge (b \in A^c \vee b \in (A^c)')$
if and only if $b \in A' \wedge b \in (A^c)'$
if and only if $u_b \cap (A/\{b\}) \neq \emptyset \wedge u_b \cap (A^c/\{b\}) \neq \emptyset$
But we have $b \subset u_b$*, so we obtain* $u_b \cap A \neq \emptyset$ *and* $u_b \cap A^c \neq \emptyset$*.*

Definition 9. *[7] A tri-topological space* $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ *functions as a tri-*T_0*-space when it contains either a* ϑ_i*-open set* u_a *containing* a *but excluding* b *or it possesses a* ϑ_j*-open set* v_b *with element* b *inside but element* a *kept outside from* v_b *where* $i \neq j$ *and* $i, j = 1, 2, 3$ *for every pair of distinct elements* a

Definition 10. *[8] If, for each of the two distinct elements* a *and* b *in* X*, there exists* ϑ_i*-open set* u_a *such that* $a \in u_a$ *and* $b \notin u_a$*, or* ϑ_j*-open set* v_b *such that* $b \in v_b$ *and* $a \notin v_b$*, where* $i \neq j$ *and* $i, j = 1, 2, 3$*, then* $(X, \vartheta_1, \vartheta_2, \vartheta 3)$ *is a* $tri - T_0$*-space.*

Theorem 6. *Let* $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ *be a tri-topological space . Then thefollowing statements are equivalent:*

(i) X *represents a tri-*T_0*-space*

(ii) For any two distinct elements a *and* b*,* $a \notin \overline{\{b\}}$ *or* $b \notin \overline{\{a\}}$

(iii) If a *and* b *are two distinct elements, we have* $\overline{\{a\}} \neq \overline{\{b\}}$

Proof.

(i) $\Rightarrow$ *(ii): Let* $a \neq b$ *be distinct elements of* X*, and suppose* X *is a tri-*T_0*-space. By definition of a tri-*T_0*-space, there exists a* ϑ_i*-open set* U_a *such that* $a \in U_a$ *and* $b \notin U_a$ *for some* $i \in \{1, 2, 3\}$*, or a* ϑ_j*-open set* V_b *such that* $b \in V_b$ *and* $a \notin V_b$ *for some* $j \in \{1, 2, 3\}$*.*

In the first case, since $a \in U_a$ *and* $U_a \cap \{b\} = \emptyset$*, this implies* $b \notin \overline{\{a\}}$*. In the second case, since* $b \in V_b$ *and* $V_b \cap \{a\} = \emptyset$*, this implies* $a \notin \overline{\{b\}}$*. As a result,* $a \notin \overline{\{b\}}$ *or* $b \notin \overline{\{a\}}$*.*

(ii) $\Rightarrow$ *(iii): For distinct elements* a *and* b*, we have* $a \notin \overline{\{b\}}$ *or* $b \notin \overline{\{a\}}$*. Without loss of generality, assume that* $a \notin \overline{\{b\}}$*. Since* $a \in \overline{\{a\}}$*, we have* $\overline{\{a\}} \neq \overline{\{b\}}$*.*

(iii) $\Rightarrow$ *(i): Assume* a *and* b *are distinct elements such that* $\overline{\{a\}} \neq \overline{\{b\}}$*. Then, there exists either* $c \in \overline{\{a\}} \setminus \overline{\{b\}}$ *or* $d \in \overline{\{b\}} \setminus \overline{\{a\}}$*.*

Without loss of generality, assume $b \notin \overline{\{a\}}$ *(and obviously* $b \in \overline{\{b\}}$*). Since* $\overline{\{a\}}$ *is a* ϑ_i*-closed set in* X *for each* $i \in \{1, 2, 3\}$*, the set* $X \setminus \overline{\{a\}} = V_b$ *is* ϑ_i*-open in* X *for each* i*. We have* $b \in V_b$ *and* $a \notin V_b$*. As a result,* X *is a tri-*T_0*-space.*

Definition 11. *[7] $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-T_1-space if, for each of the two distinct elements a and b in X, there exists ϑ_i-open set u_a such that $a \in u_a$, $b \notin u_a$, and $b \in v_b$ such that $b \in v_b$ and $a \notin v_b$, where $i \neq j$ such that $i, j = 1, 2, 3$.*

Definition 12. *[9] $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-T_2-space if, for each of the two distinct elements a and b in X, there exists ϑ_i-open set u_a such that $a \in u_a$ and ϑ_j-open set v_b such that $b \in v_b$ and $u_a \cap v_b = \phi$, where $i \neq j$ such that $i, j = 1, 2, 3$.*

Definition 13. *[7] For $i = 1, 2, 3$, a topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-$T_{2\frac{1}{2}}$-space if, for each of the two distinct elements a and b in X, there exists a ϑ_i-closed set A_a, $b \in B_b$, and $A_a \cap B_b = \phi$.*

Definition 14. *[8] $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-regular space. If A is a ϑ_i-closed set and $a \in u_a$, $A \subset v_A$, and $u_a \cap v_A = \phi$, then $i \neq j$, for $i, j = 1, 2, 3$.*

Theorem 7. *A tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-regular space if and only if, for every point $a \in X$ and ϑ_i-open set u_a containing a, there exists a ϑ_i-open set w_a such that $a \in w_a \subset \overline{w_a} \subset u_a$.*

Proof. ($\Rightarrow$) If $a \in u_a$, then $a \notin u_a^c$. Since u_a^c is a ϑ_i-closed set and $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-regular space, there exist ϑ_i-open sets w_a and $v_{u_a^c}$ such that $a \in w_a$, $u_a^c \subset v_{u_a^c}$, and $w_a \cap v_{u_a^c} = \emptyset$. Thus, $w_a \subset v_{u_a^c}^c$. Since $v_{u_a^c}$ is ϑ_i-open, $v_{u_a^c}^c$ is ϑ_i-closed, which means $\overline{w_a} \subset v_{u_a^c}^c$. Moreover, $u_a^c \subset v_{u_a^c}$ implies $v_{u_a^c}^c \subset u_a$. Consequently, $a \in w_a \subset \overline{w_a} \subset v_{u_a^c}^c \subset u_a$.

($\Leftarrow$) Let $a \in X$ and F be a ϑ_i-closed set such that $a \notin F$. Then $a \in F^c$, and F^c is a ϑ_i-open set containing a. By our hypothesis, there exists a ϑ_i-open set w_a such that $a \in w_a \subset \overline{w_a} \subset F^c$. This implies $a \in w_a$ and $F \subset (\overline{w_a})^c$. Since $\overline{w_a}$ is ϑ_i-closed, $(\overline{w_a})^c$ is ϑ_i-open. We also have $w_a \cap (\overline{w_a})^c = \emptyset$. Therefore, $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-regular space.

Definition 15. *[8] A tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-T_3-space if it is both a tri-T_1-space and tri-regular.*

Definition 16. *[7] A tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-normal space if for any two disjoint ϑ_i-closed sets A and B, there exist ϑ_i-open sets u_A and v_B such that $A \subset u_A$, $B \subset v_B$, and $u_A \cap v_B = \emptyset$.*

Theorem 8. *A tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-normal space if and only if for every ϑ_i-closed set F and ϑ_i-open set U containing F, there exists a ϑ_i-open set V such that $F \subset V \subset \overline{V} \subset U$.*

Proof. ($\Rightarrow$) Let F be a ϑ_i-closed set and U be a ϑ_i-open set containing F. Then U^c is a ϑ_i-closed set and $F \cap U^c = \emptyset$. Since $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is tri-normal, there exist ϑ_i-open sets V and W such that $F \subset V$, $U^c \subset W$, and $V \cap W = \emptyset$. Thus, $V \subset W^c$. Since W is ϑ_i-open, W^c is ϑ_i-closed, which means $\overline{V} \subset W^c$. Additionally, $U^c \subset W$ implies $W^c \subset U$. Therefore, $F \subset V \subset \overline{V} \subset W^c \subset U$.

($\Leftarrow$) Let F and G be disjoint ϑ_i-closed sets. Then $F \subset G^c$ and G^c is a ϑ_i-open set. By our hypothesis, there exists a ϑ_i-open set V such that $F \subset V \subset \overline{V} \subset G^c$. This implies $F \subset V$ and $G \subset (\overline{V})^c$. Since $\overline{V}$ is ϑ_i-closed, $(\overline{V})^c$ is ϑ_i-open. We also have $V \cap (\overline{V})^c = \emptyset$. Therefore, $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is tri-normal.

Definition 17. *[7] A tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-T_4-space if it is both a tri-T_1-space and tri-normal.*

Theorem 9. *[8] It is tri-T_{k-1}-space if $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is tri-T_k-space.*

Theorem 10. *[9] A space is tri-T_3-space if $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is tri-T_4-space.*

Proof. *By considering $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ to be tri-T_4-space, it is tri-T_1-space and tri-normal space. This suggests that for any of the two disjoint ϑ_i-closed sets A and B, there are ϑ_i-open sets u_A and v_B such that $A \subset u_A$ and $B \subset v_B$ with*

$$u_A \cap v_B = \phi. \tag{1}$$

Let $b \in B$ now, followed by $b \in v_B$. On the other hand, $A \cap B = \phi$. Thus,

$$b \notin A is obtained. \tag{2}$$

Thus, $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-regular space according to (1) and (2). Thus, $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is obtained. $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is tri-T_3-space since is tri-T_1-space and tri-regular space.

Theorem 11. *[8] A space is tri-normal if $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is tri-completely normal.*

Proof. *Consider the space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ to be tri-completely normal. By definition, there is a ϑ_i-continuous function*

$$f_i : X \to [0, 1]$$

for any two disjoint ϑ_i-closed sets A and B. such that

$$f_i(B) = \{1\}, \quad f_i(A) = \{0\} \tag{3}$$

We now define the inverse images of the open intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ under f_i, yielding two ϑ_i-open sets u_A and v_B such that

$$A \subset u_A, \quad B \subset v_B, \quad u_A \cap v_B = \emptyset. \tag{4}$$

The tri-normality of $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ may be inferred from (3) and (4).

Definition 18. *Given a tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$, a set D in $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is referred to as a tri-dense set if $\overline{D} = X$. Conversely, for every ϑ_i-open set u, we obtain $u \cap D \neq \phi$ if D is dense in $(X, \vartheta_1, \vartheta_2, \vartheta_3)$.*

Definition 19. *Let $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ and $(Y, \sigma_1, \sigma_2, \sigma_3)$ be tri-topological spaces. If $f(u) = v$, where u is ϑ_i-open set and v is σ_i-open set, then $f : (X, \vartheta_1, \vartheta_2, \vartheta_3)$ is a tri-open function.*

3. Tri-locally compact spaces

We then introduce the idea of tri-locally compact spaces and establish some of their most important properties in this section.

Definition 20. *A subset A of a tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is tri-compact if every cover of A by ϑ_i-open sets (for $i = 1, 2, 3$) has a finite subcover.*

Definition 21. *A tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is tri-locally compact if for every point $a \in X$, there exists a ϑ_i-open set U_a containing a such that $\overline{U_a}$ is tri-compact.*

Theorem 12. *A tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is tri-locally compact if and only if for every point $a \in X$ and every ϑ_i-open set U containing a, there exists a ϑ_i-open set V such that $a \in V \subset \overline{V} \subset U$ and $\overline{V}$ is tri-compact.*

Proof. ($\Rightarrow$) *Assume $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is tri-locally compact. Let $a \in X$ and U be a ϑ_i-open set containing a. By definition, there exists a ϑ_j-open set W_a containing a such that $\overline{W_a}$ is tri-compact. Let $V = W_a \cap U$, which is a ϑ_i-open set containing a. Then $\overline{V} \subset \overline{W_a} \cap \overline{U} \subset \overline{W_a} \cap U$. Since $\overline{V}$ is a closed subset of the tri-compact set $\overline{W_a}$, $\overline{V}$ is tri-compact. Therefore, $a \in V \subset \overline{V} \subset U$ and $\overline{V}$ is tri-compact.*

($\Leftarrow$) *This direction follows directly from the definition of tri-locally compact spaces.*

Theorem 13. *Let $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ be a tri-T_2-space. If X is tri-locally compact, then for every tri-compact set K and every ϑ_i-open set U containing K, there exists a ϑ_i-open set V such that $K \subset V \subset \overline{V} \subset U$ and $\overline{V}$ is tri-compact.*

Proof. Let K be a tri-compact set and U be a ϑ_i-open set containing K. For each $a \in K$, there exists a ϑ_i-open set V_a such that $a \in V_a \subset \overline{V_a} \subset U$ and $\overline{V_a}$ is tri-compact. The collection $\{V_a : a \in K\}$ forms an open cover of K. Since K is tri-compact, there exists a finite subset $\{a_1, a_2, \ldots, a_n\} \subset K$ such that $K \subset \bigcup_{j=1}^{n} V_{a_j}$.

Let $V = \bigcup_{j=1}^{n} V_{a_j}$. Then V is a ϑ_i-open set and $K \subset V \subset \overline{V} \subset \overline{\bigcup_{j=1}^{n} V_{a_j}} \subset \bigcup_{j=1}^{n} \overline{V_{a_j}} \subset U$. Since $\overline{V} \subset \bigcup_{j=1}^{n} \overline{V_{a_j}}$ and each $\overline{V_{a_j}}$ is tri-compact, their finite union $\bigcup_{j=1}^{n} \overline{V_{a_j}}$ is tri-compact. Therefore, $\overline{V}$ is tri-compact as a closed subset of a tri-compact set.

Theorem 14. *Let $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ be a tri-topological space. If X is tri-locally compact and tri-T_2, then for any tri-compact set K and any closed set F such that $K \cap F = \emptyset$, there exist ϑ_i-open sets U and V such that $K \subset U$, $F \subset V$, and $U \cap V = \emptyset$.*

Proof. Let K be a tri-compact set and F be a closed set such that $K \cap F = \emptyset$. For each $a \in K$, we have $a \notin F$, which means $a \in F^c$. Since X is tri-T_2, for each $a \in K$ and $b \in F$, there exist ϑ_i-open sets U_a and V_b such that $a \in U_a$, $b \in V_b$, and $U_a \cap V_b = \emptyset$.

For each $a \in K$, the collection $\{V_b : b \in F\}$ forms an open cover of F. Since X is tri-locally compact, there exists a finite subset $\{b_1, b_2, \ldots, b_{n_a}\} \subset F$ such that $F \subset \bigcup_{j=1}^{n_a} V_{b_j}$. Let $V_a = \bigcup_{j=1}^{n_a} V_{b_j}$. Then V_a is a ϑ_i-open set containing F.

Now, let $W_a = X \setminus V_a$. Then W_a is a ϑ_i-closed set and $a \in W_a$. By the previous theorem, there exists a ϑ_i-open set U_a such that $a \in U_a \subset \overline{U_a} \subset W_a$ and $\overline{U_a}$ is tri-compact.

The collection $\{U_a : a \in K\}$ forms an open cover of K. Since K is tri-compact, there exists a finite subset $\{a_1, a_2, \ldots, a_m\} \subset K$ such that $K \subset \bigcup_{j=1}^{m} U_{a_j}$. Let $U = \bigcup_{j=1}^{m} U_{a_j}$ and $V = \bigcap_{j=1}^{m} V_{a_j}$. Then U is a ϑ_i-open set containing K, and V is a ϑ_i-open set containing F. Furthermore, $U \cap V = \emptyset$ because $U_{a_j} \cap V_{a_j} = \emptyset$ for each $j = 1, 2, \ldots, m$.

Theorem 15. *Let $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ be a tri-T_2-space. If X is tri-locally compact, then for any tri-compact set K and any ϑ_i-open set U containing K, there exists a ϑ_i-open set V with compact closure such that $K \subset V \subset \overline{V} \subset U$.*

Proof. *Let K be a tri-compact set and U be a ϑ_i-open set containing K. For each $a \in K$, by tri-local compactness, there exists a ϑ_i-open set V_a such that $a \in V_a \subset \overline{V_a} \subset U$ and $\overline{V_a}$ is tri-compact.*

The collection $\{V_a : a \in K\}$ forms an open cover of K. Since K is tri-compact, there exists a finite subset $\{a_1, a_2, \ldots, a_n\} \subset K$ such that $K \subset \bigcup_{j=1}^{n} V_{a_j}$. Let $V = \bigcup_{j=1}^{n} V_{a_j}$. Then V is a ϑ_i-open set and $K \subset V \subset \overline{V} \subset \bigcup_{j=1}^{n} \overline{V_{a_j}} \subset U$. Since each $\overline{V_{a_j}}$ is tri-compact, their finite union $\bigcup_{j=1}^{n} \overline{V_{a_j}}$ is tri-compact. Therefore, $\overline{V}$ is tri-compact as a closed subset of a tri-compact set.

Theorem 16. *Let $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ be a tri-topological space. If X is tri-locally compact and tri-T_3, then X is tri-regular.*

Proof. *Let $a \in X$ and F be a ϑ_i-closed set such that $a \notin F$. Since X is tri-T_3, it is tri-T_1, which means $\{a\}$ is a tri-compact set. By the previous theorem, there exist ϑ_i-open sets U and V such that $\{a\} \subset U$, $F \subset V$, and $U \cap V = \emptyset$. Therefore, X is tri-regular.*

Theorem 17. *Let $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ be a tri-topological space. If X is tri-locally compact and tri-T_2, then for any two disjoint tri-compact sets K_1 and K_2, there exist ϑ_i-open sets U_1 and U_2 such that $K_1 \subset U_1$, $K_2 \subset U_2$, and $U_1 \cap U_2 = \emptyset$.*

Proof. *Let K_1 and K_2 be disjoint tri-compact sets. For each $a \in K_1$ and $b \in K_2$, since X is tri-T_2, there exist ϑ_i-open sets U_a and V_b such that $a \in U_a$, $b \in V_b$, and $U_a \cap V_b = \emptyset$.*

For each $a \in K_1$, the collection $\{V_b : b \in K_2\}$ forms an open cover of K_2. Since K_2 is tri-compact, there exists a finite subset $\{b_1, b_2, \ldots, b_{n_a}\} \subset K_2$ such that $K_2 \subset \bigcup_{j=1}^{n_a} V_{b_j}$. Let $V_a = \bigcup_{j=1}^{n_a} V_{b_j}$. Then V_a is a ϑ_i-open set containing K_2 and $U_a \cap V_a = \emptyset$.

The collection $\{U_a : a \in K_1\}$ forms an open cover of K_1. Since K_1 is tri-compact, there exists a finite subset $\{a_1, a_2, \ldots, a_m\} \subset K_1$ such that $K_1 \subset \bigcup_{j=1}^{m} U_{a_j}$. Let $U_1 = \bigcup_{j=1}^{m} U_{a_j}$ and $U_2 = \bigcap_{j=1}^{m} V_{a_j}$. Thus K_1 is an element of U_1 which is a ϑ_i open set and similarly for K_2 and U_2. Furthermore, $U_1 \cap U_2 = \emptyset$ because $U_{a_j} \cap V_{a_j} = \emptyset$ for each $j = 1, 2, \ldots, m$.

4. Tri-locally metacompact spaces

In this part, we raise the idea of metacompactness to tri topological space and discuss the connection of tri locally compact and tri locally metacompact.

Definition 22. *A family $\mathcal{U}$ of subsets of a tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is called point-finite if each point of X belongs to at most finitely many members of $\mathcal{U}$.*

Definition 23. *A tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is tri-metacompact if every open cover of X has a point-finite open refinement.*

Definition 24. *A tri-topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is tri-locally metacompact if for every point $a \in X$, there exists a ϑ_i-open set U_a containing a such that $\overline{U_a}$ is tri-metacompact.*

Theorem 18. *Every tri-locally compact space is tri-locally metacompact.*

Proof. Let $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ be a tri-locally compact space. For every point $a \in X$, there exists a ϑ_i-open set U_a containing a such that $\overline{U_a}$ is tri-compact. Since every tri-compact space is tri-metacompact, $\overline{U_a}$ is tri-metacompact. Therefore, $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ is tri-locally metacompact.

Theorem 19. *Let $(X, \vartheta_1, \vartheta_2, \vartheta_3)$ be a tri-topological space. If X is tri-locally metacompact and tri-T_3, then for any tri-metacompact set M and any ϑ_i-open set U containing M, there exists a ϑ_i-open set V such that $M \subset V \subset \overline{V} \subset U$ and $\overline{V}$ is tri-metacompact.*

Proof. Let M be a tri-metacompact set and U be a ϑ_i-open set containing M. For each $a \in M$, by tri-local metacompactness, there exists a ϑ_i-open set V_a such that $a \in V_a \subset \overline{V_a} \subset U$ and $\overline{V_a}$ is tri-metacompact.

The collection $\{V_a : a \in M\}$ forms an open cover of M. Since M is tri-metacompact, there exists a point-finite open refinement $\{W_\alpha : \alpha \in \Lambda\}$ of $\{V_a : a \in M\}$. For each $\alpha \in \Lambda$, there exists $a_\alpha \in M$ such that $W_\alpha \subset V_{a_\alpha}$.

Let $V = \bigcup_{\alpha \in \Lambda} W_\alpha$. Then V is a ϑ_i-open set and $M \subset V \subset \overline{V} \subset \overline{\bigcup_{\alpha \in \Lambda} W_\alpha} \subset \overline{\bigcup_{\alpha \in \Lambda} V_{a_\alpha}} \subset U$. Since each $\overline{V_{a_\alpha}}$ is tri-metacompact and the collection $\{W_\alpha : \alpha \in \Lambda\}$ is point-finite, $\overline{V}$ is tri-metacompact.

5. Conclusion

In this paper, we have introduced and studied the concept of tri-locally compact spaces in the context of tri-topological spaces. Several fundamental properties and theorems regarding tri-locally compact spaces are proved and their connections with tri regular spaces and tri T_3 spaces are established. For that, we have also shown that every tri-locally compact space is tri-locally metacompact, and then we have explored the connection between tri-locally compact spaces and tri-locally metacompact spaces.

The behavior of tri-locally compact spaces under various continuous operations, such as continuous mappings, products, and quotients could be further researched. One could also *explore the relationship between tri-locally compact spaces and other types of tri-topological spaces in general, i.e. between tri-locally compact spaces and tri-paracompact spaces and tri-Lindelf spaces.*

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