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Quantization of Singular Systems using Fractional Calculus

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Abstract. In this paper, we examined the theory of singular systems using fractional calculus. We quantized these systems using the fractional **WKB** approximation. We applied the Hamilton–Jacobi treatment for these systems. We obtained equations of motion. We constructed the fractional Hamilton–Jacobi partial differential equations (**FHJPDEs**) to obtain the action functions **S**. The action function enables us to obtain the wave function for these systems. We achieved that the quantum results agree with the classical results. Finally, we examined two mathematical examples to demonstrate the theory.

2020 Mathematics Subject Classifications: 34A08, 70H20, 70H45, 81Q05, 81Q20

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1. Introduction

The quantization of singular systems has been treated with more interest by Dirac's work for quantizing the gravitational field [1][2]. Following Dirac's work, researchers developed the canonical method for investigating these systems [3][4][5][6], they used this method to quantize these systems using path integral technique and WKB approximation [4][5][6]. A general theory has been investigated for quantizing higher-order singular systems using WKB approximation by Hasan et al [6]. In this theory, researchers have achieved that the quantum results approach the classical results.

In this paper, we would like to apply the fractional derivatives for this theory of singular systems. The quantization of fractional singular Lagranians has been studied for physical systems and fractional Lagrangians systems with second-order derivatives have been treated with more interest and importance [7][8]. Researchers have investigated Hamilton-Jacobi formalism for these systems within fractional derivatives. They constructed the Euler-Lagrange equations and analyzed Hamilton's equations [8]. More

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recently, researchers have constructed formalism using the canonical method for quantization singular systems using different techniques as path integral approach and WKB approximation for first-order derivatives [7]. The equations of motion are calculated in fractional form as total differential equations. In addition, the solution of this set HJPDEs in fractional form and the fractional Hamilton-Jacobi function or action function (HJF) S are obtained.

In this paper, we would like to extend the work for fractional singular Lagrangians systems with second-order derivatives and quantize these systems using a new approach which it is called WKB approximation. We constructed a formalism for investigating fractional singular Lagrangian systems and Hamilton-Jacobi formalism for second-order derivatives. Besides, we constructed the fractional (HJF) **S** to build the appropriate wave function to quantize the singular systems.

Now, we will define the most important formula of fractional calculus as Left Riemann–Liouville derivatives [9].

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \tag{1} $$

and the Right Riemann–Liouville derivative is given by:

$$ {}_tD_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau \tag{2} $$

and these derivatives have properties as follows:

$$ {}_aD_t^{\alpha} f(t) = \left(\frac{d}{dt}\right)^{\alpha} f(t) \tag{3} $$

$$ {}_tD_b^{\alpha} f(t) = \left(-\frac{d}{dt}\right)^{\alpha} f(t) \tag{4} $$

In this work, we aim to construct the formalism for quantizing singular **Lagrangians** systems with second-order derivatives within the framework of fractional derivatives.

2. Fractional Singular Lagrangian and Hamilton-Jacobi Formalism with Second-Order Derivatives

In this section, we will formulate the second-order singular Lagrangian within framework of fractional calculus. We will start with fractional derivative Lagrangian is defined by [7].

$$ L = L(D^{\alpha-1}q_i, D^{\alpha}q_i, D^{2\alpha}q_i, t) \tag{5} $$

Thus, the fractional of the Hessian matrix is defined as

$$ W_{ij} = \frac{\partial^2 L}{\partial D^{2\alpha}q_i \partial D^{2\alpha}q_j}, \quad i,j = 1,2,\ldots,N, \quad i,j = 1,2,\ldots,N \tag{6} $$

If **its** rank is N, the **Lagrangian** is called regular otherwise the **Lagrangian** is singular

$$N - R, R < N.$$

Now, we can define the momenta π_i conjugate to the coordinates $D^{\alpha}q_i$ as:

$$\pi_a = \frac{\partial L}{\partial D^{2\alpha} q_a} \tag{7}$$

$$\pi_\mu = \frac{\partial L}{\partial D^{2a} q_\mu} \tag{8}$$

The rank of the Hessian matrix is $N - R$, therefore, Eq. (7) can be solved to obtain $N - R$ accelerations $D^{2\alpha}q_a$ in terms of $D^{\alpha-1}q_i, D^{\alpha}q_i, \pi_a$ and $D^{2\alpha}q_\mu$ as follows:

$$D^{2\alpha} q_a = W_a(D^{\alpha-1}q_i, D^{\alpha}q_i, \pi_a, D^{2\alpha}q_\mu) \tag{9}$$

Substituting (9) in (8), we can obtain:

$$\pi_\mu = -H_\mu^{\pi}(D^{\alpha-1}q_i, D^{\alpha}q_i, \rho_a, \pi_a) \tag{10}$$

A similar expression for the momenta p_μ can be obtained as:

$$p_\mu = -H_\mu^{p}(D^{\alpha-1}q_i, D^{\alpha}q_i, p_a, \pi_a) \tag{11}$$

and the momenta p_i corresponding to the coordinate $D^{\alpha-1}q_i$ can be **written as**:

$$p_a = \frac{\partial L}{\partial D^{\alpha} q_a} - \frac{d}{dt}\left(\frac{\partial L}{\partial D^{2\alpha} q_a}\right) \tag{12a}$$

$$p_\mu = \frac{\partial L}{\partial D^{\alpha} q_\mu} - \frac{d}{dt}\left(\frac{\partial L}{\partial D^{2\alpha} q_\mu}\right) \tag{12b}$$

where $a = 1, 2, \ldots, N - R$, $\mu = 1, \ldots, R$.
we can write equations (10) and (11) as follows

$$H_\mu^{\prime p}(D^{\alpha-1}q_i, D^{\alpha}q_i, p_i, \pi_i) = p_\mu + H_\mu^{p} = 0 \tag{13a}$$

$$H_\mu^{\prime \pi}(D^{\alpha-1}q_i, D^{\alpha}q_i, p_i, \pi_i) = \pi_\mu + H_\mu^{\pi} = 0 \tag{13b}$$

Thus, equations (13) represent primary constraints [1][2].

The Hamiltonian formalism for higher-order derivatives has been studied by Ostrogradski [10].

He treated the derivatives as coordinates. Therefore, one can treat $D^{\alpha-1}q_i$ and $D^{\alpha}q_i$ as coordinates. So, the Poisson bracket for second-order derivatives can be defined as:

$$\{A, B\} = \frac{\partial A}{\partial D^{\alpha-1} q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial D^{\alpha-1} q_i} + \frac{\partial A}{\partial D^{\alpha} q_i}\frac{\partial B}{\partial \pi_i} - \frac{\partial A}{\partial \pi_i}\frac{\partial B}{\partial D^{\alpha} q_i}$$

Where A and B are functions described in terms of canonical variables $D^{\alpha-1}q_i, D^{\alpha}q_i, p_i$, and π_i.

Here, the generalized momenta p_i and π_i are conjugated to the generalized coordinates $D^{\alpha-1}q_i$ and $D^{\alpha}q_i$ respectively. Thus, the fundamental Poisson brackets are:

$$\{D^{\alpha-1}q_i, D^{\alpha-1}q_j\} = \{D^{\alpha}q_i, D^{\alpha}q_j\} = 0 = \{D^{\alpha}q_i, D^{\alpha-1}q_j\} = \{p_i, \pi_i\}$$

$$\{D^{\alpha-1}q_i, p_j\} \equiv \delta_{ij} \quad \text{and} \quad \{D^{\alpha}q_i, \pi_j\} \equiv \delta_{ij}$$

where $\quad i, j = 1, \ldots, N$

It is well known from Dirac's formalism that the number of degrees of freedom can be reduced due to the constraints [1][2].

Thus, the Hamiltonian $H_\circ$ can be defined as

$$H_\circ = -L(D^{\alpha-1}q_i, D^{\alpha}q_\mu, D^{2\alpha}q_\mu, W_a) + P_{\text{a}}D^{\alpha}q_a + \pi_a D^{2\alpha}q_a - D^{\alpha}q_\mu H^p_\mu - D^{2\alpha}q_\mu H^\pi_\mu \quad (14)$$

where $\quad \mu = 1, \ldots, R \quad$ and $\quad a = R+1, \ldots, N.$

Because of the nature of singular Lagrangian, the momenta p_μ and π_μ are not independent of p_a and π_a.

Thus, the set of fractional FHJPDEs is written as [11].

$$H'_\circ(D^{\alpha-1}q_i, D^{\alpha}q_i, \frac{\partial S}{\partial D^{\alpha-1}q_a}, \frac{\partial S}{\partial D^{\alpha-1}q_\mu}, \frac{\partial S}{\partial D^{\alpha}q_a}, \frac{\partial S}{\partial D^{\alpha}q_\mu}) = p_\circ + H_\mu = 0 \quad (15a)$$

$$H'^p_\mu(D^{\alpha-1}q_i, D^{\alpha}q_i, \frac{\partial S}{\partial D^{\alpha-1}q_a}, \frac{\partial S}{\partial D^{\alpha-1}q_\mu}, \frac{\partial S}{\partial D^{\alpha}q_a}, \frac{\partial S}{\partial D^{\alpha}q_\mu}) = p_\mu + H^p_\mu = 0 \quad (15b)$$

$$H'^\pi_\mu(D^{\alpha-1}q_i, D^{\alpha}q_i, \frac{\partial S}{\partial D^{\alpha-1}q_a}, \frac{\partial S}{\partial D^{\alpha-1}q_\mu}, \frac{\partial S}{\partial D^{\alpha}q_a}, \frac{\partial S}{\partial D^{\alpha}q_\mu}) = \pi_\mu + H^\pi_\mu = 0 \quad (15c)$$

Here, the fractional Hamilton's function can be written as

$$S = S(D^{\alpha-1}q_a, D^{\alpha-1}q_\mu, D^{\alpha}q_a, D^{\alpha}q_\mu, t) \quad (16)$$

and we can define:

$$p_a = \frac{\partial S}{\partial D^{\alpha-1}q_a}, \quad p_\mu = \frac{\partial S}{\partial D^{\alpha-1}q_\mu}, \quad \pi_a = \frac{\partial S}{\partial D^{\alpha}q_a}, \quad \pi_\mu = \frac{\partial S}{\partial D^{\alpha}q_\mu}, \quad P_o = \frac{\partial S}{\partial t}$$

Thus, we can write the fractional equations of motion as follows [11]:

$$dD^{\alpha-1}q_a = \frac{\partial H'_\circ}{\partial p_a}dt + \frac{\partial H'^p_\mu}{\partial p_a}dD^{\alpha-1}q_\mu + \frac{\partial H'^\pi_\mu}{\partial p_a}dD^\alpha q_\mu \tag{17a}$$

$$dD^\alpha q_a = \frac{\partial H'_\circ}{\partial \pi_a}dt + \frac{\partial H'^p_\mu}{\partial \pi_a}dD^{\alpha-1}q_\mu + \frac{\partial H'^\pi_\mu}{\partial \pi_a}dD^\alpha q_\mu \tag{17b}$$

$$-dp_i = \frac{\partial H'_\circ}{\partial D^{\alpha-1}q_i}dt + \frac{\partial H'^p_\mu}{\partial D^{\alpha-1}q_i}dD^{\alpha-1}q_\mu + \frac{\partial H'^\pi_\mu}{\partial D^{\alpha-1}q_i}dD^\alpha q_\mu \tag{17c}$$

$$-d\pi_i = \frac{\partial H'_\circ}{\partial D^\alpha q_i}dt + \frac{\partial H'^p_\mu}{\partial D^\alpha q_i}dD^{\alpha-1}q_\mu + \frac{\partial H'^\pi_\mu}{\partial D^\alpha q_i}dD^\alpha q_\mu \tag{17d}$$

If the total derivative of equation (15) is zero [3],

$$dH'_\circ = 0\,; \quad dH'^p_\mu = 0; \quad dH'^\pi_\mu = 0 \tag{18}$$

This means that equations (17) are integrable, and the rank of Hessian matrix is $N - R$. Because of constraints, the degrees of freedom are reduced from N to $N - R$, thus, the canonical coordinates transform from

$$\{D^{\alpha-1}q_i, p_i, D^\alpha q_i, \pi_i\}$$

to

$$\{D^{\alpha-1}q_a, P_a, D^\alpha q_a, \pi_a\}.$$

Thus, we can write Eqs. (15) as follows:

$$\frac{\partial S}{\partial t} + H_\circ(D^{\alpha-1}q_i, D^\alpha q_i, \rho_a, \pi_a) = 0 \tag{19a}$$

$$\frac{\partial S}{\partial D^{\alpha-1}q_\mu} + H^p_\mu(D^{\alpha-1}q_i, D^\alpha q_i, P_a, \pi_a) = 0 \tag{19b}$$

$$\frac{\partial S}{\partial D^\alpha q_\mu} + H^\pi_\mu(D^{\alpha-1}q_i, D^\alpha q_i, P_a, \pi_a) = 0 \tag{19c}$$

3. Hamilton-Jacobi Function and Quantization using WKB approximation

Following refs. [6] for investigating the Hamilton-Jacobi function and quantization using WKB approach for higher-order singular Lagrangian systems, the fractional Hamilton-Jacobi function can be written as:

$$\begin{aligned} S(D^{\alpha-1}q_a, D^{\alpha-1}q_\mu, D^\alpha q_a, D^\alpha q_\mu, t) = f(t) + W_a(D^{\alpha-1}q_a, E_a) + W'_a(D^\alpha q_a, E_a, E'_a) \\ + f_\mu(D^{\alpha-1}q_\mu) + f'_\mu(D^\alpha q_\mu) + A. \end{aligned} \tag{20}$$

In this case, we would like to write a general solution for Eqs. (19) in a separable form of Eq. (20).

Here, $f(t)$ has the solution

$$f(t) = -\sum_{a=1}^{N-R} E_a t,$$

where E'_a are representing as constants of integration; $D^{\alpha-1}q_\mu$ and $D^{\alpha}q_\mu$ are independent variables, and the remaining functions $W_a(D^{\alpha-1}q_a, E_a)$, $W'_a(D^{\alpha}q_a, E_a, E'_a)$, $f_\mu(D^{\alpha-1}q_\mu)$, and $f'_\mu(D^{\alpha}q_\mu)$ are time-independent.

Now, we can use the canonical transformations [12] to get the motion equations as follows:

$$\eta_a = \frac{\partial S}{\partial E'_a} \tag{21a}$$

$$\lambda_a = \frac{\partial S}{\partial E_a} \tag{21b}$$

$$p_i = \frac{\partial S}{\partial D^{\alpha-1}q_i} \tag{21c}$$

$$\pi_i = \frac{\partial S}{\partial D^{\alpha}q_i} \tag{21d}$$

η_a and λ_a are constants.

Because of constraints, the fractional wave function Ψ for these systems can be written as [6].

$$\Psi(D^{\alpha-1}q_a, D^{\alpha-1}q_\mu, D^{\alpha}q_a, D^{\alpha}q_\mu, t) = \left[\prod_{a=1}^{N-R} \psi_{0a}(D^{\alpha-1}q_a)\varphi_{0a}(D^{\alpha}q_a)\right] \times \exp\left(\frac{iS(D^{\alpha-1}q_a, D^{\alpha-1}q_\mu, D^{\alpha}q_a, D^{\alpha}q_\mu, t)}{\hbar}\right). \tag{22a}$$

$$\psi_{0a} = \frac{1}{\sqrt{P(D^{\alpha-1}q_a)}} \tag{22b}$$

$$\varphi_{0a} = \frac{1}{\sqrt{\pi(D^{\alpha}q_a)}} \tag{22c}$$

and the fractional wave function Eq. (22a) must satisfy the following conditions:

$$\hat{H}'_0\Psi = 0,\ \hat{H}'^{P}_{\mu}\Psi = 0,\ and\ \hat{H}'^{\pi}_{\mu}\Psi = 0 \tag{23}$$

Here, the generalized coordinates and momenta are written in fractional form as operators:

$$D^{\alpha-1}q_i \to D^{\alpha-1}q_i \tag{24a}$$

$$D^{\alpha}q_i \to D^{\alpha}q_i \tag{24b}$$

$$p_i \to \hat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial D^{\alpha-1}q_i} \tag{24c}$$

$$\pi_i \to \hat{\pi}_i = \frac{\hbar}{i}\frac{\partial}{\partial D^{\alpha}q_i} \tag{24d}$$

$$p_0 \longrightarrow \hat{p}_0 = \frac{\hbar}{i}\frac{\partial}{\partial t} \tag{24e}$$

4. Examples

Example 1: Fractional Regular Lagrangian

$$L = \frac{1}{2}\left((D^{2\alpha}q)^2 - (D^{\alpha}q)^2\right) \tag{25}$$

The momenta are:

$$p = -D^{\alpha}q - D^{3\alpha}q \tag{26a}$$

$$\pi = D^{2\alpha}q \tag{26b}$$

The Hamiltonian H_0 is written as

$$H_\circ = pD^{\alpha}q + \frac{1}{2}\pi^2 + \frac{1}{2}(D^{\alpha}q)^2 \tag{27}$$

The corresponding set of fractional HJPDE are:

$$H'_\circ = p_\circ + H_\circ = p_\circ + pD^{\alpha}q + \frac{1}{2}\pi^2 + \frac{1}{2}(D^{\alpha}q)^2 \tag{28}$$

We note in this example that there are no primary constraints [1][2].
The motion equations are written as

$$dD^{\alpha-1}q = \frac{\partial H'_\circ}{\partial p}dt = D^{\alpha}qdt \tag{29a}$$

$$dD^{\alpha}q = \frac{\partial H'_\circ}{\partial \pi}dt = \pi dt \tag{29b}$$

$$-dp = \frac{\partial H'_\circ}{\partial D^{\alpha-1}q}dt = 0 \tag{29c}$$

$$-d\pi = \frac{\partial H'_{\circ}}{\partial D^{\alpha}q}dt = (p + D^{\alpha}q)dt \tag{29d}$$

The **HJPDE**, Eq. (19a), reads

$$H'_{\circ} = p_{\circ} + H_{\circ} = \frac{\partial S}{\partial t} + D^{\alpha}q\frac{\partial S}{\partial D^{\alpha-1}q} + \frac{1}{2}\left(\frac{\partial S}{\partial D^{\alpha}q}\right)^2 + \frac{1}{2}(D^{\alpha}q)^2 = 0 \tag{30}$$

Substituting Eq. (20) into (30), we have:

$$\frac{\partial f}{\partial t} + D^{\alpha}q\frac{\partial W}{\partial D^{\alpha-1}q} + \frac{1}{2}\left(\frac{\partial W}{\partial D^{\alpha}q}\right)^2 + \frac{1}{2}(D^{\alpha}q)^2 = 0 \tag{31}$$

Since H_0 is time-independent, we can write $f(t) = -E't$. Eq. (31) can then be written as

$$-E' + D^{\alpha}q\frac{\partial W}{\partial D^{\alpha-1}q} + \frac{1}{2}\left(\frac{\partial W}{\partial D^{\alpha}q}\right)^2 + \frac{1}{2}(D^{\alpha}q)^2 = 0 \tag{32}$$

From Eq. (32), the function W depends only on $D^{\alpha-1}q$ and W' depends only on $D^{\alpha}q$. This means that

$$\frac{\partial W}{\partial D^{\alpha-1}q} = E \tag{33a}$$

$$W = D^{\alpha-1}qE \tag{33b}$$

Substituting Eq. (33b) into (32), one can obtain:

$$-E' + D^{\alpha}qE + \frac{1}{2}\left(\frac{\partial W'}{\partial D^{\alpha}q}\right)^2 + \frac{1}{2}(D^{\alpha}q)^2 = 0 \tag{34}$$

This equation gives

$$W'(D^{\alpha}q, E, E') = \int \sqrt{2E' + E^2 - (D^{\alpha}q + E)^2}\, dD^{\alpha}q \tag{35}$$

Thus, one can obtain the Hamilton-Jacobi function as:

$$S(D^{\alpha-1}q, D^{\alpha}q, E, E') = -E't + D^{\alpha-1}qE + \int \sqrt{2E' + E^2 - (D^{\alpha}q + E)^2}\, dD^{\alpha}q + A \tag{36}$$

By using Eqs. (21a, 21b), we can obtain the solutions for the coordinates.

$$\eta = \frac{\partial S}{\partial E'} = -t + \int \frac{dD^{\alpha}q}{\sqrt{2E' + E^2 - (D^{\alpha}q + E)^2}} \tag{37a}$$

$$\lambda = \frac{\partial S}{\partial E} = D^{\alpha-1}q + \int \frac{[E - (D^{\alpha}q + E)]}{\sqrt{2E' + E^2 - (D^{\alpha}q + E)^2}} dD^{\alpha}q \tag{37b}$$

The generalized momenta can be determined by using Eqs. (21c, 21d).

$$p = \frac{\partial S}{\partial D^{\alpha-1}q} = E \tag{38a}$$

$$\pi = \frac{\partial S}{\partial D^{\alpha}q} = \sqrt{2E' + E^2 - (D^{\alpha}q + E)^2} \tag{38b}$$

Thus, the wave function is calculated as:

$$\Psi(D^{\alpha-1}q, D^{\alpha}q, t) = \left[\psi_{01}(D^{\alpha-1}q)\varphi_{01}(D^{\alpha}q)\right] \exp\left(\frac{iS(D^{\alpha-1}q, D^{\alpha}q, t)}{\hbar}\right) \tag{39}$$

Where

$$\psi_{01} = \frac{1}{\sqrt{p(D^{a-1}q)}} = [E]^{-\frac{1}{2}} \tag{40a}$$

$$\varphi_{01} = \frac{1}{\sqrt{\pi(D^{\alpha}q)}} = \left[2E' + E^2 - (D^{\alpha}q + E)^2\right]^{-\frac{1}{4}} \tag{40b}$$

and the function S is calculated by Eq. (36).

Now, we can apply the fractional HJPDE, Eq. (28), to the function Ψ, we represent the generalized coordinates and momenta as operators:

$$\hat{H}'_{\circ}\Psi = \left[\frac{\hbar}{i}\frac{\partial}{\partial t} + D^{\alpha}q\frac{\hbar}{i}\frac{\partial}{\partial D^{\alpha-1}q} - \frac{\hbar^2}{2}\frac{\partial^2}{\partial(D^{\alpha}q)^2} + \frac{1}{2}(D^{\alpha}q)^2\right]\Psi \tag{41}$$

After some algebra, we have

$$\frac{\hbar}{i}\frac{\partial}{\partial t}\Psi = -E'\Psi \tag{42a}$$

$$\frac{\hbar}{i}\frac{\partial}{\partial D^{\alpha-1}q}\Psi = E\Psi \tag{42b}$$

$$\begin{aligned} -\frac{\hbar^2}{2}\frac{\partial}{\partial(D^{\alpha}q)}\Psi = \Bigg[&-\frac{5\hbar^2}{8}(D^{\alpha}q + E)^2(2E' + E^2 - (D^{\alpha}q + E)^2)^{-2} \\ &-\frac{\hbar^2}{4}(2E' + E^2 - (D^{\alpha}q + E)^2)^{-1} + \frac{1}{2}(2E' + E^2 - (D^{\alpha}q + E)^2)\Bigg]\Psi. \end{aligned} \tag{42c}$$

Thus, Eq. (41) becomes

$$\hat{H}'_{\circ}\Psi = \begin{bmatrix} -E' + D^{a}qE - \frac{5\hbar^2}{8}(D^{a}q + E)^2(2E' + E^2 - (D^{a}q + E)^2)^{-2} - \\ \frac{\hbar^2}{4}(2E' + E^2 - (D^{a}q + E)^2)^{-1} + \frac{1}{2}(2E' + E^2 - (D^{a}q + E)^2) + \frac{1}{2}(D^{a}q)^2 \end{bmatrix}\Psi \tag{43}$$

Taking the semiclassical limit $\hbar \to 0$ in Eq. (43), we obtain

$$\hat{H}'_{\circ}\Psi = \left[-E' + D^{\alpha}qE + \frac{1}{2}\left(2E' + E^2 - (D^{\alpha}q + E)^2\right) + \frac{1}{2}(D^{\alpha}q)^2\right]\Psi = 0 \tag{44}$$

Example 2: We will discuss the following mathematical singular Lagrangian with two primary first-class constraints

$$L = \frac{1}{2}\left((D^{2\alpha}q_1)^2 + (D^{2\alpha}q_2)^2\right) + D^{\alpha}q_3D^{2\alpha}q_3 + D^{\alpha}q_3D^{\alpha-1}q_3 + D^{\alpha-1}q_2D^{\alpha}q_2 \tag{45}$$

The corresponding generalized momenta, Eqs. (7,8) and (12) are:

$$p_1 = -D^{3\alpha}q_1 \tag{46a}$$

$$p_2 = D^{\alpha-1}q_2 - D^{3\alpha}q_2 \tag{46b}$$

$$p_3 = D^{\alpha-1}q_3 = -H_3^p \tag{46c}$$

$$\pi_1 = D^{2\alpha}q_1 \tag{46d}$$

$$\pi_2 = D^{2\alpha}q_2 \tag{46e}$$

$$\pi_3 = D^{\alpha}q_3 = -H_3^{\pi} \tag{46f}$$

Here, Equations (46c) and (46f) can be written as:

$$\hat{H}_3^p = p_3 - D^{\alpha-1}q_3 = 0 \tag{47a}$$

$$\hat{H}_3^{\pi} = \pi_3 - D^{\alpha}q_3 = 0 \tag{47b}$$

and represent as primary constraints [1][2].
We calculate the Hamiltonian H_0 as

$$H_{\circ} = p_1D^{\alpha}q_1 + (p_2 - D^{\alpha-1}q_2)D^{\alpha}q_2 + \frac{1}{2}(\pi_1^2 + \pi_2^2) \tag{48}$$

The set of fractional HJPDEs, Eqs. (15), reads:

$$H'_{\circ} = p_{\circ} + H_{\circ} = p_1D^{\alpha}q_1 + (p_2 - D^{\alpha-1}q_2)D^{\alpha}q_2 + \frac{1}{2}(\pi_1^2 + \pi_2^2) \tag{49a}$$

$$H_3^{\prime p} = p_3 - D^{\alpha-1}q_3 = 0. \tag{49b}$$

$$H_3'^{\pi} = \pi_3 - D^{\alpha} q_3 = 0. \tag{49c}$$

Here, the Poisson brackets

$$\{H_3'^p, H_\circ'\} = 0, \quad \{H_3'^\pi, H_\circ'\} = 0 \quad \text{and} \quad \{H_3'^p, H_3'^\pi\} = 0$$

The set of fractional HJPDEs, Eqs. (19), reads:

$$H_\circ' = p_\circ + H_\circ = \frac{\partial S}{\partial t} + D^{\alpha} q_1 \frac{\partial S}{\partial D^{\alpha-1} q_1} + D^{\alpha} q_2 \left(\frac{\partial S}{\partial D^{\alpha-1} q_2} - D^{\alpha-1} q_2 \right) + \frac{1}{2} \left(\frac{\partial S}{\partial D^{\alpha} q_1} \right)^2 + \frac{1}{2} \left(\frac{\partial S}{\partial D^{\alpha} q_2} \right)^2 = 0. \tag{50a}$$

$$H_3'^p = \frac{\partial S}{\partial D^{a-1} q_3} - D^{a-1} q_3 = 0. \tag{50b}$$

$$H_3'^\pi = \frac{\partial S}{\partial D^{a} q_3} - D^{a} q_3 = 0. \tag{50c}$$

The function S Eq. (20) can be written as

$$S(D^{a-1}q_1, D^{a-1}q_2, D^{a-1}q_3, D^a q_1 D^a q_2, D^a q_3, t) = f(t) + W_1(D^{a-1}q_1, E_1) + W_2(D^{a-1}q_2, E_2) + W_1'(D^a q_1, E_1, E_1') + W_2'(D^a q_2, E_2, E_2') + f_3(D^{a-1}q_3) + f_3'(D^a q_3) + A \tag{51}$$

Since H_0 is time-independent and the coordinates $D^{\alpha-1}q_3$ and $D^{\alpha}q_3$ are treated as independent variables, one can write

$$f(t) = -\left(E_1' + E_2'\right) t$$

Substituting S into Eq. (50a), we have

$$-E_1' + D^{\alpha} q_1 \frac{\partial W_1}{\partial D^{\alpha-1} q_1} + \frac{1}{2} \left(\frac{\partial W_1}{\partial D^{\alpha} q_1} \right)^2 - E_2' + D^{\alpha} q_2 \left(\frac{\partial W_2}{\partial D^{\alpha-1} q_2} - D^{\alpha} q_2 \right) + \frac{1}{2} \left(\frac{\partial W_2}{\partial D^{\alpha} q_2} \right)^2 = 0 \tag{52}$$

We note that W_1 depends only on $D^{\alpha-1}q_1$ and W_2 depends only on $D^{\alpha-1}q_2$. We can then write

$$\frac{\partial W_1}{\partial D^{\alpha-1} q_1} = E_1$$

so that

$$W_1 = D^{\alpha-1}q_1 E_1 \tag{53a}$$

and

$$\frac{\partial W_2}{\partial D^{\alpha-1}q_2} - D^{\alpha-1}q_2 = E_2$$

so that

$$W_2 = D^{\alpha-1}q_2 E_2 + \frac{1}{2}(D^{\alpha-1}q_2)^2 \tag{53b}$$

Substituting Eqs. (53) into (52), we get:

$$-E_1' + D^{\alpha}q_1 E_1 + \frac{1}{2}\left(\frac{\partial W_1'}{\partial D^{\alpha}q_1}\right)^2 - E_2' + D^{\alpha}q_2 E_2 + \frac{1}{2}\left(\frac{\partial W_2'}{\partial D^{\alpha}q_2}\right)^2 = 0 \tag{54}$$

Separation of variables in this equation yields

$$\frac{1}{2}\left(\frac{\partial W_1'}{\partial D^{\alpha}q_1}\right)^2 + D^{\alpha}q_1 E_1 - E_1' = 0 \tag{55a}$$

$$\frac{1}{2}\left(\frac{\partial W_2'}{\partial D^{\alpha}q_2}\right)^2 + D^{\alpha}q_2 E_2 - E_2' = 0 \tag{55b}$$

We solve the above two Eqs. (55) to obtain:

$$W_1'(D^{\alpha}q_1, E_1, E_1') = \int \sqrt{2E_1' - 2D^{\alpha}q_1 E_1}\, dD^{\alpha}q_1 \tag{56a}$$

$$W_2'(D^{\alpha}q_2, E_2, E_2') = \int \sqrt{2E_2' - 2D^{\alpha}q_2 E_2}\, dD^{\alpha}q_2 \tag{56b}$$

Using Eq. (50b), we find:

$$f_3(D^{\alpha-1}q_3) = \frac{1}{2}(D^{\alpha-1}q_3)^2$$

and using Eq. (50c), we find:

$$f_3'(D^{\alpha}q_3) = \frac{1}{2}(D^{\alpha}q_3)^2$$

Thus, the function S can be written as

$$S(D^{\alpha-1}q_1, D^{\alpha-1}q_2, D^{\alpha-1}q_3, D^{\alpha}q_1, D^{\alpha}q_2, D^{\alpha}q_3, t) = (-E_1' - E_2')t + D^{\alpha-1}q_1 E_1 + \frac{1}{2}(D^{\alpha-1}q_2)^2$$
$$+ \int \sqrt{2E_1' - 2D^{\alpha}q_1 E_1}\, dD^{\alpha}q_1 + \int \sqrt{2E_2' - 2D^{\alpha}q_2 E_2}\, dD^{\alpha}q_2$$

$$+\frac{1}{2}(D^{\alpha-1}q_3)^2+\frac{1}{2}(D^{\alpha}q_3)^2+A. \quad (57)$$

Now, we can use the transformations Eqs. (21a, 21b) to obtain the solutions for the coordinates as:

$$\eta_1=\frac{\partial S}{\partial E_1'}=-t+\int\frac{dD^{a}q_1}{\sqrt{2E_1'-2D^{a}q_1E_1}} \quad (58a)$$

$$\eta_2=\frac{\partial S}{\partial E_2'}=-t+\int\frac{dD^{a}q_2}{\sqrt{2E_2'-2D^{a}q_2E_2}} \quad (58b)$$

$$\lambda_1=\frac{\partial S}{\partial E_1}=D^{\alpha-1}q_1+\int\frac{D^{\alpha}q_1}{\sqrt{2E_1'-2D^{\alpha}q_1E_1}}dD^{\alpha}q_1 \quad (58c)$$

$$\lambda_2=\frac{\partial S}{\partial E_2}=D^{\alpha-1}q_2+\int\frac{D^{\alpha}q_2}{\sqrt{2E_2'-2D^{\alpha}q_2E_2}}dD^{\alpha}q_2 \quad (58d)$$

Using Eqs. (21c, 21d) to find the generalized momenta:

$$p_1=\frac{\partial S}{\partial D^{\alpha-1}q_1}=E_1 \quad (59a)$$

$$p_2=\frac{\partial S}{\partial D^{\alpha-1}q_2}=E_2+D^{\alpha-1}q_2 \quad (59b)$$

$$p_3=\frac{\partial S}{\partial D^{\alpha-1}q_3}=D^{\alpha-1}q_3 \quad (59c)$$

$$\pi_1=\frac{\partial S}{\partial D^{\alpha}q_1}=\sqrt{2E_1'-2D^{\alpha}q_1E_1} \quad (59d)$$

$$\pi_2=\frac{\partial S}{\partial D^{\alpha}q_2}=\sqrt{2E_2'-2D^{\alpha}q_2E_2} \quad (59e)$$

$$\pi_3=\frac{\partial S}{\partial D^{\alpha}q_3}=D^{\alpha}q_3 \quad (59f)$$

where $D^{\alpha-1}q_3$ and $D^{\alpha}q_3$ are arbitrary parameters.

Also, we can determine the equations of motion using Eqs. (17).

Now, our purpose is to quantize our singular system. We can write the fractional wave function Eq. (22a) for this example as:

$$\Psi(D^{\alpha-1}q_1,D^{\alpha-1}q_2,D^{\alpha-1}q_3,D^{\alpha}q_1,D^{\alpha}q_2,D^{\alpha}q_3,t)=\left[\psi_{01}(D^{\alpha-1}q_1)\psi_{02}(D^{\alpha-1}q_2)\right] \times\left[\varphi_{01}(D^{\alpha}q_1)\varphi_{02}(D^{\alpha}q_2)\right]\exp\left(\frac{iS}{\hbar}\right) \quad (60)$$

Where

$$\psi_{01}(D^{\alpha-1}q_1) = \frac{1}{\sqrt{P_1(D^{\alpha-1}q_1)}} = [E_1]^{-\frac{1}{2}} \tag{61a}$$

$$\psi_{02}(D^{\alpha-1}q_2) = \frac{1}{\sqrt{P_2(D^{\alpha-1}q_2)}} = [E_2 + D^{\alpha-1}q_2]^{-\frac{1}{2}} \tag{61b}$$

$$\varphi_{01}(D^{\alpha}q_1) = \frac{1}{\sqrt{\pi_1(D^{\alpha}q_1)}} = \left[2E_1' - 2D^{\alpha}q_1E_1\right]^{-\frac{1}{4}} \tag{61c}$$

$$\varphi_{02}(D^{\alpha}q_2) = \frac{1}{\sqrt{\pi_2(D^{\alpha}q_2)}} = \left[2E_2' - 2D^{\alpha}q_2E_2\right]^{-\frac{1}{4}} \tag{61d}$$

Thus, the function S can be given by Eq. (57).

Now, the fractional HJPDEs, Eqs. (49) can be applied to the wave function Ψ, after representing the generalized coordinates and momenta as operators:

$$\begin{aligned}\hat{H}_{\circ}'\Psi = \Bigg[&\frac{\hbar}{i}\frac{\partial}{\partial t} + D^{\alpha}q_1\frac{\hbar}{i}\frac{\partial}{\partial D^{\alpha-1}q_1} + D^{\alpha}q_2\left(\frac{\hbar}{i}\frac{\partial}{\partial D^{\alpha-1}q_2} - D^{\alpha-1}q_2\right) \\ &-\frac{\hbar^2}{2}\frac{\partial^2}{\partial(D^{\alpha}q_1)^2} - \frac{\hbar^2}{2}\frac{\partial^2}{\partial(D^{\alpha}q_2)^2}\Bigg]\Psi\end{aligned} \tag{62a}$$

$$\hat{H}_3'^{p}\Psi = \left[\frac{\hbar}{i}\frac{\partial}{\partial D^{\alpha-1}q_3} - D^{\alpha-1}q_3\right]\Psi \tag{62b}$$

$$\hat{H}_3'^{\pi}\Psi = \left[\frac{\hbar}{i}\frac{\partial}{\partial D^{\alpha}q_3} - D^{\alpha}q_3\right]\Psi \tag{62c}$$

After some algebra, we have

$$\frac{\hbar}{i}\frac{\partial}{\partial t}\Psi = (-E_1 - E_2)\Psi \tag{63a}$$

$$\frac{\hbar}{i}\frac{\partial}{\partial D^{\alpha-1}q_1}\Psi = E_1\Psi \tag{63b}$$

$$\frac{\hbar}{i}\frac{\partial}{\partial D^{\alpha-1}q_2}\Psi = \left[(D^{\alpha-1}q_2 + E_2) - \frac{\hbar}{2i}(D^{\alpha-1}q_2 + E_2)^{-1}\right]\Psi \tag{63c}$$

$$-\frac{\hbar^2}{2}\frac{\partial}{\partial(D^{\alpha}q_1)^2}\Psi = \left[-\frac{5\hbar^2}{8}E_1^2(2E_1' - 2D^{\alpha}q_1E_1)^{-2} + \frac{1}{2}(2E_1' - 2D^{\alpha}q_1E_1)\right]\Psi \tag{63d}$$

$$-\frac{\hbar^2}{2}\frac{\partial}{\partial(D^{\alpha}q_2)^2}\Psi = \left[-\frac{5\hbar^2}{8}E_2^2(2E_2' - 2D^{\alpha}q_2E_2)^{-2} + \frac{1}{2}(2E_2' - 2D^{\alpha}q_2E_2)\right]\Psi \tag{63e}$$

$$\frac{\hbar}{i}\frac{\partial}{\partial D^{\alpha-1}q_3}\Psi = D^{\alpha-1}q_3\Psi \tag{63f}$$

$$\frac{\hbar}{i}\frac{\partial}{\partial D^{\alpha}q_3}\Psi = D^{\alpha}q_3\Psi \tag{63g}$$

Substituting the results of Eqs. (63) in Eqs. (62), we obtain:

$$\hat{H}'_{\circ}\Psi = \begin{bmatrix} -E'_1 - E'_2 + D^a q_1 E_1 + D^a q_2 \left[(E_2 + D^{a-1}q_2) - \frac{\hbar}{2i}(E_2 + D^{a-1}q_2)^{-1} - D^{a-1}q_2 \right] - \\ \frac{5\hbar^2}{8}E_1^2(2E'_1 - 2D^a q_1 E_1)^{-2} + \frac{1}{2}(2E'_1 - 2D^a q_1 E_1) - \frac{5\hbar^2}{8}E_2^2(2E'_2 - 2D^a q_2 E_2)^{-2}) + \\ \frac{1}{2}(2E'_2 - 2D^a q_2 E_2) \end{bmatrix}\Psi \tag{64a}$$

$$\hat{H}'^{p}_{3}\Psi = \left[\frac{\hbar}{i}\frac{\partial}{\partial D^{a-1}q_3} - D^{a-1}q_3\right]\Psi = \left[D^{a-1}q_3 - D^{a-1}q_3\right]\Psi = 0. \tag{64b}$$

$$\hat{H}'^{\pi}_{3}\Psi = \left[\frac{\hbar}{i}\frac{\partial}{\partial D^{a}q_3} - D^{a}q_3\right]\Psi = \left[D^{a}q_3 - D^{a}q_3\right]\Psi = 0. \tag{64c}$$

Taking the limit $\hbar \to 0$ in Eq. (64a), we get:

$$\hat{H}'_{\circ}\Psi = \begin{bmatrix} -E'_1 - E'_2 + D^a q_1 E_1 + D^a q_2 \left[E_2 + D^{a-1}q_2 - D^{a-1}q_2\right] + \\ \frac{1}{2}(2E'_1 - 2D^a q_1 E_1) + \frac{1}{2}(2E'_2 - 2D^a q_2 E_2) \end{bmatrix}\Psi = 0. \tag{65}$$

5. Conclusion

In our work, we have extended a general theory for singular Lagrangian systems using fractional calculus. In this work, we solved the set of the fractional HJPDEs for these systems. In this paper, the canonical method is used for these systems to obtain fractional HJPDEs. We determined the fractional function S to obtain the equations of motion. Also, function S enables us to determine the appropriate fractional wave function for these systems. We achieved that the constraints in singular systems become conditions on fractional wave function. These conditions are achieved in semiclassical limit. Also, in this limit, Schrödinger equation is satisfied. In other words, we have approved that the quantum results agree with the classical results. Finally, we have examined two mathematical examples.

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