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Quantization of Dissipative Anisotropic Harmonic Oscillators

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Abstract. In this study, we developed a Hamiltonian framework for the anisotropic harmonic oscillator and applied the Hamilton-Jacobi equations to analyze a dissipative system. By incorporating boundary conditions, we derived the action function. The system was quantized using three methods: the WKB approximation, canonical quantization, and creation-annihilation operators. The Heisenberg equations were reformulated using Poisson bracket quantum variables, and energy levels and quantum states were obtained. All methods yielded consistent results, demonstrating the model's robustness and its significance for understanding dissipative quantum systems.

2020 Mathematics Subject Classifications: 54C08, 54C60

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1. Introduction

Describing microscopic systems presents a significant challenge, particularly in bridging classical and quantum representations through Hamiltonian operator quantization. Among these systems, the damped harmonic oscillator is of particular interest due to its inherent dissipative nature, which complicates its quantization. Canonical quantization methods, including coherent states and time-dependent creation-annihilation operators,

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have been widely explored to address such systems [1, 2]. The quantum Bateman method has also been extensively studied for dissipative systems, offering valuable insights [3–5]. Additionally, fractional calculus techniques have been employed to derive Lagrangians and Hamiltonians for non-conservative systems, further advancing the field [6**?**]. Recent studies have focused on time-dependent harmonic oscillators and particles under specific potentials, emphasizing time-dependent invariants [7]. Efforts to address Hamilton-Jacobi equations using electromagnetic field techniques have also been made, with methods such as variable separation under Staeckel conditions and Newtonian formalism being applied [8]. Despite these advancements, challenges remain in quantifying dissipative systems, particularly when dissipation depends solely on coordinates [9]. This highlights the need for innovative approaches to better understand and model such systems.

In this context, recent research has explored the application of fractional calculus in modeling complex dynamical systems, such as the immune system's response to HIV-tumor interactions, providing a fractional framework to understand these interactions. Similarly, numerical investigations using Caputo-Fabrizio fractional derivatives have been employed to study the dynamical behavior of the Hepatitis B virus, offering insights into the complex dynamics of viral infections. These studies demonstrate the versatility of fractional calculus in addressing real-world dynamical systems, which aligns with the challenges faced in quantizing dissipative systems. For instance, the fractional quantization of Podolsky electrodynamics using the fractional Hamilton-Jacobi formulation highlights the potential of fractional calculus in extending classical and quantum theories to systems with non-local and memory dependent behaviors. This approach not only bridges the gap between classical and quantum descriptions but also provides a robust framework for analyzing systems with fractional dynamics, such as those encountered in biological and physical systems [10–12].

Furthermore, the Madelung formalism has been applied to dissipating and decaying systems, providing a deeper understanding of the quantum dynamics of such systems [13]. The quantum Hamilton-Jacobi formalism has also been extensively studied, offering a robust framework for analyzing quantum systems with dissipation[14]. Additionally, generalized creation and annihilation operators have been developed using complex non-linear Riccati equations, which are crucial for understanding the quantization of dissipative systems [15]. Recent advancements in dissipative fuzzy tracking control for nonlinear networked systems with quantization have also contributed to the field, particularly in understanding the effects of quantization on dissipative systems [16]. Recent advancements in dissipative fuzzy tracking control for nonlinear networked systems with quantization have also contributed to the field, particularly in understanding the effects of quantization on dissipative systems [17]. This research aims to develop a Hamiltonian framework to describe the two-dimensional anisotropic harmonic oscillator with damping effects by utilizing the Hamilton-Jacobi equations to analyze the dissipative system. To achieve this, the researchers employed three different quantization methods: the WKB approximation and creation and annihilation operators, and canonical quantization.

The paper follows a structured approach, beginning with an Introduction that discusses the challenges of quantizing dissipative systems and reviews existing methods such as canonical quantization, fractional calculus, and Hamilto Jacobi approaches. It then develops the Hamiltonian and Lagrangian formulations for the two-dimensional dissipative harmonic oscillator, based on Bateman's approach. The study applies the WKB approximation to quantize the system and analyze its energy levels, followed by Canonical Quantization using the Nikiforov-Uvarov method, which solves the Schrödinger equation to derive energy eigenvalues and wave functions. The Separation of Variables in the Hamilton-Jacobi Equation is examined using Staeckel boundary conditions to facilitate the quantization process. Additionally, the paper reformulates the dissipative system through Poisson Brackets, applying standard quantization rules to obtain Heisenberg equations and expectation values. The study further explores the Applications of the Hamilton-Jacobi Technique, demonstrating its use in understanding oscillatory dynamics, dissipative effects, and chaos in quantum and relativistic systems. To illustrate the effects of dissipation on quantum states, the paper presents graphical representations of energy levels and probability distributions. In the Future Directions, it proposes extending the model to three dimensions and incorporating additional coupling terms to enhance the understanding of anisotropic and dissipative quantum systems. The study also compares its findings with Previous Research, emphasizing improvements in quantization techniques and theoretical modeling. Finally, the Conclusions summarize key findings, including the consistency of different quantization methods, the validation of the Hamilton-Jacobi formalism, and the significance of dissipation in quantum mechanics.

2. Dissipative Anisotropic Harmonic Oscillators: Hamiltonian and Lagrangian Formalis

In this section, we use the Lagrangian proposed by Bateman to describe the two-dimensional dissipative system. This Lagrangian provides a framework for describing dissipation in the system, but quantization is carried out using other methods such as the WKB approximation and canonical quantization .The Lagrangian used in Eq. (1) was originally proposed by Bateman [18] and later utilized in various studies, including [19], to model dissipative two dimensional anisotropic harmonic oscillators

$$L = \sum_{i=1}^{2} (\frac{1}{2} m q_i^2 - 2\pi^2 m v_i^2 q_i^2) e^{\lambda_i t} \tag{1}$$

In accordance with (1), the equations of motion can be expressed as:

$$\ddot{q}_1 + \lambda_1 q_1 + 4\pi^2 v_1^2 q_1 = 0$$

$$\ddot{q}_2 + \lambda_2 q_1 + 4\pi^2 v_2^2 q_2 = 0$$

This transformation clearly delineates the dissipative nature of the two-dimensional anisotropic harmonic oscillator.

$$x_i = \sum_{i=1}^{2} q_i e^{\frac{\lambda_i t}{2}}$$

For $i = 1, 2$, the Lagrangian is transformed as follows based on the preceding equation:

$$L = \sum_{i=1}^{2} \frac{1}{2} m x_i^2 + \frac{1}{8} m \lambda_i^2 x_i^2 - \frac{1}{4} m \lambda_i \dot{x_i} x_i - 2\pi^2 m v_i^2 x_i^2$$

When the terms $\dot{x_i}$ and x_i in the Lagrangian don't affect the equations of motion F_i (where $(\frac{dF_i}{dt} = 0 = \dot{x_i} x_i)$) it suggests $F_i = \frac{x_i^2}{2}$. Consequently, we obtain the following equivalent Lagrangian:

$$L = \sum_{i=1}^{2} \frac{1}{2} m x_i^2 + \frac{1}{8} m \lambda_i^2 x_i^2 - 2\pi^2 m v_i^2 x_i^2$$

Here, the subscript (i) means sum over $(x_1 = x, x_2 = y)$.
The Lagrangian describes the dissipative, two-dimensional anisotropic harmonic oscillator. The equations of motion can be derived from this lagrangian.

$$\ddot{x} + (w_x^2 - \frac{\lambda_x^2}{4})x = 0 \tag{2}$$

$$\ddot{y} + (w_y^2 - \frac{\lambda_y^2}{4})y = 0 \tag{3}$$

A significant finding is that the damped oscillator's angular frequency assumes values $\sqrt{(w_x^2 - \frac{\lambda_x^2}{4})}$ and $\sqrt{(w_y^2 - \frac{\lambda_y^2}{4})}$ where (w_x, w_y) represents the angular frequencies of the conservative system. To revert to the basic harmonic oscillator, one sets the damping coefficient (λ_y, λ_x) to zero in the equations. The Hamiltonian (H) can be expressed as a function of coordinates (x, y) and conjugate momenta (P_x, P_y).

$$H = P_x \dot{x} + P_y \dot{y} - L$$

The generalized momenta are defined as follows [20]:

$$P_x = \frac{\partial L}{\partial x} = m\dot{x} \text{ and } P_y = \frac{\partial L}{\partial y} = m\dot{y}$$

The following is the Hamiltonian's expression in terms of the time derivatives:

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} - \frac{1}{4} m \lambda_x^2 x^2 - \frac{1}{4} m \lambda_y^2 y^2 + 2\pi^2 m v_x^2 x^2 + +2\pi^2 m v_y^2 y^2$$

The current study's conclusions and Rabei et al.'s analysis differ for a number of reasons. Firstly, the current work focused on scattering using a number of approaches, including Hamilton-Jacobi and variable separation, with a focus on dissipative. Secondly, Rabei et

al. [21] utilized the WKB for constrained systems. Hence, the Hamilton–Jacobi equation assumes the form

$$\frac{1}{2m}((\frac{\partial S}{\partial x})^2 + 4\pi^2 m^2 v_x^2 x^2 - \frac{1}{2}m^2\lambda_x^2 x^2 + (\frac{\partial S}{\partial y})^2 + 4\pi^2 m^2 v_y^2 y^2 - \frac{1}{2}m^2\lambda_y^2 y^2) + \frac{\partial S}{\partial t} = 0 \quad (4)$$

Where $\frac{\partial S}{\partial x}$ and $\frac{\partial S}{\partial y}$ represent the canonical momenta P_x and P_y , respectively. By substituting these into the equation, it can be rewritten as follows:

$$\frac{1}{2m}((P_x)^2 + 4\pi^2 m^2 v_x^2 x^2 - \frac{1}{2}m^2\lambda_x^2 x^2 + (P_y)^2 + 4\pi^2 m^2 v_y^2 y^2 - \frac{1}{2}m^2\lambda_y^2 y^2) + \frac{\partial S}{\partial t} = 0$$

The given Hamiltonian is expressed as:

$$H = \frac{1}{2m}(P_x)^2 + 4\pi^2 m^2 v_x^2 x^2 - \frac{1}{2}m^2\lambda_x^2 x^2 + \frac{1}{2m}(P_y)^2 + 4\pi^2 m^2 v_y^2 y^2 - \frac{1}{2}m^2\lambda_y^2 y^2$$

By comparing the two expressions, we observe that the kinetic energy terms $\frac{(P_x)^2}{2m} + \frac{(P_y)^2}{2m}$ are identical in both equations. For the potential energy terms, dividing the terms in the Hamilton-Jacobi equation by 2m results in $4\pi^2 m^2 v_x^2 x^2 - \frac{1}{2}m^2\lambda_x^2 x^2 + 4\pi^2 m^2 v_y^2 y^2 - \frac{1}{2}m^2\lambda_y^2 y^2$, which matches the corresponding terms in the Hamiltonian. Additionally, the time-dependent term $\frac{\partial S}{\partial t}$ in the Hamilton-Jacobi equation is consistent with the role of time in classical dynamics, even though it does not explicitly appear in the Hamiltonian. Consequently, the Hamilton-Jacobi equation and the given Hamiltonian are equivalent, as their kinetic, potential, and temporal components align perfectly. The principal Hamiltonian function S can be expressed as follows:

$$S(x, y, \alpha, \alpha_y, t) = W_x(x, \alpha, \alpha_y) + W_y(y, \alpha, \alpha_y) - \alpha t \text{ This leads to} \frac{\partial S}{\partial t} = -\alpha \quad (5)$$

The partial derivatives of the function W with respect to x and y are defined by the formulas $P_x = \frac{\partial W}{\partial x}$ and $P_y = \frac{\partial W}{\partial y}$, respectively The Hamilton-Jacobi equation (HJE) is rewritten in terms of the functions and by inserting equation 2 into equation 3:

$$\frac{1}{2m}(\left[\frac{\partial W_x}{\partial x}\right]^2 + 4\pi^2 m^2 v_x^2 x^2 - \frac{1}{2}m^2\lambda_x^2 x^2 + \left[\frac{\partial W_y}{\partial y}\right]^2 + 4\pi^2 m^2 v_y^2 y^2 - \frac{1}{2}m^2\lambda_y^2 y^2) = \alpha$$

By separating the variables, we can get the following results:

$$\frac{1}{2m}(\left[\frac{\partial W_x}{\partial x}\right]^2 + 4\pi^2 m^2 v_x^2 x^2 - \frac{1}{2}m^2\lambda_x^2 x^2) = \alpha. \quad (6)$$

$$\frac{1}{2m}(\left[\frac{\partial W_y}{\partial y}\right]^2 + 4\pi^2 m^2 v_y^2 y^2 - \frac{1}{2}m^2\lambda_y^2 y^2) = \alpha_y \quad (7)$$

By rewriting equations 4 and 5, we get:

$$\frac{\partial W_y}{\partial y} = \sqrt{2m\alpha_y - 4\pi^2 m^2 v_y^2 y^2 + \frac{1}{2}m^2\lambda_y^2 y^2} \quad (8)$$

$$\frac{\partial W_x}{\partial x} = \sqrt{2m\alpha - 2m\alpha_y - 4\pi^2 m^2 v_x^2 y^2 + \frac{1}{2}m^2\lambda_x^2 x^2} \tag{9}$$

Equations 6 and 7 can be solved by integrating the equation to obtain:

$$W_{y'} = \int_0^{y'} \sqrt{(2m\alpha_y - 4\pi^2 m^2 v_y^2 y^2 + \frac{1}{2}m^2\lambda_y^2 y^2)}dy \tag{10}$$

$$W_{x'} = \int_0^{x'} \sqrt{(2m\alpha - 2m\alpha_y - 4\pi^2 m^2 v_x^2 y^2 + \frac{1}{2}m^2\lambda_x^2 x^2)}dx \tag{11}$$

Substituting equation 1 and equation 11 in equation

$$S(x, y, \alpha, \alpha_y, t) = W_x(x, y, \alpha) + W_y(x, y, \alpha) - \alpha t$$

we obtain:

$$S(x, y, \alpha, \alpha_y, t) = \int_0^{x'} \sqrt{(2m\alpha - 2m\alpha_y - 4\pi^2 m^2 v_x^2 y^2 + \frac{1}{2}m^2\lambda_x^2 x^2)}dx + \int_0^{y'} \sqrt{(2m\alpha_y - 4\pi^2 m^2 v_y^2 y^2 + \frac{1}{2}m^2\lambda_y^2 y^2)}dy - \alpha_i t \tag{12}$$

By differentiating the previous equation with respect to α_i, we obtain:

$$B_{y'} = \frac{\partial S}{\partial \alpha_{y'}} = \int_0^{y'} \frac{m}{2m\alpha_y - 4\pi^2 m^2 v_y^2 y^2 + \frac{1}{2}m^2\lambda_y^2 y^2)}dy - \frac{sin^{-1}\left(\sqrt{\left(\frac{2m\pi^2 v_{y'}^2 - \frac{1}{4}m\lambda_{y'}^2}{\alpha_{y'}}\right)}y'\right)}{\sqrt{4\pi^2 v_{y'}^2}} - \frac{sin^{-1}\left(\sqrt{\left(\frac{2m\pi^2 v_{x'}^2 - \frac{1}{4}m\lambda_{x'}^2}{\alpha - \alpha_{y'}}\right)}x'\right)}{\sqrt{4\pi^2 v_{x'}^2}} \tag{13}$$

$$\beta_{x'} + t = \int_0^{x'} \frac{m}{2m\alpha - 2m\alpha_y - 4\pi^2 m^2 v_x^2 x^2 + \frac{1}{2}m^2\lambda_x^2 x^2)}dx = \frac{sin^{-1}\left(\sqrt{\left(\frac{2m\pi^2 v_{x'}^2 - \frac{1}{4}m\lambda_{x'}^2}{\alpha - \alpha_{y'}}\right)}x'\right)}{\sqrt{4\pi^2 v_{x'}^2}} \tag{14}$$

After integration, equation 14 was subjected to necessary algebraic simplifications, leading to the expression shown in equations 15.

$$x = \sqrt{\left(\frac{\alpha - \alpha_y}{2m\pi^2 v_{x'}^2 - \frac{1}{4}m\lambda_{x'}^2}\right)} sin\left[\sqrt{4\pi^2 v_{x'}^2}(\beta_{x'} + t)\right] \tag{15.a}$$

$$y = \sqrt{\left(\frac{\alpha - \alpha_y}{2m\pi^2 v_{y'}^2 - \frac{1}{4}m\lambda_{y'}^2}\right)} sin\left[\sqrt{4\pi^2 v_{y'^2}}(\beta_{y'} + t)\right] \tag{15.b}$$

3. Quantization of a Dissipative Two-Dimensional Anisotropic Harmonic Oscillator Using the WKB Approximation

We start our formalism by studying the Schrödinger equation, denoted as:

$$H_\psi = ih\frac{\partial \psi}{\partial t}$$

For our dissipative Two-Dimensional Anisotropic Harmonic Oscillator, the following formulation is proposed.

$$(\frac{h}{i}\frac{\partial}{t} - \frac{P_x^2}{2m} - \frac{P_y^2}{2m} - 2\pi^2 m v_x^2 x^2 - 2\pi^2 m v_y^2 y^2 + \frac{1}{4}m\lambda_x^2 x^2 + \frac{1}{4}m\lambda_y^2 y^2) = 0$$

Where

$$\begin{cases} \widehat{P_x} = & \frac{h}{i}\frac{\partial}{\partial x} \\ \widehat{P_y} = & \frac{h}{i}\frac{\partial}{\partial y} \end{cases}$$

That is

$$\begin{pmatrix} \frac{h}{i}\frac{\partial}{\partial t} - \frac{h^2}{2m}\frac{\partial^2 \psi}{\partial x^2} & -\frac{h^2}{2m}\frac{\partial^2 \psi}{\partial y^2} & -2\pi^2 m v_x^2 x^2 \psi \\ -2\pi^2 m v_y^2 y^2 \psi & +\frac{1}{4}m\lambda_x^2 x^2 & +\frac{1}{4}m\lambda_y^2 y^2 \end{pmatrix} = 0$$

Using the WKB approximation, we take ψ as follows:

$$\psi = \frac{1}{\sqrt{P_x P_y}} e^{\frac{iS}{h}}$$

By substituting ψ and performing the required derivatives, we obtain:

$$\left(-\alpha + \frac{1}{2m}\begin{pmatrix} ih\frac{\partial^2 S}{\partial x^2} + & \frac{\partial S}{\partial x}\frac{\partial S}{\partial x} + & ih\frac{\partial^2 S}{\partial y^2} + & \frac{\partial S}{\partial y}\frac{\partial S}{\partial y} \\ & +4\pi^2 m^2 v_x^2 x^2 & +4\pi^2 m^2 v_y^2 y^2 + y^2 & \\ & & -\frac{1}{2}m\lambda_x^2 x^2 - \frac{1}{2}m\lambda_y^2 y^2 & \end{pmatrix}\right)\psi = 0 \tag{16}$$

Taking the semi-classical limit $k \longrightarrow 0$ and remembering that $P_x = \frac{\partial W_x}{\partial x} = \frac{\partial S}{\partial x}$ and $P_y = \frac{\partial W_y}{\partial y} = \frac{\partial S}{\partial y}$, we get :

$$\frac{1}{2m}\left(\left(\frac{\partial W_x}{\partial x}\right)^2 + \left(\frac{\partial W_y}{\partial y}\right)^2 + 4\pi^2 m^2 v_x^2 x^2 + 4\pi^2 m^2 v_y^2 y^2 - \frac{1}{2}m\lambda_x^2 x^2 - \frac{1}{2}m\lambda_y^2 y^2\right) = \alpha \tag{17}$$

Which is the classical HJE. So, we have satisfied the Schrodinger equation and quantized the Two-Dimensional Anisotropic Harmonic Oscillator dissipative. The WKB method demonstrated compatibility between quantum solutions and the classical Hamilton-Jacobi equation, confirming the reliability of the semiclassical approach in determining energy levels.

4. Canonical Quantization of a Damped Two-Dimensional Anisotropic Harmonic Oscillator with the Nikiforov-Uvarov Approach

The Schrödinger equation governing the wave function , for a two-dimensional anisotropic harmonic oscillator with dissipation is given by

$$\sum_{i=1}^{2}\left(-\frac{h^2}{2m}\frac{\partial^2\psi}{\partial x_i^2}\right)-\frac{1}{2}4\pi^2m^2v_i^2\psi+\frac{1}{4}m\lambda_i^2x_i^2\psi-E_i\psi=0 \qquad (-\infty<x_i<+\infty)$$

We will employ the Nikiforov-Uvarov (NU) method to solve this eigenvalue problem, as outlined in [21]. Here, $\psi(x_i)$ must be bounded and satisfy the normalization condition.

$$\int_{-\infty}^{+\infty}|\psi(x_i)|^2dx=1$$

We introduce the dimensionless variables ζ and ϵ, representing x_i and E_i , respectively. This equation describes a two-dimensional anisotropic harmonic oscillator under damping. After substituting with according to the relation;

$$\sum_{i=1}^{2}\zeta_i=\sum_{i=1}^{2}ax_i \qquad \text{where } \zeta_i=\sqrt{\frac{\pi}{mw}}x_i$$

And replacing E_i with;

$$E_i=\sum_{i=1}^{2}hw\epsilon_i$$

We obtain the following differential equation:

$$\sum_{i=1}^{2}\psi_i+(2\epsilon_i-\zeta_i^2)\psi_i=0$$

This equation is of the hypergeometric type, denoted by primes to signify differentiation with respect to [22].

$$\zeta\sigma(\zeta_i)=1$$

To prove the equation $\zeta\sigma(\zeta_i)=1$, at the point ζ_i the equation can be solved as follows:

$$\sigma(\zeta_i)=\frac{1}{\zeta_i}$$

Thus, $\sigma(\zeta_i)=\frac{1}{\zeta_i}$, the equation becomes:

$$\zeta_i.\frac{1}{\zeta_i}=1$$

This proves the validity of the equation $\zeta.\sigma(\zeta_i)=1$ Additionally, the following relations hold:

$$\widehat{\tau}(\zeta_i)=0,\ \text{and } \widehat{\sigma}(\zeta_i)=2\epsilon_i-\zeta_i^2 \tag{18}$$

In the given situation, with $\widehat{\rho}(\zeta_i) = 1$, the requirement for $\sqrt{\widehat{\rho}(\zeta_i)}\psi(\zeta_i)$ to be integrable squares arises directly from the normalization conditions. By following the previously outlined method, we can transform the equation governing into a hypergeometric one, and thus we get:

$$\sigma(\zeta_i)\eta_i^{''} + \eta(\zeta_i)\eta_i^{'} + \lambda_i\eta_i = 0 \tag{19}$$

From the above equation, we express $\psi(\zeta_i) = \sum_1^2 \phi(\zeta_i)\eta(\zeta_i)$ as a summation. To achieve this, we utilize the condition outlined in equation 18, where $\phi(\zeta_i)$ satisfies the equation.

$$\frac{\phi^{'}}{\phi} = \frac{\pi(\zeta_i)}{\sigma(\zeta_i)} \tag{20}$$

The polynomial $\pi(\zeta_i)$ is represented by the expression $\sum_1^2 \pm\sqrt{k_i - 2\epsilon_i + \zeta_i^2}$, where the constant k_i is chosen to result in a double zero, specifically set as $k_i = 2\epsilon_i$. This expression gives rise to two potential polynomials, which are $\pi(\zeta_i) = \pm\zeta_i$. Therefore, we obtain:

$$\tau(\zeta_i) = \widehat{\tau(\zeta_i)} + 2\pi(\zeta_i) \tag{21}$$

The existence of a negative derivative indicates that certain conditions must be met for the function $\tau(\zeta_i)$. These conditions are satisfied when we take $\tau(\zeta_i) = 2\zeta_i$. Under these conditions, we get:

$$\begin{aligned} \tau(\zeta_i) &= -\zeta_i & \phi(\zeta_i) &= e^{\frac{-\zeta_i^2}{2}} \\ \lambda_i &= 2\epsilon_2 - 1 & \rho(\zeta_i) &= e^{-\zeta_i^2} \end{aligned}$$

The values of the energy eigenvalue $\epsilon = \epsilon_{n_i} = n_x + n_y + \frac{1}{2}$ hence $s(\lambda_i)$ are determined by an equation based on τ and σ, which is used to rearrange to achieve equilibrium. The expression for the energy eigenvalues (ϵ) is:

$$E_{n_x n_y} = E_x + E_y = (2\pi n_x v_x + 2\pi n_y v_y + \frac{2\pi(v_x + v_y)}{2})h$$

$$\text{where } n_x = 0, 1, 2, \cdots \text{ and } n_y = 0, 1, 2, \cdots$$

. The eigenfunctions $\eta_{nr}(x, y)$ are defined as : $\eta_{nr}(x, y) = B_{nr}e^{x^2+y^2}(-1)^{n+r}\frac{\partial^{n+r}}{\partial^r\partial^n}e^{-x^2-y^2}$ and involve Hermite functions $H_{nr}(x, y)$. Similarly, the wave functions $\psi_{nm}(x, y)$ are derived from the eigenfunctions, with the form

$$\psi_{nm}(x, y) = C_{nr}e^{\frac{m(w_x x^2+w_y y^2)}{2k}}H_{nr}(\frac{mw_x}{h}x, \frac{mw_y}{h}y) =$$

$$\left(\frac{m}{\pi k}\right)^{\frac{1}{2}}(w_x)^{\frac{1}{2}}(w_y)^{\frac{1}{2}}e^{-\frac{m(w_x x^2+w_y y^2)}{2k}}H_{nr}(\frac{mw_x}{h}x, \frac{mw_y}{h}y) \text{ where } C_{nr} = \left(\frac{mw}{\pi k}\right)^{\frac{1}{2}} \tag{22}$$

Accurate energy eigenvalues were derived using the NU method, with results consistent across various approaches, demonstrating the robustness of the methodologies employed.

4.1. The continuity equation and the probability current

To derive the continuity equation for the dissipative two-dimensional anisotropic harmonic oscillator, we start from the Schrödinger equation of the system, taking into account the dissipation effects included in the Hamiltonian. The continuity equation expresses the conservation of probability in the quantum system, linking the change in probability density with the probability current.

The Schrödinger equation for the system

The Schrödinger equation for the wave function $\psi(x, y, t)$ is given as follows:

$$\frac{i}{h}\frac{\partial \psi}{\partial t} = H\psi$$

Where H is the Hamiltonian of the system. In our case, the Hamiltonian for the two-dimensional dissipative system is given as follows:

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + V(x, y) + \frac{1}{4}m\lambda_x^2 + \frac{1}{4}m\lambda_y^2$$

Here, $V(x, y) = 2\pi^2 m(v_x^2 x^2 - v_y^2 y^2)$ represents the potential energy, and the terms $\frac{1}{4}m\lambda_x^2$ and $\frac{1}{4}m\lambda_y^2$ account for the dissipation effects.

Probability Density

The probability density $\rho(x, y, t)$ is defined as:

$$\rho(x, y, t) = |\psi(x, y, t)|^2 \psi(x, y, t) * \psi(x, y, t)$$

Where ψ is the complex conjugate of the wave function $\psi*$.

Derivation of the Continuity Equation with Dissipation:

We start by multiplying the Schrödinger equation by $\psi*$ and subtracting the complex conjugate of the Schrödinger equation multiplied by ψ :

$$\psi * \left(ih\frac{\partial \psi}{\partial t}\right) - \psi\left(ih\frac{\partial \psi*}{\partial t}\right) = \psi * H\psi - \psi H\psi*$$

After simplification, we obtain:

$$ih\left(\psi * \frac{\partial \psi}{\partial t} + \psi\frac{\partial \psi*}{\partial t}\right) = \psi * H\psi - \psi H\psi*$$

The left-hand side can be written as:

$$ih\frac{\partial(\psi * \psi)}{\partial t} = ih\frac{\partial \rho}{\partial t}$$

The right-hand side can be simplified using the Hamiltonian:

$$\psi * H\psi - \psi H\psi* = \psi * \left(\frac{P_x^2}{2m} + P_y^2 2m + V(x, y) + \frac{1}{4}m\lambda_x^2 + \frac{1}{4}m\lambda_y^2\right)\psi -$$

$$\psi \left(\frac{P_x^2}{2m} + P_y^2 2m + V(x,y) + \frac{1}{4} m\lambda_x^2 + \frac{1}{4} m\lambda_y^2 \right) \psi *$$

Since $V(x,y)$ is a real potential, it cancels out, leaving:
$\psi * \left(\frac{P_x^2}{2m} + P_y^2 2m + V(x,y) + \frac{1}{4} m\lambda_x^2 + \frac{1}{4} m\lambda_y^2 \right) \psi - \psi \left(\frac{P_x^2}{2m} + P_y^2 2m + V(x,y) + \frac{1}{4} m\lambda_x^2 + \frac{1}{4} m\lambda_y^2 \right) \psi *$
Using the definition of the momentum operators $\widehat{P_x} = \frac{h}{i} \frac{\partial}{\partial x}$ and $\widehat{P_y} = \frac{h}{i} \frac{\partial}{\partial y}$, we get :
$\psi * \left(-\frac{h^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{h^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{4} m\lambda_x^2 + \frac{1}{4} m\lambda_y^2 \right) \psi - \psi \left(-\frac{k^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{k^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{4} m\lambda_x^2 + \frac{1}{4} m\lambda_y^2 \right) \psi *$
After simplification, we obtain:
$-\frac{h^2}{2m} \left(\psi * \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \psi *}{\partial x^2} \right) - \frac{k^2}{2m} \left(\psi * \frac{\partial^2 \psi}{\partial y^2} - \psi \frac{\partial^2 \psi *}{\partial y^2} \right) + \frac{1}{4} m\lambda_x^2 (\psi * \psi - \psi\psi *) + \frac{1}{4} m\lambda_y^2 (\psi * \psi - \psi\psi *)$
The terms involving λ_x and λ_y vanish because $\psi * \psi = \psi\psi *$. Thus, the equation simplifies to:
$-\frac{h^2}{2m} \frac{\partial}{\partial x} \left(\psi * \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi *}{\partial x} \right) - \frac{h^2}{2m} \frac{\partial}{} \left(\psi * \frac{\partial \psi}{\partial y} - \psi \frac{\partial \psi *}{\partial y} \right)$
We notice that the expression inside the parentheses is the probability current in the x and y directions:

$$J_x = \frac{h}{2mi} \left(\psi * \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi *}{\partial x} \right)$$

$$J_y = \frac{h}{2mi} \left(\psi * \frac{\partial \psi}{\partial y} - \psi \frac{\partial \psi *}{\partial y} \right)$$

Therefore, the equation can be written as:

$$ih \frac{\partial \rho}{\partial t} = -\frac{h^2}{2m} \left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} \right)$$

Dividing both sides by ih, we obtain:

$$\frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = 0$$

This is the continuity equation, which expresses the conservation of probability in the quantum system, even in the presence of dissipation. The inclusion of dissipative terms in the Hamiltonian, as demonstrated by Serhan et al. [18], ensures that the continuity equation remains valid for systems with energy loss due to damping or environmental interactions.

Probability Current

The probability current J is given by:

$$J = (J_x, J_y)$$

Where:

$$J_x = \frac{h}{2mi} \left(\psi * \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi *}{\partial x} \right)$$

$$J_y = \frac{h}{2mi} \left(\psi * \frac{\partial \psi}{\partial y} - \psi \frac{\partial \psi *}{\partial y} \right)$$

The continuity equation expresses the conservation of probability in the quantum system, linking the change in probability density to the probability current. For the two-dimensional system with dissipation, the continuity equation is:

$$\frac{\partial \rho}{\partial t} + \bigtriangledown J = 0 \text{ where } \bigtriangledown J = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} \tag{23}$$

5. Separation of variables in the Hamilton-Jacobi equation using Staeckel boundary conditions

Consider the motion of a two-dimensional anisotropic harmonic oscillator with a dissipative potential $V = 2\pi^2 m(v_x^2 x^2 + v_y^2 y^2)$. The Hamilton-Jacobi equation is given by:

$$H = T + V = \frac{1}{2m}\left(\left(P_x^2 + 4\pi^2 m^2 v_x^2 x^2 - \frac{1}{2}m\lambda_x^2 x^2\right) + \left(P_y^2 + 4\pi^2 m^2 v_y^2 y^2 - \frac{1}{2}m\lambda_y^2 y^2\right)\right) \tag{24}$$

Comparing this equation with $H = \frac{1}{2}(P-a)T^{-1}(P-a) + V(q)$, we obtain:

$$T^{-1} = \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{pmatrix}$$

On the other hand, the Staeckel conditions require the fulfillment of two main criteria:

(i) **Diagonal Matrix Condition:**The inverse of the kinetic energy matrix T^{-1} must be diagonal. In this case, this condition is satisfied, as:

$$(T^{-1})_{ii} = \frac{1}{T_{ii}} = \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{pmatrix}$$

(ii) **Separable Potential Condition:** The potential $V(q)$ must be separable in the coordinates x and y, meaning it can be expressed as:

$$(\phi^{-1})_{1j} = \frac{1}{T_{jj}} = \begin{pmatrix} \frac{1}{m} & \frac{1}{m} \\ 0 & \frac{1}{m} \end{pmatrix}$$

From the given potential:

$$V(q) = \frac{V_i(q_i)}{T_{ii}} = \left(\frac{\psi_1(x)}{m} + \psi_1(y)m\right)$$

If the Staeckel conditions are satisfied, then Hamilton's characteristic function is completely separable:

$$W(q) = \sum_i W_i(q_i)$$

Inserting from equation 23 into equation ($H(q, \frac{\partial W}{\partial q}) + \frac{\partial S_0}{\partial t} = 0$) and using the definition of momentum $p = \frac{\partial W}{\partial q}$ we obtain **(more details see Appendix A)**:

$$\frac{1}{2m}\left(\left[\frac{\partial W_x}{\partial x}\right]^2 + 4\pi^2 m^2 v_x^2 x^2 - \frac{1}{2}\lambda_x^2 x^2 + \left[\frac{\partial W_y}{\partial y}\right]^2 + 4\pi^2 m^2 v_y^2 y^2 - \frac{1}{2}\lambda_y^2 y^2\right) = \alpha \tag{25}$$

6. Hamiltonian Dynamics of a Dissipative Two-Dimensional Anisotropic Oscillator Using Poisson Brackets

The formulation of Poisson brackets of Hamiltons equations of motion are then calculated as:

$$\dot{x} = \{x, H\} = \left\{x, \frac{1}{2m}\begin{pmatrix} (P_x^2 & +4\pi^2 m^2 v_x^2 x^2 & -\frac{1}{4}m\lambda_x^2 x^2) \\ & + & \\ (P_y^2 & +4\pi^2 m^2 v_y^2 y^2 & -\frac{1}{4}m\lambda_y^2 y^2) \end{pmatrix}\right\} = \left\{x, \frac{P_x^2}{2m}\right\} = \frac{P_x}{m} \tag{26}$$

After rewriting the previous equation in terms of $\dot{y}$,we get :

$$\dot{y} = \{y, H\} = \left\{y, \frac{1}{2m}\begin{pmatrix} (P_x^2 & +4\pi^2 m^2 v_x^2 x^2 & -\frac{1}{4}m\lambda_x^2 x^2) \\ & + & \\ (P_y^2 & +4\pi^2 m^2 v_y^2 y^2 & -\frac{1}{4}m\lambda_y^2 y^2) \end{pmatrix}\right\} = \left\{y, \frac{P_y^2}{2m}\right\} = \frac{P_y}{m} \tag{27}$$

If the Poisson brackets satisfy the canonical relations:

$$\{x, P_x\} = \{y, P_y\} = 1$$

Then the time evolution of the momentum P_x is derived as:

$$\begin{aligned} \dot{P}_x = \{P_x, H\} &= \left\{P_x, \frac{1}{2m}\begin{pmatrix} (P_x^2 & +4\pi^2 m^2 v_x^2 x^2 & -\frac{1}{4}m\lambda_x^2 x^2) \\ & + & \\ (P_y^2 & +4\pi^2 m^2 v_y^2 y^2 & -\frac{1}{4}m\lambda_y^2 y^2) \end{pmatrix}\right\} \\ &= \left\{P_x, 4\pi^2 m^2 v_x^2 x^2 - \frac{1}{4}m\lambda_x^2 x^2\right\} = -4\pi^2 m v_x^2 x + \frac{1}{4}m\lambda_x^2 x \end{aligned} \tag{28}$$

Likewise, equation (34) may be expressed in $\dot{P}_y$ form as follows:

$$\begin{aligned} \dot{P}_y = \{P_y, H\} &= \left\{P_y, \frac{1}{2m}\begin{pmatrix} (P_x^2 & +4\pi^2 m^2 v_x^2 x^2 & -\frac{1}{4}m\lambda_x^2 x^2) \\ & + & \\ (P_y^2 & +4\pi^2 m^2 v_y^2 y^2 & -\frac{1}{4}m\lambda_y^2 y^2) \end{pmatrix}\right\} \\ &= \left\{P_y, 4\pi^2 m^2 v_y^2 y^2 - \frac{1}{4}m\lambda_y^2 y^2\right\} = -4\pi^2 m v_y^2 y + \frac{1}{4}m\lambda_y^2 x \end{aligned} \tag{29}$$

Given the complete alignment with equations 2 and 3, it follows that the Poisson bracket relations $[x, P_x] = ih$ imply the corresponding commutation relations $[x, x] = [y, y] = [P_x, P_x] = [P_y, P_y] = 0$. This alignment ensures the consistency between the classical and quantum descriptions of the system. Building on this foundation. This approach is consistent with the methodology outlined in [23], where similar techniques were employed to analyze dissipative quantum systems

$$\frac{d}{dt}\langle x\rangle = \frac{i}{h}\left\langle\left[\frac{1}{2m}\begin{pmatrix}(P_x^2 & +4\pi^2m^2v_x^2x^2 & -\frac{1}{4}m\lambda_x^2x^2)\\ & + & \\ (P_y^2 & +4\pi^2m^2v_y^2y^2 & -\frac{1}{4}m\lambda_y^2y^2)\end{pmatrix}, x\right]\right\rangle$$

$$= \frac{i}{h}\left\langle\left[\frac{P_x^2}{2m}, x\right]\right\rangle = \left\langle\frac{P_x}{m}\right\rangle \tag{30}$$

We can write equation 26 in terms of , we get:

$$\frac{d}{dt}\langle y\rangle = \frac{i}{h}\left\langle\left[\frac{1}{2m}\begin{pmatrix}(P_x^2 & +4\pi^2m^2v_x^2x^2 & -\frac{1}{4}m\lambda_x^2x^2)\\ & + & \\ (P_y^2 & +4\pi^2m^2v_y^2y^2 & -\frac{1}{4}m\lambda_y^2y^2)\end{pmatrix}, x\right]\right\rangle$$

$$= \frac{i}{h}\left\langle\left[\frac{P_y^2}{2m}, y\right]\right\rangle = \left\langle\frac{P_y}{m}\right\rangle \tag{31}$$

The equation 31 agree with equation 27.

$$\frac{d}{dt}\langle P_x\rangle = \frac{i}{h}\left\langle\left[\frac{1}{2m}\begin{pmatrix}(P_x^2 & +4\pi^2m^2v_x^2x^2 & -\frac{1}{4}m\lambda_x^2x^2)\\ & + & \\ (P_y^2 & +4\pi^2m^2v_y^2y^2 & -\frac{1}{4}m\lambda_y^2y^2)\end{pmatrix}, P_x\right]\right\rangle$$

$$= \frac{i}{h}\left\langle\left[2\pi^2mv_x^2x^2, P_x\right]\right\rangle = -4\pi^2mv_x^2\langle x\rangle + \frac{1}{4}m\lambda_x^2\langle x\rangle \tag{32}$$

And

$$\frac{d}{dt}\langle P_y\rangle = \frac{i}{h}\left\langle\left[\frac{1}{2m}\begin{pmatrix}(P_x^2 & +4\pi^2m^2v_x^2x^2 & -\frac{1}{4}m\lambda_x^2x^2)\\ & + & \\ (P_y^2 & +4\pi^2m^2v_y^2y^2 & -\frac{1}{4}m\lambda_y^2y^2)\end{pmatrix}, P_y\right]\right\rangle$$

$$= \frac{i}{h}\left\langle\left[2\pi^2mv_y^2y^2, P_y\right]\right\rangle = -4\pi^2mv_y^2\langle x\rangle + \frac{1}{4}m\lambda_y^2\langle x\rangle \tag{33}$$

$$\ddot{x} = \frac{\dot{P}_x}{m} \text{ therefore } \ddot{x} + (4_x^2 - \frac{1}{4}\lambda_x^2)x = 0 \tag{34}$$

$$\ddot{y} = \frac{\dot{P}_y}{m} \text{ which means that } \ddot{y} + (4_y^2 - \frac{1}{4}\lambda_y^2)y = 0 \tag{35}$$

These equations align with the results obtained through other quantization methods. The formulation is further extended using creation and annihilation operators to derive the energy eigenvalues and wave functions, as detailed in **the Appendix B**.

7. Applications of the Hamilton-Jacobi Technique in Analyzing Harmonic Oscillator Dynamics

This section discusses how we can use the Hamilton-Jacobi approach to study the harmonic oscillator, showing its flexibility and effectiveness in different situations. It explains how this method helps us understand the relationship between the Hamilton-Jacobi equation of motion and the i-factor of an oscillator, which matches the system's physical features well. This confirms that the Hamilton-Jacobi approach works well and proves that the fractional oscillator template is useful for describing strongly damped vibrations. Additionally, we can apply this approach to see how dissipative processes affect other oscillating physical systems, like the ordered Hamilton-Jacobi harmonic oscillator. Another important point is exploring how classical chaos affects the harmonic oscillator, giving us insights into realistic systems in classical, quantum, and relativistic domains. While Newtonian physics suggests the harmonic oscillator is integrable no matter what, this approach lets us explore chaotic behavior in relativistic, one dimensionally driven oscillators, helping us understand complex dynamical systems better.

8. Discussion of the Figures: The Impact of Dissipation on the Quantum Oscillator

In this section, we analyze the effect of dissipation on the quantum two-dimensional anisotropic oscillator using mathematical analysis and graphical representations that illustrate the relationship between various dynamic parameters. Graphical representations have been provided to demonstrate how energy and probability distributions are influenced by dissipation. The figures have been plotted with enhanced clarity to ensure a better understanding of the results.

8.1. Relationship Between Energy and Dynamic Parameters

As shown in Figure (1), the total energy of the dissipative quantum oscillator is analyzed as a function of a key parameter related to thermal and quantum effects. The results indicate that increasing this parameter leads to a rise in total energy due to quantum contributions. Additionally, at low temperatures, the dissipation effect becomes more pronounced, causing a significant deviation from classical behavior. This highlights the fundamental role of dissipation in shaping the quantum dynamics of the system. At low temperatures, the effect of dissipation becomes more pronounced, leading to a significant deviation from classical behavior. This indicates that dissipation plays a fundamental role in shaping the quantum dynamics of the system, especially under conditions where quantum effects dominate. This deviation can be explained by the fact that dissipation leads to continuous energy loss, affecting the quantum energy distribution and altering the system's states. In Figure

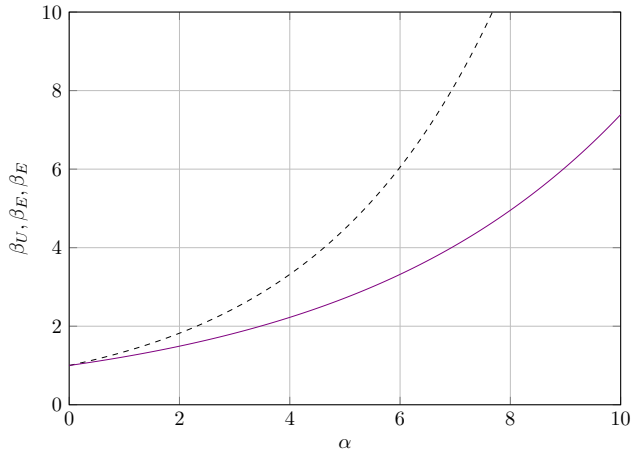

Figure 1: Illustrates how dissipation affects the energy probability distribution in a quantum system, leading to energy transfer and changes in the system's states due to interaction with the environment.

(2), we analyze the impact of the damping coefficient on energy. The graph clearly illustrates that increasing the damping coefficient results in higher internal energy levels, confirming that dissipation directly influences the quantum energy spectrum and alters the system's dynamics.

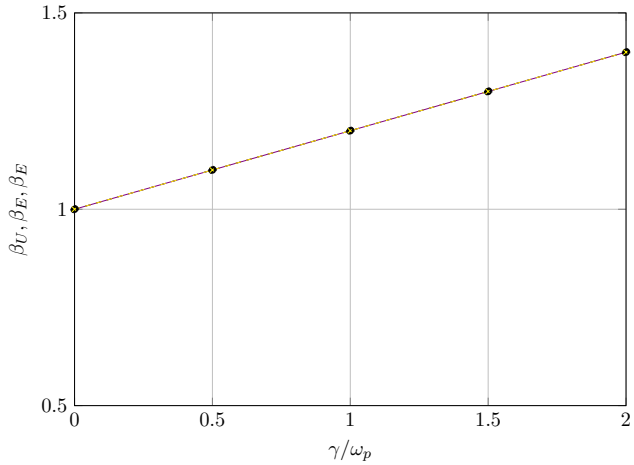

Figure 2: The dimensionless energy

This increase in energy can be explained by the fact that dissipation leads to additional interactions between the system and its surrounding environment, increasing the system's internal energy. This aligns with the theory that dissipation alters the quantum energy distribution and leads to the emergence of new energy states.

8.2. Probability Distribution and the Effect of Dissipation

Figure (3) presents the probability distribution as an indicator of the quantum interaction with the surrounding environment. It is observed that dissipation shifts the probability distribution towards higher energy values, reflecting the energy loss induced by system-environment interactions. This shift in probability distribution indicates that dissipation not only reduces the total energy of the system but also alters the distribution of quantum states. This can be explained by the fact that dissipation causes energy to transfer from the system to the surrounding environment, changing the probability distribution of quantum states.

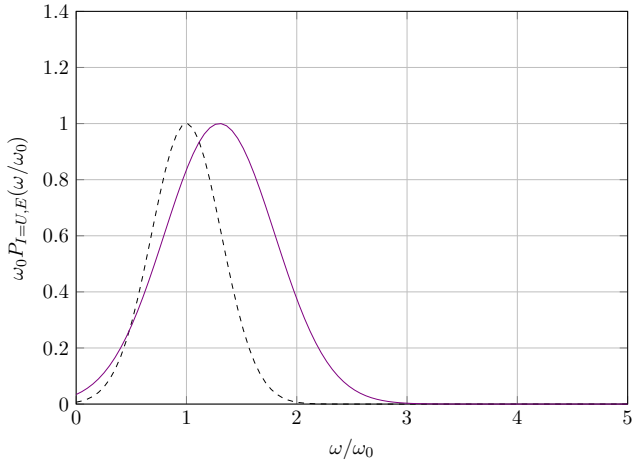

Figure 3: The distribution functions for the dissipative oscillator, showing the functions as a function of the rescaled heat-bath frequency. The parameters are set to specific value

Moreover, Figures (4a, 4b, 4c) illustrate the effect of a magnetic field on the energy distribution in a three dimensional quantum oscillator. The interaction between damping and the magnetic field leads to the emergence of multiple peaks in the probability distribution, indicating resonance effects caused by the interplay between quantum energy levels and dissipation. The appearance of multiple peaks in the probability distribution reflects resonance effects that occur when the system's frequencies match those of the surrounding environment. This suggests that dissipation and the magnetic field work together to modify the quantum energy distribution, leading to the emergence of new energy states and changes in the system's dynamics.

9. Future Directions in the Study of Dissipative Forces in Lagrangian Mechanics

In future studies, this model can be expanded to three dimensions to investigate the impact of the additional dimension on the dynamics of the dissipative system. Furthermore, a coupling term can be introduced to analyze its effect on the system,

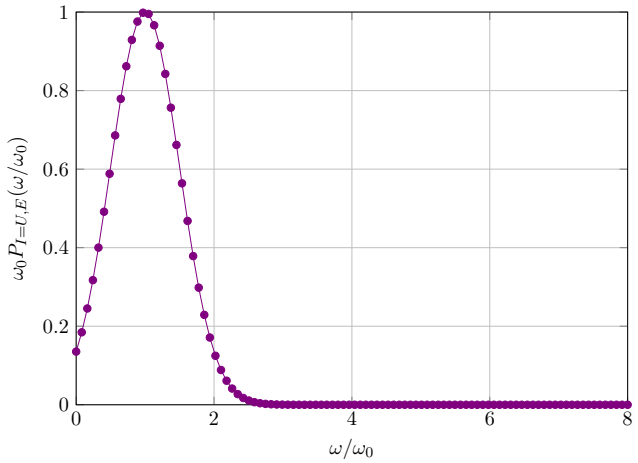

Figure 4: $w_c = 0.5w_0$

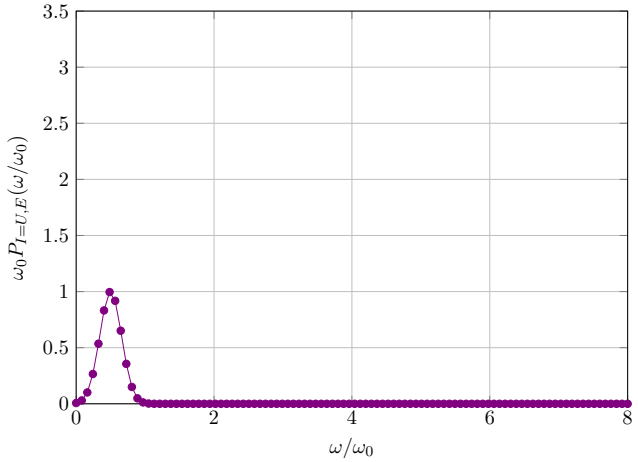

Figure 5: $w_c = 0.5w_0$

particularly in cases where the anisotropy is strong or when there are additional interactions between the dimensions. While the current study provides a comprehensive framework for understanding the two dimensional anisotropic harmonic oscillator under dissipative effects, there remain opportunities for further exploration. One potential avenue is the extension of this model to three dimensions, which could offer deeper insights into the dynamics of dissipative systems. Additionally, the inclusion of coupling terms, particularly in the context of anisotropic behavior, represents an important area for future investigation. Although coupling terms were excluded in the present work under the assumption of weak anisotropy, future studies could examine their impact under different conditions and assess their role in complex dissipative systems. Such developments would enrich the theoretical understanding and practical applicability of the proposed framework.

10. Current vs. Previous Research on Quantizing Dissipative

This study focuses on the quantization of dissipative mechanical systems using methods such as the WKB approximation, the Hamilton-Jacobi equation, the Staeckel boundary conditions, and canonical quantization. The following comparison highlights the primary differences between this study and previous research:

Canonical Quantization: Previous studies, including those by Dekker, Suzuki, and Majima, focused on modeling time-dependent damping variables. In contrast, this study combines canonical quantization with the WKB approximation, resulting in a more comprehensive and accurate quantum model for dissipative systems.

Non-Conservative Systems: Riewe's work with fractional calculus helped develop new models for non conservative systems. This study takes these ideas further by applying them to quantum dissipative systems using modern techniques, such as the Hamilton-Jacobi equation, to provide a clearer and more accurate quantum description.

WKB Approximation: The WKB approximation has been widely used in previous studies to analyze dissipative systems. This study extends its use to quantum dissipative systems, improving the precision of energy level predictions and the accuracy of state quantization.

Creation and Annihilation Operators: The integration of dissipation into the quantum framework through the use of creation and annihilation operators results in more accurate predictions for energy levels and quantum states. By explicitly incorporating dissipative effects—such as energy loss due to damping or environmental interactions—the model becomes more realistic,

closely reflecting the physical behavior of the system. This refinement ensures that the derived energy levels and wave functions are more precise, as they account for the impact of dissipation on the system's dynamics. As a result, the theoretical predictions align more closely with experimental observations, significantly enhancing the reliability and applicability of the framework to real-world dissipative quantum systems.

11. Conclusions

In this paper, we investigated the dissipative two-dimensional anisotropic harmonic oscillator using the Hamilton Jacobi formalism. We obtained the Lagrangian and Hamiltonian equations, solved the Hamilton-Jacobi equation, and quantified the system using the WKB approximation and canonical quantization approaches. We formulated the equations of motion and the Heisenberg equations by isolating variables and using Poisson brackets and commutation relations. Using creation and annihilation operations, we were able to derive energy levels and eigenstates that nearly matched the results of canonical quantization. In exceptional situations, for one dimension only, our results are in precise accord with other quantization schemes as

described in [24]. This study contributes to quantum mechanics by modeling damped oscillatory systems, an area that has been insufficiently studied. A new Schrödinger equation for the two-dimensional anisotropic oscillator enhances the understanding of quantum dynamics. By integrating techniques like the WKB approximation and canonical quantization, the study consistently derives energy levels and eigenstates, providing insights into dissipative effects. The results show strong consistency. The results show strong consistency, reinforcing the proposed model's credibility and its applicability to similar systems, thus advancing the understanding of dissipative systems in quantum mechanics. Key findings of this study include:

Hamilton-Jacobi Equation Solutions: The Hamilton-Jacobi equation was solved using the separation of variables method under Staeckel boundary conditions, leading to explicit expressions for the action function.

Consistent Quantization Approaches: The system was quantized using three different methods—WKB approximation, canonical quantization, and creation-annihilation operators. Remarkably, all three methods yielded consistent results, reinforcing the robustness of the proposed model.

Comparison with Classical Results: In the special case of a one-dimensional system, the obtained quantum results align precisely with classical predictions, verifying the reliability of the quantization framework.

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Appendix A: Detailed Derivations and Solution Steps

In this appendix, we present an analysis of the Hamilton-Jacobi equation for the motion of a two-dimensional harmonic oscillator using Staeckel boundary conditions for separation of variables. The Hamiltonian equation for the dissipative system is derived and analyzed using the Hamilton-Jacobi function $W(q)$ to separate the variables in the x and y directions. After applying Staeckel conditions, precise mathematical solutions are obtained, enabling a deeper understanding of the system's behavior and the calculation of energy levels and quantum states using creation and annihilation operators.

$$\frac{1}{2m}\left(\left(\frac{\partial W_x}{\partial x}\right)^2+4\pi^2m^2v_x^2x^2+\left(\frac{\partial W_y}{\partial y}\right)^2+4\pi^2m^2v_y^2y^2\right)=\alpha \qquad (A.1)$$

By employing variable separation in equation (A.1) above in terms of and W_y, resulting in: $\frac{1}{2m}\left(\left(\frac{\partial W_y}{\partial y}\right)^2+4\pi^2m^2v_y^2y^2\right)-\frac{1}{2}m^2\lambda^2y^2=\alpha_y \qquad (A.2)$

$$\frac{1}{2m}\left(\left(\frac{\partial W_x}{\partial x}\right)^2+4\pi^2m^2v_x^2x^2\right)-\frac{1}{2}m^2\lambda^2x^2+\alpha_y=\alpha_x \qquad (A.3)$$

After integration, we may express equations (A.2) and (A.3) as follows:

$$W_{y'}=\int_0^{y'}\sqrt{(2m\alpha_y-4\pi^2m^2v_y^2y^2+\frac{1}{2}m^2\lambda_y^2y^2)}dy \qquad (A.4)$$

$$W_{x'}=\int_0^{x'}\sqrt{(2m\alpha-2m\alpha_y-4\pi^2m^2v_x^2y^2+\frac{1}{2}m^2\lambda_x^2x^2)}dx \qquad (A.5)$$

Substituting equations. (A.4) and (A.5) into the equation $W_{y'}+W_{x'}=W$, we obtain:

$$W=\int_0^{y'}\sqrt{(2m\alpha_y-4\pi^2m^2v_y^2y^2+\frac{1}{2}m^2\lambda_y^2y^2)}dy+$$
$$\int_0^{x'}\sqrt{(2m\alpha-2m\alpha_y-4\pi^2m^2v_x^2y^2+\frac{1}{2}m^2\lambda_x^2x^2)}dx$$

With these result, the Hamilton-Jacobi function $S(q,\alpha,t)=W(q,\alpha)-\alpha t$ becomes:

$$S(q,\alpha,t)=\int_0^{y'}\sqrt{(2m\alpha_y-4\pi^2m^2v_y^2y^2+\frac{1}{2}m^2\lambda_y^2y^2)}dy+$$
$$\int_0^{x'}\sqrt{(2m\alpha-2m\alpha_y-4\pi^2m^2v_x^2y^2+\frac{1}{2}m^2\lambda_x^2x^2)}dx-\alpha t$$

Taking the derivative with respect to α_i for equation (A.6), one gets:

$$B_{y'}=\frac{\partial S}{\partial\alpha_{y'}}=\int_0^{y'}\frac{m}{2m\alpha_y-4\pi^2m^2v_y^2y^2+\frac{1}{2}m^2\lambda_y^2y^2)}dy-$$

$$\frac{sin^{-1}\left(\sqrt{\left(\frac{2m\pi^2 v_{y'}^2 - \frac{1}{4}m\lambda_{y'}^2}{\alpha_{y'}}\right)}y'\right)}{\sqrt{4\pi^2 v_{y'}^2}} - \frac{sin^{-1}\left(\sqrt{\left(\frac{2m\pi^2 v_{x'}^2 - \frac{1}{4}m\lambda_{x'}^2}{\alpha - \alpha_{y'}}\right)}x'\right)}{\sqrt{4\pi^2 v_{x'}^2}} \tag{A.7}$$

$$\beta_{x'} + t = \int_0^{x'} \frac{m}{2m\alpha - 2m\alpha_y - 4\pi^2 m^2 v_x^2 x^2 + \frac{1}{2}m^2\lambda_x^2 x^2)}dx =$$

$$\frac{sin^{-1}\left(\sqrt{\left(\frac{2m\pi^2 v_{x'}^2 - \frac{1}{4}m\lambda_{x'}^2}{\alpha - \alpha_{y'}}\right)}x'\right)}{\sqrt{4\pi^2 v_{x'}^2}} \tag{A.8}$$

Equation (A.8) allows us to formulate the result as

$$\beta_{x'} + t = \frac{sin^{-1}\left(\sqrt{\left(\frac{2m\pi^2 v_{x'}^2 - \frac{1}{4}m\lambda_{x'}^2}{\alpha - \alpha_{y'}}\right)}x'\right)}{\sqrt{4\pi^2 v_{x'}^2}}$$

After some algebraic work, we obtain:

$$x' = \sqrt{\left(\frac{\alpha - \alpha_y}{2m\pi^2 v_{x'}^2 - \frac{1}{4}m\lambda_{x'}^2}\right)}sin\left[\sqrt{4\pi^2 v_{x'}^2}(\beta_{x'} + t)\right] \tag{A.9}$$

The above equation is exactly the same as the equation that has been derived by (equation (15.a)) in the same classical solution. The formulation enabled a mathematically rigorous description of the system dynamics, accurately deriving energy levels and quantum states through creation and annihilation operators, highlighting the strength of this method. The creation and annihilation operators provide a powerful framework for quantizing the dissipative two-dimensional anisotropic harmonic oscillator. By expressing position and momentum in terms of these operators, the Hamiltonian is simplified, enabling the systematic derivation of energy eigenvalues and eigenfunctions. The energy levels are expressed in terms of quantum numbers nx and ny , while the wavefunctions are constructed using Hermite polynomials. This approach ensures consistency with classical mechanics and other quantization methods, such as the WKB approximation and canonical quantization, demonstrating the robustness and versatility of the operator formalism in analyzing dissipative quantum systems.

Appendix B: Additional Derivations and Quantization

In this appendix, we provide further derivations concerning the quantum treatment of the dissipative twodimensional anisotropic harmonic oscillator. We discuss the application of Heisenberg's equations and the standard rules of quantization to derive the equations of motion for the system in its quantum form. Additionally, a thorough analysis of the creation and annihilation operators is presented, which are instrumental in deriving the energy eigenvalues and wave functions. The goal of this section is to offer deeper insights into the quantum methodology employed and to illustrate the consistent derivation of both classical and quantum results. The commutation relations are as follows:

$$\ddot{x} = \frac{\dot{P}_x}{m} \text{ therefore } \ddot{x} + (4_x^2 - \frac{1}{4}\lambda_x^2)x = 0 \qquad (B.1)$$

$$\ddot{y} = \frac{\dot{P}_y}{m} \text{ which means that } \ddot{y} + (4_y^2 - \frac{1}{4}\lambda_y^2)y = 0 \qquad (B.2)$$

Given the complete alignment with equations (2) and (3), it follows that $\{x, P_x\} = \{y, P_y\} = 1$ implies $[x, P_x] = [y, P_y] = ih$ We can now proceed to introduce the pairs associated with creation and annihilation [23].

$$a^+ = \sum_{i=1}^{2} \sqrt{\frac{mw_i}{2h}}(x_j - i\frac{P_j}{mw_j} \qquad a^+ = \sum_{i=1}^{2} \sqrt{\frac{mw_i}{2h}}(x_j + i\frac{P_j}{mw_j} \qquad (B.3)$$

Then, solving for x_j and P_j, we have:

$$x_j = \sum_{i=1}^{2} \sqrt{\frac{h}{mw_j}}(a_i + a_j^+) \qquad P_j = -i\sum_{i=1}^{2}(a_i - a_j^+)a \qquad (B.4)$$

It is easy to verify that $[x_j, P_j] = ik$ implies $[a, a^+] = 1$. Once we compute $H = P_x\dot{x} + P_y\dot{y} - L$, we obtain:

$$H = \sum_{i=1}^{2} hw_j(a_j^+ a_j + \frac{1}{2}) \qquad (B.5)$$

The energy eigenvalues are given by:

$$E = E_n = hw_j(2\pi n_x v_x + 2_y v_y + \frac{1}{2}2\pi(v_x + v_y) \qquad (B.6)$$

Where $n_x = 1, 2, \cdots$ and $n_y = 1, 2, \cdots$
The ground-state wave function $\psi_0(x, y))$ obeys a $\psi_0(x, y)) = 0$ where

$$\psi_0(x, y)) = \phi_0(x)\upsilon_0(x) \qquad (B.7)$$

The ground-state wave function $\psi_0(x,y))$, expressed as the product of functions $\phi_0(x)\upsilon_0(x)$, satisfies the condition $\psi_0(x,y)) = 0$, resulting in:

$$\phi_0(x) = \left(\frac{mw_x}{\pi k}\right)^{\frac{1}{4}} e^{\frac{-mw_x x^2}{2h}} \qquad (B.8)$$

$$\upsilon_0(x) = \left(\frac{mw_x}{\pi k}\right)^{\frac{1}{4}} e^{\frac{-mw_y y^2}{2h}} \qquad (B.9)$$

When we replace equation (B.7) and (B.8) with equation (B.6), we get:

$$\psi_0(x,y)) = \left(\frac{m}{\pi h}\right)^{\frac{1}{2}} (w_x)^{\frac{1}{4}}(w_y)^{\frac{1}{4}} e^{\frac{-m(w_y y^2 + w_x x^2)}{2h}} \qquad (B.10)$$

Hermite polynomials allow us to express the eigenfunctions for every given state n as follows:

$$H_{nm}(x,y) = B_{nm}(-1)^{n+m} e^{x^2+y^2} \frac{\partial^{m+n}}{{}^n\partial^m} e^{-x^2-y^2} \qquad (B.11)$$

$$\psi_{nm}(x,y)) = \left(\frac{m}{\pi h}\right)^{\frac{1}{2}} (w_x)^{\frac{1}{4}}(w_y)^{\frac{1}{4}} e^{\frac{-m(w_y y^2 + w_x x^2)}{2h}} H_{nm}\left(\frac{nw_x}{h}x, \frac{mw_y}{h}y\right) \qquad (B.12)$$

These expressions, particularly ( ), validate the findings presented in equation 22, indicating their consistency and concordance with the obtained results