EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 5832
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

An Investigation of q-Rung Orthopair Fuzzy Subgroups and Fundamental Isomorphism Theorems

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Abstract. Abstract: The q-rung orthopair fuzzy set (q-ROFS) has been developed as an extension of the Pythagorean fuzzy set (PFS) to address ambiguity in various decision-making contexts. Group theory, a significant area of mathematics, has extensive applications across diverse scientific fields, including cryptography, pattern recognition, and network analysis. This paper examines q-rung orthopair fuzzy group theory, emphasizing the importance of q-ROFS and group theory. The concept of a q-rung orthopair fuzzy subgroup (q-ROFSG) is introduced, and its various algebraic properties are examined. A comprehensive investigation into q-rung orthopair fuzzy cosets (q-ROFCs) and q-rung orthopair fuzzy normal subgroups (q-ROFNSGs) has been conducted. The definitions of q-rung orthopair fuzzy homomorphism and isomorphism are presented. We extend the concept of the quotient group of a classical group V in relation to its normal subgroup U by introducing a q-ROFSG of V/U. The q-rung orthopair fuzzy variant of the three fundamental isomorphism theorems has been demonstrated.

2020 Mathematics Subject Classifications: 03E72, 20N25

Key Words and Phrases: q-ROFS, q-ROFSG, q-ROFCs, q-rung orthopair fuzzy isomorphism

1. Introduction

1.1. Background

Zadeh [1] proposed the notion of a fuzzy set (FS) which has many applications on decision making and various other fields. A fuzzy subset Q of a conventional set V is a function μ_Q from V to a unit interval. In other words, if each element ϵ of V is associated with a real number from $[0,1]$ then this setting is called fuzzy subset Q of V and is written as $Q = \epsilon, \mu_Q(\epsilon) : \epsilon \in V$. In this context, μ_Q is designated as the membership function, whereas $\mu_Q\ (\epsilon) \in [0,1]$ represents the membership degree of $\epsilon \in V$ with regard to FS Q. The theory of FS is a potent instrument that organically extends the boundaries of crisp set theory. The characteristic function of a crisp set, which assigns

DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.5832

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a value of 1 if $\epsilon \in V$, and 0 otherwise, is equivalent with the membership function of the set. In response to the challenges posed by ambiguity and uncertainty, a multitude of innovative concepts have surfaced since the inception of fuzzy sets. To address vagueness and ambiguity, numerous theories have been devised, some of which are extensions of the FS theory. Certain data types might not be adequately described by membership value alone. As a result, the "non-membership value" has been incorporated in order to accurately represent nuanced information. In 1986, Atanassov [2] introduced the notion of intuitionistic fuzzy sets (IFSs) and presented an extension of classical fuzzy sets (FSs). An IFS Q, associated with a crisp set V, can be represented as an entity $(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V$, where $\mu_Q : V \to [0,1]$ and $\nu_Q : V \to [0,1]$, are membership and non-membership functions satisfying $\mu_Q(\epsilon) + \nu_Q(\epsilon) \leq 1$ for all $s \in V$. IFSs, contrary to classical FSs, integrate membership and non-membership functions in order to more effectively manage ambiguity and uncertainty in practice, particularly in the context of decision-making [3–6]. Yager[7] generalized IFSs in 2013 by proposing the concept of a Pythagorean fuzzy set (PFS) $Q = (\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V$, where $\mu_Q : V \to [0,1]$ and $\nu_Q : V \to [0,1]$, are membership and non-membership functions, respectively, that satisfy the condition $(\mu_Q(\epsilon))^2 + (\nu_Q(\epsilon))^2 \leq 1$ for all $s \in V$. This conceptual framework aims to turn uncertainty and ambiguity into a mathematical domain to help solve complicated physical problems [8–11]. Pythagorean fuzzy subsets (PFSs) have solved many real-world issues, however they have limits that require further refining. Pythagorean fuzzy subsets cannot accommodate decision-makers who advocate membership and non-membership values like 0.85 and 0.65. This constraint arises from the condition $(0.85)^2 + (0.65)^2 > 1$. Consequently, Pythagorean fuzzy subsets are inadequate for addressing such situations. In response to this limitation, Yager [12] introduces the concept of the q-rung orthopair fuzzy set (q-ROFS), wherein q is a natural number. This innovative concept aims to provide a rational solution for addressing scenarios of the aforementioned nature. The q-ROFS is formally denoted as $Q = (\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V$, where $\mu_Q : V \to [0,1]$ and $\nu_Q : V \to [0,1]$, are membership and non-membership functions, respectively, that ensure the condition that $(\mu_Q(\epsilon))^q + (\nu_Q(\epsilon))^q \leq 1$ for all $\epsilon \in V$. For further insights into the practical uses of q-ROFSs, we suggest referring to the sources [13–15].

1.2. Literature Review and Present Study

Rosenfeld [16] introduced the idea of fuzzy subgroups (FSs) and extended the idea of traditional subgroups. Since the advent of FSs, multiple efforts have been made to extend the ideas of conventional groups in a variety of fuzzy settings. The concept of a level subgroup of an FS was developed by Das in [17]. The study investigated several algebraic implications of this topic. Liu [18], presented the idea of fuzzy invariant subgroups and established several key features of this term. In [19], a proof of a fuzzy variant of Lagrange's theorem was established by Mukherjee and Bhattacharya. The same authors established a variety of results that are analogues to several fundamental theorems of traditional group theory [20]. In [21], the idea of a semidirect product of FSs is set up and studied, along with a number of findings about this topic. In [22–24], significant research was carried out on level fuzzy subgroups and normal FSs. Biswas [25], started work on intuitionistic fuzzy subgroups (IFSGs). In [26], several group theoretic findings pertaining to cosets of a subgroup are explored in an IFS framework. In [27], the concept of the direct product of IFSGs is established. The concept of ω-fuzzy subrings was first explored by Altassan et al. in [28]. Alharbi and Alghazzawi [29] introduced the notion of (ρ, η) complex fuzzy subgroups. For more information on IFSGs, the readers should consult [30–32]. Bhunia et al. [33], have defined the concept of Pythagorean fuzzy subgroups (PFSG). The scholars provided some of the fundamental characteristics of this idea. Razaq et al. [34] proved many results related to Pythagorean normal subgroup and Pythagorean fuzzy quotient groups. In [35], the notion of q-rung orthopair fuzzy subgroups is introduced. Addis et al. [36], established fuzzy homomorphism theorems, bridging fuzzy kernels and ideals. The aforementioned summary of the literature presents a few of the most significant findings from studies of FSG, IFSG and PFSG. In addition, some results concerning q-rung orthopair fuzzy cosets, q-rung orthopair fuzzy normal subgroups, and q-rung orthopair fuzzy level subgroups have been investigated. However, numerous unsolved inquiries remain. Keeping in mind a few of those questions, the present research has been carried out. Our main contributions are as follows;

(i) Considering the importance of cosets in classical group theory, we have proved many results related to q-ROFCs of a q-ROFSG.

(ii) Since the normal subgroups and quotient groups are central notions in the theory of groups, therefore we have defined Pythagorean fuzzy normal subgroup of a q-ROFSG and generalized the notion of quotient group in q-rung orthopair fuzzy environment. Furthermore, several theorems associated with these concepts have been proved.

(iii) The concepts of q-rung orthopair fuzzy homomorphism and isomorphism have been defined.

(iv) Last but not the least, all three fundamental theorems of group isomorphisms have been generalized in q-rung orthopair fuzzy format.

The rest of the paper is structured as follows: the second section covers the basic definitions and concepts essential to demonstrate our key findings. The qq-ROFCs of a q-ROFSG and q-ROFNSGs of a group are discussed in the third section. We also demonstrate many theorems relevant with these concepts. In section 4, we define the q-rung orthopair fuzzy normal subgroup of a q-rung orthopair fuzzy subgroup and examine this idea in detail. In the fifth section, the concepts of q-rung orthopair fuzzy homomorphism and isomorphism are defined. By introducing a q-ROFSG of V/U we thus extend the idea of quotient group of a classic group V related to its normal subgroup U. In addition, the q-rung orthopair fuzzy formulation of fundamental isomorphism theorems have also been demonstrated.

2. Preliminaries

Definition 1. *[16] An FS $Q = \{(\epsilon, \mu_Q(\epsilon)) : \epsilon \in V\}$ of a conventional group V is called a*

FSG of V if the preceding conditions are satisfied:

(i) $\mu_Q(\epsilon_1\epsilon_2) \geq \min\{\mu_Q(\epsilon_1), \mu_Q(\epsilon_2)\}$ *for all* $\epsilon_1, \epsilon_2 \in V$.

(ii) $\mu_Q(\epsilon^{-1}) \geq \mu_Q(\epsilon)$ *for all* $\epsilon \in V$.

Definition 2. *[25] An IFS $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V\}$ of V referred to be an IFSG of V if the following conditions are met:*

(i) $\mu_Q(\epsilon_1\epsilon_2) \geq \min\{\mu_Q(\epsilon_1), \mu_Q(\epsilon_2)\}$ *and*
$\nu_Q(\epsilon_1\epsilon_2) \leq \max\{\nu_Q(\epsilon_1), \nu_Q(\epsilon_2)\}$ *for all* $\epsilon_1, \epsilon_2 \in V$.

(ii) $\mu_Q(\epsilon^{-1}) \geq \mu_Q(\epsilon)$ *and* $\nu_Q(\epsilon^{-1})) \leq \nu_Q(\epsilon)$ *for all* $\epsilon \in V$.

Definition 3. *[33] A PFS $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V\}$ of V referred to be a PFSG of V if the following conditions are met:*

(i) $(\mu_Q(\epsilon_1\epsilon_2))^2 \geq \min\{(\mu_Q(\epsilon_1))^2, (\mu_Q(\epsilon_2))^2\}$ *and*
$(\nu_Q(\epsilon_1\epsilon_2))^2 \leq \max\{(\nu_Q(\epsilon_1))^2, (\nu_Q(\epsilon_2))^2\}$ *for all* $\epsilon_1, \epsilon_2 \in V$.

(ii) $(\mu_Q(\epsilon^{-1}))^2 \geq (\mu_Q(\epsilon))^2$ *and* $(\nu_Q(\epsilon^{-1}))^2 \leq (\nu_Q(\epsilon))^2$ *for all* $\epsilon \in V$.

Definition 4. *[35] A q-ROFS $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V\}$ of V referred to be a q-ROFSG of V if the following conditions are met:.*

(i) $(\mu_Q(\epsilon_1\epsilon_2))^q \geq \min\{(\mu_Q(\epsilon_1))^q, (\mu_Q(\epsilon_2))^q\}$ *and*
$(\mu_Q(\epsilon_1\epsilon_2))^q \leq \max\{(\nu_Q(\epsilon_1))^q, (\nu_Q(\epsilon_2))^q\}$ *for all* $\epsilon_1, \epsilon_2 \in V$.

(ii) $(\mu_Q(\epsilon^{-1}))^q \geq (\mu_Q(\epsilon))^q$ *and* $(\nu_Q(\epsilon^{-1}))^q \leq (\nu_Q(\epsilon))^q$ *for all* $\epsilon \in V$.

Theorem 1. *[35] A q-ROFS Q of V is a q-ROFSG of $V \Leftrightarrow (\mu_Q(\epsilon_1\epsilon_2^{-1}))^q \geq \min\{(\mu_Q(\epsilon_1))^q, (\mu_Q(\epsilon_2))^q\}$ and $(\nu_Q(\epsilon_1\epsilon_2^{-1}))^q \leq \max\{(\nu_Q(\epsilon_1))^q, (\nu_Q(\epsilon_2))^q\}$ for all $\epsilon_1, \epsilon_2 \in V$.*

Theorem 2. *[35] Let Q and K be two q-ROFSGs of V. Then $Q \cap K$ is a q-ROFSG of V.*

The notion of q-rung orthopair fuzzy level subset (q-ROFLS) is presented in [35]. Herein, we repeat the definition of q-ROFLS together with some associated findings.

Definition 5. *[35] Let $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V\}$ be a q-ROFS of V and $\theta, \tau \in [0, 1]$ satisfying $\theta^q + \tau^q \leq 1$. Then the set $Q_{(\theta,\tau)} = \{\epsilon \in V : (\mu_Q(\epsilon))^q \geq \theta, (\nu_Q(\epsilon))^q \leq \tau\}$ is known as q-ROFLS of Q.*

The succeeding Theorem presented in [36] is subsequently utilized to demonstrate our findings.

Theorem 3. *[35] Let Q and K be two q-ROFSs of V and $Q \subseteq K$, then $Q_{(\theta,\tau)}) \subseteq K_{(\theta,\tau)}$ for all $\theta, \tau \in [0, 1]$ satisfying $\theta^q + \tau^q \leq 1$.*

Theorem 4. *[35] A q-ROFS Q of V is a q-ROFSG of $V \Leftrightarrow Q_{(\theta,\tau)}$ is a subgroup of V for all $\theta \in [0, (\mu_Q(e))^q]$ and $\tau \in [(\nu_Q(e))^q, 1]$.*

Now, we present the concepts of q-rung orthopair fuzzy left and right cosets of a q-ROFSG of a group that is defined in [36].

Definition 6. *[35] Let $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V\}$ be a q-ROFSG of V and $\lambda \in V$. Then q-rung orthopair fuzzy left coset (q-ROFLC) of Q associated to λ is denoted by λQ such that $\lambda Q = \{(\epsilon, \mu_{\lambda Q}(\epsilon), \nu_{\lambda Q}(\epsilon)) : \epsilon \in V\}$, where $(\mu_{\lambda Q}(\epsilon))^q = (\mu_Q(\lambda^{-1}\epsilon))^q$ and $(\nu_{\lambda Q}(\epsilon))^q = (\nu_Q(\lambda^{-1}\epsilon))^q$. Similarly, $Q\lambda = \{(\epsilon, \mu_{Q\lambda}(\epsilon), \nu_{Q\lambda}(\epsilon)) : \epsilon \in V\}$, where $(\mu_{Q\lambda}(\epsilon))^q = (\mu_Q(\epsilon\lambda^{-1}))^q$ and $(\nu_{Q\lambda}(\epsilon))^q = (\nu_Q(\epsilon\lambda^{-1}))^q$, is called q-rung orthopair fuzzy right coset (q-ROFRC) of Q associated to λ.*

Next, we present the definition of q-rung orthopair fuzzy normal subgroup (q-ROFNSG) of V that is demonstrated in [35].

Definition 7. *[35] A q-ROFSG $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V\}$ of V is called q-ROFNSG of V if $\lambda Q = Q\lambda$ for all $\lambda \in V$.*

Theorem 5. *[35] Let $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V\}$ be a q-ROFSG of V. Then Q is PFNS of V if and only if $(\mu_Q(\epsilon_1\epsilon_2))^q = (\mu_Q(\epsilon_2\epsilon_1))^q$ and $(\nu_Q(\epsilon_1\epsilon_2))^q = \nu_Q(\epsilon_2\epsilon_1))^q$ for all $\epsilon_1, \epsilon_2 \in V$.*

Theorem 6. *[35] Let $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V\}$ be a q-ROFSG of V. Then Q is PFNS of V if and only if $(\mu_Q(\epsilon\lambda\epsilon^{-1}))^q = (\mu_Q(\lambda))^q$ and $(\nu_Q(\epsilon\lambda\epsilon^{-1}))^q = \nu_Q(\lambda))^q$ for all $\lambda, \in V$.*

3. More on q-rung orthopair fuzzy cosets of a q-rung orthopair fuzzy subgroup and q-rung orthopair fuzzy normal subgroups of a group

In this section, we establish some fundamental facts about q-ROFSG that serve as the foundation for proving the theorems discussed in Sections 4 and 5. This section is dedicated to expanding the ideas of q-rung orthopair fuzzy cosets of a q-ROFSG and q-ROFNSGs of a crisp group by establishing a number of related theorems.
The example below describes the notion of q-ROFLCs of a q-ROFSG of a group.

Example 1. *Let us take a finite group*

$$V = S_3 = \langle \alpha, \beta : \alpha^2 = \beta^3 = (\alpha\beta)^1 = 1 \rangle = \{e, \alpha, \beta, \beta^2, \alpha\beta, \alpha\beta^2\}$$

containing symmetries of equilateral triangle. The generators α and β of V represent diagonal reflection and rotation of 120° anti-clockwise respectively. This yields a total of six symmetries $e, \alpha, \beta, \beta^2, \alpha\beta, \alpha$ and β^2 of equilateral triangle. Next, we design a 3-ROFSG of V as follows;

$$Q = \left\{ \begin{array}{l} (e, 0.85, 0.65), (\beta, 0.85, 0.65), (\beta^2, 0.85, 0.65), \\ (\alpha, 0.80, 0.75), (\alpha\beta, 0.80, 0.75), (\alpha\beta^2, 0.80, 0.75) \end{array} \right\}$$

Then, we compute the q-ROFLCs of Q for all $\epsilon \in V$.

(i) The q-ROFLC of Q for $e \in V$ is

$$eQ = \left\{ \begin{array}{l} \left(e, \mu_Q(e^{-1}e), \nu_Q(e^{-1}e)\right), \left(\beta, \mu_Q(e^{-1}\beta), \nu_Q(e^{-1}\beta)\right), \\ \left(\beta^2, \mu_Q(e^{-1}\beta^2), \nu_Q(e^{-1}\beta^2)\right), \left(\alpha, \mu_Q(e^{-1}\alpha), \nu_Q(e^{-1}\alpha)\right), \\ \left(\alpha\beta, \mu_Q(e^{-1}\alpha\beta), \nu_Q(e^{-1}\alpha\beta)\right), \left(\alpha\beta^2, \mu_Q(e^{-1}\alpha\beta^2), \nu_Q(e^{-1}\alpha\beta^2)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} (e, 0.85, 0.65), (\beta, 0.85, 0.65), (\beta^2, 0.85, 0.65), \\ (\alpha, 0.80, 0.75), (\alpha\beta, 0.80, 0.75), (\alpha\beta^2, 0.80, 0.75) \end{array} \right\}$$

(ii) The q-ROFLC of Q for $\beta \in V$ is

$$\beta Q = \left\{ \begin{array}{l} \left(e, \mu_Q(\beta^{-1}e), \nu_Q(\beta^{-1}e)\right), \left(\beta, \mu_Q(\beta^{-1}\beta), \nu_Q(\beta^{-1}\beta)\right), \\ \left(\beta^2, \mu_Q(\beta^{-1}\beta^2), \nu_Q(\beta^{-1}\beta^2)\right), \left(\alpha, \mu_Q(\beta^{-1}\alpha), \nu_Q(\beta^{-1}\alpha)\right), \\ \left(\alpha\beta, \mu_Q(\beta^{-1}\alpha\beta), \nu_Q(\beta^{-1}\alpha\beta)\right), \left(\alpha\beta^2, \mu_Q(\beta^{-1}\alpha\beta^2), \nu_Q(\beta^{-1}\alpha\beta^2)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left(e, \mu_Q(\beta^2), \nu_Q(\beta^2)\right), \left(\beta, \mu_Q(e), \nu_Q(e)\right), \left(\beta^2, \mu_Q(\beta), \nu_Q(\beta)\right), \\ \left(\alpha, \mu_Q(\alpha\beta), \nu_Q(\alpha\beta)\right), \left(\alpha\beta, \mu_Q(\alpha\beta^2), \nu_Q(\alpha\beta^2)\right), \left(\alpha\beta^2, \mu_Q(\alpha), \nu_Q(\alpha)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} (e, 0.85, 0.65), (\beta, 0.85, 0.65), (\beta^2, 0.85, 0.65), \\ (\alpha, 0.80, 0.75), (\alpha\beta, 0.80, 0.75), (\alpha\beta^2, 0.80, 0.75) \end{array} \right\}$$

(iii) The q-ROFLC of Q for $\beta^2 \in V$ is

$$\beta^2 Q = \left\{ \begin{array}{l} \left(e, \mu_Q((\beta^2)^{-1}e), \nu_Q((\beta^2)^{-1}e)\right), \left(\beta, \mu_Q((\beta^2)^{-1}\beta), \nu_Q((\beta^2)^{-1}\beta)\right), \\ \left(\beta^2, \mu_Q((\beta^2)^{-1}\beta^2), \nu_Q((\beta^2)^{-1}\beta^2)\right), \left(\alpha, \mu_Q((\beta^2)^{-1}\alpha), \nu_Q((\beta^2)^{-1}\alpha)\right), \\ \left(\alpha\beta, \mu_Q((\beta^2)^{-1}\alpha\beta), \nu_Q((\beta^2)^{-1}\alpha\beta)\right), \left(\alpha\beta^2, \mu_Q((\beta^2)^{-1}\alpha\beta^2), \nu_Q((\beta^2)^{-1}\alpha\beta^2)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left(e, \mu_Q(\beta), \nu_Q(\beta)\right), \left(\beta, \mu_Q(\beta^2), \nu_Q(\beta^2)\right), \left(\beta^2, \mu_Q(e), \nu_Q(e)\right), \\ \left(\alpha, \mu_Q(\alpha\beta^2), \nu_Q(\alpha\beta^2)\right), \left(\alpha\beta, \mu_Q(\alpha), \nu_Q(\alpha)\right), \left(\alpha\beta^2, \mu_Q(\alpha\beta), \nu_Q(\alpha\beta)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} (e, 0.85, 0.65), (\beta, 0.85, 0.65), (\beta^2, 0.85, 0.65), \\ (\alpha, 0.80, 0.75), (\alpha\beta, 0.80, 0.75), (\alpha\beta^2, 0.80, 0.75) \end{array} \right\} = Q$$

(iv) The q-ROFLC of Q for $\alpha \in V$ is

$$\alpha Q = \left\{ \begin{array}{l} \left(e, \mu_Q(\alpha^{-1}e), \nu_Q(\alpha^{-1}e)\right), \left(\beta, \mu_Q(\alpha^{-1}\beta), \nu_Q(\alpha^{-1}\beta)\right), \\ \left(\beta^2, \mu_Q(\alpha^{-1}\beta^2), \nu_Q(\alpha^{-1}\beta^2)\right), \left(\alpha, \mu_Q(\alpha^{-1}\alpha), \nu_Q(\alpha^{-1}\alpha)\right), \\ \left(\alpha\beta, \mu_Q(\alpha^{-1}\alpha\beta), \nu_Q(\alpha^{-1}\alpha\beta)\right), \left(\alpha\beta^2, \mu_Q(\alpha^{-1}\alpha\beta^2), \nu_Q(\alpha^{-1}\alpha\beta^2)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left(e, \mu_Q(\alpha), \nu_Q(\alpha)\right), \left(\beta, \mu_Q(\alpha\beta), \nu_Q(\alpha\beta)\right), \left(\beta^2, \mu_Q(\alpha\beta^2), \nu_Q(\alpha\beta^2)\right), \\ \left(\alpha, \mu_Q(e), \nu_Q(e)\right), \left(\alpha\beta, \mu_Q(\beta), \nu_Q(\beta)\right), \left(\alpha\beta^2, \mu_Q(\beta^2), \nu_Q(\beta^2)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} (e, 0.0.80, 0.75), (\beta, 0.80, 0.75), (\beta^2, 0.80, 0.75), \\ (\alpha, 0.85, 0.65), (\alpha\beta, 0.85, 0.65), (\alpha\beta^2, 0.85, 0.65) \end{array} \right\}$$

(v) *The q-ROFLC of Q for* $\alpha\beta \in V$ *is*

$$\alpha\beta Q = \left\{ \begin{array}{l} \left(e, \mu_Q((\alpha\beta)^{-1}e), \nu_Q((\alpha\beta)^{-1}e)\right), \left(\beta, \mu_Q((\alpha\beta)^{-1}\beta), \nu_Q((\alpha\beta)^{-1}\beta)\right), \\ \left(\beta^2, \mu_Q((\alpha\beta)^{-1}\beta^2), \nu_Q((\alpha\beta)^{-1}\beta^2)\right), \left(\alpha, \mu_Q((\alpha\beta)^{-1}\alpha), \nu_Q((\alpha\beta)^{-1}\alpha)\right), \\ \left(\alpha\beta, \mu_Q((\alpha\beta)^{-1}\alpha\beta), \nu_Q((\alpha\beta)^{-1}\alpha\beta)\right), \left(\alpha\beta^2, \mu_Q((\alpha\beta)^{-1}\alpha\beta^2), \nu_Q((\alpha\beta)^{-1}\alpha\beta^2)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} (e, \mu_Q(\alpha\beta), \nu_Q(\alpha\beta)), \left(\beta, \mu_Q(\alpha\beta^2), \nu_Q(\alpha\beta^2)\right), \left(\beta^2, \mu_Q(\alpha), \nu_Q(\alpha)\right), \\ \left(\alpha, \mu_Q(\beta^2), \nu_Q(\beta^2)\right), (\alpha\beta, \mu_Q(e), \nu_Q(e)), \left(\alpha\beta^2, \mu_Q(\beta), \nu_Q(\beta)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} (e, 0.0.80, 0.75), (\beta, 0.80, 0.75), (\beta^2, 0.80, 0.75), \\ (\alpha, 0.85, 0.65), (\alpha\beta, 0.85, 0.65), (\alpha\beta^2, 0.85, 0.65) \end{array} \right\}$$

(vi) *The q-ROFLC of Q for* $\alpha\beta^2 \in V$ *is*

$$\alpha\beta^2 Q = \left\{ \begin{array}{l} \left(e, \mu_Q((\alpha\beta^2)^{-1}e), \nu_Q((\alpha\beta^2)^{-1}e)\right), \left(\beta, \mu_Q((\alpha\beta^2)^{-1}\beta), \nu_Q((\alpha\beta^2)^{-1}\beta)\right), \\ \left(\beta^2, \mu_Q((\alpha\beta^2)^{-1}\beta^2), \nu_Q((\alpha\beta^2)^{-1}\beta^2)\right), \left(\alpha, \mu_Q((\alpha\beta^2)^{-1}\alpha), \nu_Q((\alpha\beta^2)^{-1}\alpha)\right), \\ \left(\alpha\beta, \mu_Q((\alpha\beta^2)^{-1}\alpha\beta), \nu_Q((\alpha\beta^2)^{-1}\alpha\beta)\right), \\ \left(\alpha\beta^2, \mu_Q((\alpha\beta^2)^{-1}\alpha\beta^2), \nu_Q((\alpha\beta^2)^{-1}\alpha\beta^2)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left(e, \mu_Q(\alpha\beta^2), \nu_Q(\alpha\beta^2)\right), (\beta, \mu_Q(\alpha), \nu_Q(\alpha)), \left(\beta^2, \mu_Q(\alpha\beta), \nu_Q(\alpha\beta)\right), \\ (\alpha, \mu_Q(\beta), \nu_Q(\beta)), (\alpha\beta, \mu_Q(\beta), \nu_Q(\beta)), \left(\alpha\beta^2, \mu_Q(e), \nu_Q(e)\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} (e, 0.0.80, 0.75), (\beta, 0.80, 0.75), (\beta^2, 0.80, 0.75), \\ (\alpha, 0.85, 0.65), (\alpha\beta, 0.85, 0.65), (\alpha\beta^2, 0.85, 0.65) \end{array} \right\}$$

As a result, Q has two distinct q-ROFCs with regard to all elements of V, namely, $eQ = \beta Q = \beta^2 Q$ *and* $\alpha Q = \alpha\beta Q = \alpha\beta^2 Q$.

Definition 8. *Let* $Q = (\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon))$ *be a q-ROFSG of V. Then*

$$Q_* = \{\epsilon \in V : (\mu_Q(\epsilon))^2 = (\mu_Q(e))^2 \text{ and } (\nu_Q(\epsilon))^2 = (\nu_Q(e))^2\}$$

Definition 9. *Let* $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V\}$ *be a q-ROFSG of V. Then support set of Q is denoted by* Q^* *and is defined as;*

$$Q^* = \{\epsilon \in V : (\mu_Q(\epsilon))^2 > 0 \text{ and } (\nu_Q(\epsilon))^2 < 1\}$$

The following theorem addresses the question: what type of elements of V generates the same q-ROFCs.

Theorem 7. *Let* $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in\}$ *be a q-ROFSG of V. Then for all* $\lambda, \kappa \in V$

(i) $\lambda Q = \kappa Q \Leftrightarrow \lambda Q_* = \kappa Q_*$

(ii) $Q\lambda = Q\kappa \Leftrightarrow Q_*\lambda = Q_*\kappa$

Proof. Due to the similarities of the proofs for (i) and (ii), we will only prove (i) here. Assume that $\lambda Q = \kappa Q$ for some $\lambda, \kappa \in V$. Then $\mu_\lambda Q(\epsilon) = \mu_\kappa Q(\epsilon)$ and $\nu_\lambda Q(\epsilon) = \nu_\kappa Q(\epsilon)$ implies that $\mu_Q(\lambda^{-1}\epsilon) = \mu_Q(\kappa^{-1}\epsilon)$ and $\nu_Q(\lambda^{-1}\epsilon) = \nu_Q(\kappa^{-1}\epsilon)$ for all $\epsilon \in V$. Let us take $\epsilon = \kappa$, then $\mu_Q(\lambda^{-1}\kappa) = \mu_Q(e)$ and $\nu_Q(\lambda^{-1}\kappa) = \nu_Q(e)$. So,$\lambda^{-1}\kappa \in Q_*$ implying that $\lambda^{-1}\kappa Q_* = Q_*$. Consequently, $\kappa Q_* = \lambda Q_*$.
Conversely, suppose that $\kappa Q_* = \lambda Q_*$, then $\lambda^{-1}\kappa \in Q_*$ and $\kappa^{-1}\lambda \in Q_*$.
Next, for any $\epsilon \in V$, consider

$$\mu_Q(\lambda^{-1}\epsilon) = \mu_Q(\lambda^{-1}\kappa\kappa^{-1}\epsilon)$$

$$\geq \min\{\mu_Q(\lambda^{-1}\kappa), \mu_Q(\kappa^{-1}\epsilon)\}$$

$$= min \mu_Q(e), \mu_Q(\kappa^{-1}\epsilon)$$

$$= \mu_Q(\kappa^{-1}\epsilon)$$

In a similar manner, we can obtain $\mu_Q(\kappa^{-1}\epsilon) \geq \mu_Q(\lambda^{-1}\epsilon)$. Therefore,

$$\mu_Q(\lambda^{-1}\epsilon) = \mu_Q(\kappa^{-1}\epsilon) \, \text{for all} \, \epsilon \in V.$$

Furthermore, the same reasoning leads to $\nu_Q(\lambda^{-1}\epsilon) = \nu_Q(\kappa^{-1}\epsilon)$ which implies that $\mu_{\lambda Q}(\epsilon) = \mu_{\kappa Q}(\epsilon)$ and $\nu_{\lambda Q}(\epsilon) = \nu_{\kappa Q}(\epsilon)$ for all $\epsilon \in V$. Hence $\lambda Q = \kappa Q$.

Lemma 1. *If* $\left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \geq (\mu_Q(\lambda))^q$ *and* $\left(\nu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \leq (\nu_Q(\lambda))^q$ *for all* $\lambda, \epsilon \in V$, *then* Q *is a q-ROFNS of* V.

Proof. Let $\left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \geq (\mu_Q(\lambda))^q$ and $\left(\nu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \leq (\nu_Q(\lambda))^q$ for all $\lambda, \epsilon \in V$. Since $\epsilon\lambda\epsilon^{-1}, \epsilon^{-1} \in V$, therefore

$$\left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \leq \left(\mu_Q(\epsilon^{-1}(\epsilon\lambda\epsilon^{-1})(\epsilon^{-1})^{-1})\right)^q$$

$$= (\mu_Q(\lambda))^q$$

Similarly, $\left(\nu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \geq (\nu_Q(\lambda))^q$. Therefore,

$$\left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q = (\mu_Q(\lambda))^q \, and \, \left(\nu_Q(\epsilon\lambda\epsilon^{-1})\right)^q = (\nu_Q(\lambda))^q$$

Then, Theorem 6 reveals that Q is a q-ROFNS of V.

Theorem 8. *Let* $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V\}$ *be a q-ROFSG of* V. *Then* Q *is a q-ROFNSG of* $V \Leftrightarrow Q_{(\theta,\tau)} \trianglelefteq V$ *for all* $\theta \in [0, (\mu_Q(e))^q]$ *and* $\tau \in [(\nu_Q(e))^q, 1]$.

Proof. In [35], it has been proved that if Q is a q-ROFNS of V, then $Q_{(\theta,\tau)} \trianglelefteq V$ for all $\theta \in [0, (\mu_Q(e))^q]$ and $\tau \in [(\nu_Q(e))^q, 1]$.
Conversely, let $Q_{(\theta,\tau)} \trianglelefteq V$ for all $\theta \in [0, (\mu_Q(e))^q]$ and $\tau \in [(\nu_Q(e))^q, 1]$. According to Theorem 4, Q is a q-ROFSG of V. Assume that $\lambda, \epsilon \in V$ such that $(\mu_Q(\lambda))^q = a$ and $(\nu_Q(\lambda))^q = b$. Then

$$\lambda \in Q_{(a,b)} \Rightarrow \epsilon\lambda\epsilon^{-1} \in Q_{(a,b)}$$
$$\Rightarrow \left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \geq a = (\mu_Q(\lambda))^q \, and \left(\nu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \leq b = (\nu_Q(\lambda))^q$$

Then, Q is thus a q-ROFNSG of V, according to Lemma 1.

Lemma 2. *If $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V\}$ is a q-ROFSG of V. Then Q^* and Q_* are subgroups of V.*

Using the preceding concepts, the proof can be easily established.

Theorem 9. *If $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V\}$ is a q-ROFNSG of V. Then*

(i) $Q^* \trianglelefteq V$.

(ii) $Q_* \trianglelefteq V$.

Proof. Let $Q = \{(\epsilon, \mu_Q(\epsilon), \nu_Q(\epsilon)) : \epsilon \in V\}$ be a q-ROFNSG of V. Then $(\mu_Q(\lambda))^q = \left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q$ and $(\nu_Q(\lambda))^q = \left(\nu_Q(\epsilon\lambda\epsilon^{-1})\right)^q$ for all $\lambda, \epsilon \in V$.

(i) Let $\epsilon \in V$ and $\lambda \in Q^*$, then

$$\left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q = (\mu_Q(\lambda))^q \quad (Since\, Q\, is\, a\, q-ROFNSG\, of\, V)$$
$$> 0 \quad (Since\, \lambda \in Q^*)$$

Similarly, it can be derived, $\left(\nu_Q(\epsilon\lambda\epsilon^{-1})\right)^q < 1$ for all $\epsilon \in V$ and $\lambda \in Q^*$. Therefore, $\epsilon\lambda\epsilon^{-1} \in Q^*$ for all $\epsilon \in V$ and $\lambda \in Q^*$. Hence, $Q^* \trianglelefteq V$.

(ii) The proof is analogous to (i).

In the following example, the contrary of Theorem 9 is proved to be false.

Example 2. *Let us design a q-ROFSG Q of $S_3 = \{e, \lambda, \kappa, \kappa^2, \lambda\kappa, \lambda\kappa^2\}$ as follows;*

$$Q = \{(e, 1, 0), (\lambda, 0.9, 0.5)(\kappa, 0.7, 0.8), (\kappa^2, 0.7, 0.8), (\lambda\kappa, 0.7, 0.8), (\lambda\kappa^2, 0.7, 0.8)\}.$$

Then $Q^* = S_3$ and $Q_* = e$ are normal in S_3 but Q is not a q-ROFNSG of S_3 because $\mu_Q\left((\kappa)(\lambda\kappa^2)\right) = \mu_Q(\lambda\kappa) = 0.7 \neq 0.9 = \mu_Q(\lambda) = \mu_Q\left((\lambda\kappa^2)(\kappa)\right)$.

4. q-Rung Orthopair Fuzzy Normal Subgroup of a q-Rung Orthopair Fuzzy Subgroup

This section introduces the notion of q-ROFNSG of a q-ROFSG. In addition, an in-depth study of this concept is provided.

Definition 10. *Let Q and P be two q-ROFSGs of V such that $Q \subseteq P$. Then Q is a q-ROFNSG of P if $\left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \geq \min\{(\mu_Q(\lambda))^q, (\mu_P(\epsilon))^q\}$ and $\left(\nu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \leq \max\{(\nu_Q(\lambda))^q, (\nu_P(\epsilon))^q\}$ for all $\lambda, \epsilon \in V$.*

The result below demonstrates a necessary condition for a q-ROFSG Q that is a subset of a q-ROFSG P to be a q-ROFNSG of P.

Theorem 10. *Assume that Q and P are q-ROFSGs of V such that Q is q-ROFNSG of P. Then $Q^* \trianglelefteq P^*$.*

Proof. Let $\lambda \in Q^*$ and $\epsilon \in P^*$, then $(\mu_Q(\lambda))^q, (\mu_P(\epsilon))^q > 0$ and $(\nu_Q(\lambda))^q, (\nu_P(\epsilon))^q < 1$. Therefore $\min\{(\mu_Q(\lambda))^q, (\mu_P(\epsilon))^q\} > 0$ and $\max\{(\nu_Q(\lambda))^q, (\nu_P(\epsilon))^q\} < 1$.
Consequently, we yield $\left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \geq \min\{(\mu_Q(\lambda))^q, (\mu_P(\epsilon))^q\} > 0$ and $\left(\nu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \leq \max\{(\nu_Q(\lambda))^q, (\nu_P(\epsilon))^q\} < 1$, both of which indicate that $\epsilon\lambda\epsilon^{-1} \in Q^*$. Hence, $Q^* \trianglelefteq P^*$.

Theorem 11. *Suppose that Q and P are q-ROFSGs of V and $Q \subseteq P$. Then Q is a q-ROFNSG of P if and only if $(\mu_Q(\lambda\epsilon))^q \geq \min\{(\mu_Q(\epsilon\lambda))^q, (\mu_P(\lambda))^q\}$ and $(\nu_Q(\lambda\epsilon))^q \leq \max\{(\nu_Q(\epsilon\lambda))^q, (\nu_P(\lambda))^q\}$ for all $\epsilon, \lambda \in V$.*

Proof. Let Q be a q-ROFNSG of P, then $(\mu_Q(\lambda\epsilon))^q = \left(\mu_Q(\lambda\epsilon\lambda\lambda^{-1})\right)^q \geq \min\{(\mu_Q(\epsilon\lambda))^q, (\mu_P(\lambda))^q\}$ and $(\nu_Q(\lambda\epsilon))^q = \left(\nu_Q(\lambda\epsilon\lambda\lambda^{-1})\right)^q \leq \max\{(\nu_Q(\epsilon\lambda))^q, (\nu_P(\lambda))^q\}$ for all $\epsilon, \lambda \in V$.
Conversely, consider $(\mu_Q(\lambda\epsilon))^q \geq \min\{(\mu_Q(\epsilon\lambda))^q, (\mu_P(\lambda))^q\}$ and $(\nu_Q(\lambda\epsilon))^q \leq \max\{(\nu_Q(\epsilon\lambda))^q, (\nu_P(\lambda))^q\}$ for all $\epsilon, \lambda \in V$. Then

$$\left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \geq \min\{\left(\mu_Q(\lambda\epsilon^{-1}\epsilon)\right)^q, (\mu_P(\epsilon))^q\} = \min\{(\mu_Q(\lambda))^q, (\mu_P(\epsilon))^q\} \quad and$$

$$\left(\nu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \leq \max\{\left(\nu_Q(\lambda\epsilon^{-1}\epsilon)\right)^q, (\nu_P(\epsilon))^q\} = \max\{(\nu_Q(\lambda))^q, (\nu_P(\epsilon))^q\} \; forall \; \epsilon, \lambda \in V.$$

Thus, Q is a q-ROFNSG of P.

Theorem 12. *Suppose that Q and P are q-ROFSGs of V. Then Q is a q-ROFNSG of $P \Leftrightarrow Q_{(\theta,\tau)} \trianglelefteq P_{(\theta,\tau)}$, for all $\theta \in [0, (\mu_Q(e))^q]$ and $\tau \in [(\nu_Q(e))^q, 1]$.*

Proof. Let Q be a q-ROFNSG of Q' and $\theta \in [0, \mu_Q(e)]$ and $\tau \in [\nu_Q(e), 1]$, then the combination of Theorems 3 and 4 reveals that $Q_{(\theta,\tau)}$ is a subgroup of $P_{(\theta,\tau)}$. Assume that $\lambda \in Q_{(\theta,\tau)}$ and $\epsilon \in P_{(\theta,\tau)}$, then $(\mu_Q(\lambda))^q, (\mu_P(\epsilon))^q \geq \theta$ and $(\nu_Q(\lambda))^q, (\nu_P(\epsilon))^q \leq \tau$. Now

$$\left(\mu_Q\left(\epsilon\lambda\epsilon^{-1}\right)\right)^q \geq \min\{(\mu_Q(\lambda))^q, (\mu_P(\epsilon))^q\} \geq \theta \quad and$$

$$\left(\nu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \leq \max\{(\nu_Q(\lambda))^q, (\nu_P(\epsilon))^q\} \leq \tau.$$

Therefore $\epsilon\lambda\epsilon^{-1} \in Q_{(\theta,\tau)}$ which further gives $Q_{(\theta,\tau)} \trianglelefteq P_{(\theta,\tau)}$.
Conversely, let $Q_{(\theta,\tau)} \trianglelefteq P_{(\theta,\tau)}$ for all $\theta \in [0, \mu_Q(e)]$ and $\tau \in [\nu_Q(e), 1]$. Consider $\lambda, \epsilon \in V$ such that $(\mu_Q(\lambda))^q = \alpha, (\nu_Q(\lambda))^q = \beta$ and $(\mu_P(\epsilon))^q = \gamma, (\nu_P(\epsilon))^q = \delta$. Consequently, there are four possibilities;

(i) $\gamma \geq \alpha$ and $\delta \geq \beta$

(ii) $\gamma \geq \alpha$ and $\delta \leq \beta$

(iii) $\gamma \leq \alpha$ and $\delta \geq \beta$

(iv) $\gamma \leq \alpha$ and $\delta \leq \beta$

(i) $\gamma \geq \alpha$ and $\delta \geq \beta$ then $\epsilon \in P_{(\alpha,\delta)}$ and $\lambda \in Q_{(\alpha,\delta)}$. Then by assumption, $\epsilon\lambda\epsilon^{-1} \in Q_{(\alpha,\delta)}$ which gives

$$\left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \geq \alpha = \min(\alpha, \gamma) = \min((\mu_Q(\lambda))^q, (\mu_P(\epsilon))^q)$$

and

$$\left(\nu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \leq \delta = \max(\beta, \delta) = \max((\nu_Q(\lambda))^q, (\nu_P(\epsilon))^q)$$

(ii) If $\gamma \geq \alpha$ and $\delta \leq \beta$ then $\epsilon \in P_{(\alpha,\beta)}$ and $\lambda \in Q_{(\alpha,\beta)}$. Then by assumption, $\epsilon\lambda\epsilon^{-1} \in Q_{(\alpha,\beta)}$ which gives

$$\left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \geq \alpha = \min(\alpha, \gamma) = \min((\mu_Q(\lambda))^q, (\mu_P(\epsilon))^q)$$

and

$$\left(\nu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \leq \beta = \max(\beta, \delta) = \max((\nu_Q(\lambda))^q, (\nu_P(\epsilon))^q)$$

Likewise, for *(iii)* and *(iv)*, we get the same conclusion. Consequently, Q is a q-ROFNSG of P.

Theorem 13. *Let Q and P be a q-ROFSGs of V. Then $Q^* \trianglelefteq P^*$ and $Q_* \trianglelefteq P_*$ if Q is a q-ROFNSG of P.*

Proof. Assume that Q is a q-ROFNSG of P and $\lambda \in Q^*$ and $\epsilon \in P^*$, then

$$(\mu_Q(\epsilon\lambda\epsilon^{-1}))^q \geq \min((\mu_(\lambda))^q, (\mu_P(\epsilon))^q) = (\mu_Q(e))^q$$

and

$$(\nu_Q(\epsilon\lambda\epsilon^{-1}))^q \leq \max((\nu_Q(\lambda))^q, (\nu_P(\epsilon))^q) = (\nu_Q(e))^q$$

Consequently, $\epsilon\lambda\epsilon^{-1} \in Q^*$ implying that $Q^* \trianglelefteq P^*$.
The similar argument indicates that $Q_* \trianglelefteq P_*$.

Theorem 14. *Assume that Q is a q-ROFNSG and P is a q-ROFSG of V. Then $Q \cap P$ is a q-ROFNSG of P.*

Proof. In accordance with Theorem 2, $Q \cap P$ is a q-ROFSG of V and $Q \cap P \subseteq P$. Let $\lambda, \epsilon \in V$, then

$$\begin{aligned}\left(\mu_{Q\cap P}(\epsilon\lambda\epsilon^{-1})\right)^q &= \min\{\left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q, \left(\mu_P(\epsilon\lambda\epsilon^{-1})\right)^q\}\\ &\geq \min\{(\mu_Q(\lambda))^q, \min\{(\mu_P(\epsilon\lambda))^q, \left(\mu_P(\epsilon^{-1})\right)^q\}\}\\ &\geq \min\{(\mu_Q(\lambda))^q, \min\{\min\{(\mu_P(\epsilon))^q, (\mu_P(\lambda))^q\}, (\mu_P(\epsilon))^q\}\}\\ &= \min\{(\mu_Q(\lambda))^q, \min\{(\mu_P(\epsilon))^q, (\mu_P(\lambda))^q\}\}\\ &= \min\{\min\{(\mu_Q(\lambda))^q, (\mu_P(\lambda))^q\}, (\mu_P(\epsilon))^q\}\\ &= \min\{(\mu_{Q\cap P}(\lambda))^q, (\mu_P(\epsilon))^q\}\end{aligned}$$

So, $\left(\mu_{Q\cap P}(\epsilon\lambda\epsilon^{-1})\right)^q \geq \min\{(\mu_{Q\cap P}(\lambda))^q, (\mu_P(\epsilon))^q\}$ for all $\lambda, \epsilon \in V$. In a similar vein, we can demonstrate that $\left(\nu_{Q\cap P}(\epsilon\lambda\epsilon^{-1})\right)^q \leq \max\{(\nu_{Q\cap P}(\lambda))^q, (\nu_P(\epsilon))^q\}$ for all $\lambda, \epsilon \in V$. Thus, $Q \cap P$ is a q-ROFNSG of P.

Theorem 15. *Assume that Q,P and T are q-ROFSGs of V such that Q and P are q-ROFNSGs of T. Then $Q \cap P$ is q-ROFSG of T.*

Proof. It is straightforward to establish from routine calculation that $Q \cap P$ is a q-ROFSG of V and $Q \cap P \subseteq T$. Let $\lambda, \epsilon \in V$, then

$$\begin{aligned}\left(\mu_{Q\cap P}(\epsilon\lambda\epsilon^{-1})\right)^q &= \min\{\left(\mu_(Q)(\epsilon\lambda\epsilon^{-1})\right)^q, \left(\mu_P(\epsilon\lambda\epsilon^{-1})\right)^q\}\\ &\geq \min\{\min\{(\mu_Q(\lambda))^q, (\mu_T(\epsilon))^q\}, \min\{(\mu_P(\lambda))^q, (\mu_T(\epsilon))^q\}\}\\ &= \min\{\min\{(\mu_Q(\lambda))^q, (\mu_P(\lambda))^q\}, (\mu_T(\epsilon))^q\}\\ &= \min\{(\mu_{Q\cap P}(\lambda))^q, (\mu_T(\epsilon))^q\}\end{aligned}$$

Similarly,

$$\left(\nu_{Q\cap P}(\epsilon\lambda\epsilon^{-1})\right)^q \geq \max\{(\nu_{Q\cap P}(\lambda))^q, (\nu_T(\epsilon))^q\}$$

Theorem 16. *Suppose that Q and P are q-ROFSGs of V and α is a homomorphism from V to W. Then Q is a q-ROFNSG of P implies $\alpha(Q)$ is a q-ROFNSG of $\alpha(P)$.*

Proof. Suppose that Q and P are q-ROFSGs of V and α is a homomorphism from V to W. Then Q is a q-ROFNSG of P implies $\alpha(Q)$ is a q-ROFNSG of $\alpha(P)$.

$$\begin{aligned}(\mu_{\alpha(Q)}(\epsilon\lambda\epsilon^{-1}))^q &= \max\{(\mu_Q(z))^q : z \in V, \alpha(z) = \epsilon\lambda\epsilon^{-1}\}\\ &= \max\{(\mu_Q(kjk^{-1}))^q : k, j \in V, \alpha(k) = \epsilon, \alpha(j) = \lambda\}\\ &\geq \max\{\min((\mu_Q(j))^q, (\mu_P(k))^q) : k, j \in V, \alpha(k) = \epsilon, \alpha(j) = \lambda\}\\ &= \min\left(\max\{(\mu_Q(j))^q : j \in V, \alpha(j) = \lambda\}, \max\{(\mu_P(k))^q : k \in V, \alpha(k) = \epsilon\}\right)\\ &= \min\left((\mu_{\alpha(Q)}(\lambda))^q, (\mu_{\alpha(P)}(\epsilon))^q\right)\end{aligned}$$

Similarly, we can acquire

$$\left(\nu_{\alpha(Q)}(\epsilon\lambda\epsilon^{-1})\right)^q \leq \max\left((\nu_{\alpha(Q)}(\lambda))^q, (\nu_{\alpha(P)}(\epsilon))^q\right)$$

Thus, $\alpha(Q)$ is a q-ROFNSG of $\alpha(P)$.

Theorem 17. *Assume that Q and P are q-ROFSGs of W and α is a homomorphism from V to W. Then $\alpha^{-1}(Q)$ is a q-ROFNSG of $\alpha^{-1}(P)$ if Q is a q-ROFNSG of P.*

Proof. From routine computation, it can be easily proved that $\alpha^{-1}(Q)$ and $\alpha^{-1}(P)$ are q-ROFNSGs of V and $\alpha^{-1}(Q) \subseteq \alpha^{-1}(P)$. Now

$$\begin{aligned}\left(\mu_{\alpha^{-1}(Q)}(\epsilon\lambda\epsilon^{-1})\right)^q &= \left(\mu_Q(\alpha(\epsilon\lambda\epsilon^{-1}))\right)^q \\ &= \left(\mu_Q(\alpha(\epsilon), \alpha(\lambda), (\alpha(\epsilon))^{-1})\right)^q \\ &\geq \min\{(\mu_Q(\alpha(\lambda)))^q, (\mu_P(\alpha(\epsilon)))^q\} \\ &= \min\{(\mu_{\alpha^{-1}(Q)}(\lambda))^q, (\mu_{\alpha^{-1}(P)}(\epsilon))^q\}\end{aligned}$$

Similarly

$$(\nu_{\alpha^{-1}(Q)}(\epsilon\lambda\epsilon^{-1}))^q \leq \max\{(\nu_{\alpha^{-1}(Q)}(\lambda))^q, (\nu_{\alpha^{-1}(P)}(\epsilon))^q\}$$

Therefore, $\alpha^{-1}(Q)$ is a q-ROFNSG of α^{-1} (P).

5. Fundamental Theorems of q-rung orthopair Fuzzy Isomorphisms

The present section introduces the concepts of q-rung orthopair fuzzy homomorphism (q-ROFHom) and isomorphism (q-ROFIso). To extend the concept of the quotient group V/U of V with respect to its normal subgroup U, we define the q-ROFSG of V/U. In addition, we develop the q-rung orthopair fuzzy counterpart of the fundamental theorems of isomorphisms.

Definition 11. *Assume that Q_1 and Q_2 are q-ROFSGs of V and W respectively. Then a homomorphism $\alpha : V \to W$ is called q-ROFHom from Q_1 to Q_2 if $\alpha(Q_1) = Q_2$. The existence of q-ROFHom between Q_1 and Q_2 is denoted by $Q_1 \approx Q_2$.*

Definition 12. *Assume that Q_1 and Q_2 are q-ROFSGs of V and W respectively. Then an isomorphism $\alpha : V \to W$ is called q-ROFIso from Q_1 to Q_2 if $\alpha(Q_1) = Q_2$. The existence of a q-ROFIso between Q_1 and Q_2 is denoted by $Q_1 \cong Q_2$.*

Definition 13. *Let Q be a q-ROFSG of V and $U \trianglelefteq V$. Then $Q' = \{(\epsilon U, \mu_{Q'}(\epsilon U), \nu_{Q'}(\epsilon U)) : \epsilon U \in V/U\}$, where $\left(\mu_{Q'}(\epsilon U)\right)^q = \max\{(\mu_Q(t))^q : t \in \epsilon U\}$ and $\left(\nu_{Q'}(\epsilon U)\right)^q = \min\{(\nu_Q(t))^q : t \in \epsilon U\}$, forms q-ROFS of V/U.*

Theorem 18. *Q' is a q-ROFSG of V/U.*

Proof. Let $\epsilon U \in V/U$, then

$$\left(\mu_{Q'}(\epsilon U)^{-1}\right)^q = \left(\mu_{Q'}(\epsilon^{-1}U)\right)^q$$
$$= \max\{(\mu_Q(t))^q : t \in \epsilon^{-1}U\}$$
$$= \max\{(\mu_Q(t^{-1}))^q : t^{-1} \in \epsilon U\} \quad (Since\, U\, is\, normal\, in\, V)$$
$$= \left(\mu_{Q'}(\epsilon U)\right)^q$$

Similarly,

$$\left(\nu_{Q'}(\epsilon U)^{-1}\right)^q = \left(\nu_{Q'}(\epsilon U)\right)^q$$

Next, let $\epsilon_1 U, \epsilon_2 U \in V/U$, then

$$\left(\mu_{Q'}(\epsilon_1 U \dot{\epsilon}_2 U)\right)^q = \left(\mu_{Q'}(\epsilon_1 \epsilon_2 U)\right)^q$$
$$= \max\{(\mu_Q(t))^q : t \in \epsilon_1\epsilon_2 U = \epsilon_1 \dot{\epsilon}_2 U\}$$
$$= \max\{(\mu_Q(\lambda\kappa))^q : \lambda \in \epsilon_1 U, \kappa \in \epsilon_2 U\}$$
$$\geq \max\{\min((\mu_Q(\lambda))^q, (\mu_Q(\kappa))^q) : \lambda \in \epsilon_1 U, \kappa \in \epsilon_2 U\}$$
$$= \min\left(\max\{(\mu_Q(\lambda))^q : \lambda \in \epsilon_1 U\}, \max\{(\mu_Q(\kappa))^q : \kappa \in \epsilon_2 U\}\right)$$
$$= \min\left((\mu_{Q'}(\epsilon_1 U))^q, (\mu_{Q'}(\epsilon_2 U))^q\right)$$

Likewise, it can be demonstrated that

$$\left(\nu_{Q'}(\epsilon_1 U \dot{\epsilon}_2 U)\right)^q \leq \max\left((\nu_{Q'}(\epsilon_1 U))^q, (\nu_{Q'}(\epsilon_2 U))^q\right).$$

Hence, Q' is a q-ROFSG of V/U.

Remark 1. *The q-ROFSG Q' is called q-rung orthopair fuzzy quotient group (q-ROFFG) of Q with respect to U.*
Consider two q-ROFSGs Q and P of V such that P is a q-ROFNSG of Q. According to Theorem 13 P^* is normal in Q^*, so the quotient group Q^*/P^* can be formed.
Consider a q-ROFS Q'' of Q^* as follows;

(i) $\left(\mu_{Q''}(\epsilon)\right)^q = (\mu_Q(\epsilon))^q$ *for all* $\epsilon \in Q^*$

(ii) $\left(\nu_{Q''}(\epsilon)\right)^q = (\nu_Q(\epsilon))^q$ *for all* $\epsilon \in Q^*$

It is a matter of simple calculations to verify that Q'' is a q-ROFSG of Q^*. According to Definition 13 and Theorem 18, q-ROFFG of Q'' with regard to P^* exists. We refer to it as Q/P for the purpose of convenience. Obviously, Q/P is q-ROFSG of Q^*/P^* and

(i) $\left(\mu_{Q/P}(\epsilon P^*)\right)^q = \max\{(\mu_{Q''}(t))^q : t \in \epsilon P^*\}$

(ii) $\left(\nu_{Q/P}(\epsilon P^*)\right)^q = \min\{(\nu_{Q''}(t))^q : t \in \epsilon P^*\}$

Theorem 19. *Assume that Q and P be q-ROFSGs of V and P be a q-ROFNSG of Q. Then $Q'' \approx Q/P$.*

Proof. We know Q'' is q-ROFSG of Q^* and Q/P is a q-ROFSG of Q^*/P^*. Let $\alpha : Q^* \to Q^*/P^*$ be defined by $\alpha(\epsilon) = \epsilon P^*$. Then obviously α is a homomorphism. Now

$$\begin{aligned}\left(\mu_{\alpha(Q'')}(\epsilon P^*)\right)^q &= \max\{(\mu_{Q''}(t))^q : t \in Q^*, \alpha(t) = \epsilon P^*\} \\ &= \max\{(\mu_Q(u))^q : u \in \epsilon P^*\} \\ &= \left(\mu_{Q/P}(\epsilon P^*)\right)^q\end{aligned}$$

By using the same arguments, we show that

$$\left(\nu_{\alpha(Q'')}(\epsilon P^*)\right)^q = \left(\nu_{Q/P}(\epsilon P^*)\right)^q$$

Therefore, $\alpha(Q'') = Q/P$. Finally, Definition 11 results in $Q'' \approx Q/P$.

Lemma 3. *Suppose that Q is a q-ROFSG of V and $\alpha : V \to U$ is a homomorphism from V to U. Then $(\alpha(Q))^* = \alpha(Q^*)$.*

Proof. Consider $\lambda \in \alpha(Q^*)$, then there exists $\epsilon \in Q^*$ for which $\alpha(\epsilon) = \lambda$.
Now

$$\left(\mu_{\alpha(Q)}(\lambda)\right)^q = \sup\{\left(\mu_Q(z) : z \in \alpha^{-1}(\lambda)\right)^q\} \geq (\mu_Q(\epsilon))^q > 0$$

and

$$\left(\nu_{\alpha(Q)}(\lambda)\right)^q = \inf\{\left(\nu_Q(z) : z \in \alpha^{-1}(\lambda)\right)^q\} \leq (\nu_Q(\epsilon))^q < 1.$$

Therefore, $\lambda \in (\alpha(Q))^*$. Thus, $\alpha(Q^*) \subseteq (\alpha(Q))^*$.
On the other hand, suppose that $\lambda \in (\alpha(Q))^*$ then $\alpha(\epsilon) = \lambda$ and $\mu_Q(\epsilon) = \mu_{\alpha(Q)}(\lambda)$ for some $\epsilon \in V$. Since $\left(\mu_{\alpha(Q)}(\lambda)\right)^q > 0$, therefore $(\mu_Q(\epsilon))^q > 0$, which further reveals that $(\nu_Q(\epsilon))^q < 1$. Consequently, $\epsilon \in Q^*$ implying that $\lambda = \alpha(\epsilon) \in \alpha(Q^*)$. Thus, $(\alpha(Q))^* \subseteq \alpha(Q^*)$.

Theorem 20. *(First Isomorphism Theorem in q-rung orthopair Fuzzy Settings) Suppose Q and T are q-ROFSGs of V and U respectively. Then $Q \cong T$ implies $Q/P \cong T''$ for some q-ROFNSG P of Q.*

Proof. Obviously, $Q \cong T$ ensures the existence of an epimorphism $\alpha : V \to U$ satisfying $\alpha(Q) = T$. Design a q-ROFSG P of V as follows;

$$\mu_P(\epsilon) = \begin{cases} \mu_Q(\epsilon), & if\, \epsilon \in ker\alpha \\ 0, & otherwise \end{cases}$$

and

$$\nu_P(\epsilon) = \begin{cases} \nu_Q(\epsilon), & if\, \epsilon \in ker\alpha \\ 1, & otherwise \end{cases}$$

Clearly, P is a q-ROFSG of V and $P \subseteq Q$.
Let $\lambda \in V$, then we have the two following cases;

(i) If $\lambda \in ker\alpha$, then $\epsilon\lambda\epsilon^{-1} \in ker\alpha$ for all $\epsilon \in V$.

$$\left(\mu_P(\epsilon\lambda\epsilon^{-1})\right)^q = \left(\mu_Q(\epsilon\lambda\epsilon^{-1})\right)^q \geq \min\{(\mu_Q(\lambda))^q, (\mu_Q(\epsilon))^q\} = \min\{(\mu_P(\lambda))^q, (\mu_Q(\epsilon))^q\}$$

and

$$\left(\nu_P(\epsilon\lambda\epsilon^{-1})\right)^q = (\nu_Q(\epsilon\lambda\epsilon^{-1}))^q \leq \max\{(\nu_Q(\lambda))^q, (\nu_Q(\epsilon))^q\} = \max\{(\nu_P(\lambda))^q, (\nu_Q(\epsilon))^q\}.$$

(ii) If $\lambda \in V/ker\alpha$, then $\mu_P(\lambda) = 0$ and $\nu_P(\lambda) = 1$.
Now

$$\left(\mu_P(\epsilon\lambda\epsilon^{-1})\right)^q \geq \min\{(\mu_P(\lambda))^q, (\mu_Q(\epsilon))^q\}$$

and

$$\left(\nu_P(\epsilon\lambda\epsilon^{-1})\right)^q \leq \max\{(\nu_P(\lambda))^q, (\nu_Q(\epsilon))^q\}$$

Thus, P is a q-ROFNSG of Q in both instances. Also $Q \cong T$ gives $\alpha(Q) = T$, therefore $(\alpha(Q))^* = T^*$. The use of Lemma 3 yields $\alpha(Q^*) = T^*$. Let $\beta = \alpha|_{Q^*}$, then β is a homomophism from Q^* to T^* such that $Ker\beta = P^*$. The first isomorphism Theorem of conventional group theory ensures the existence an isomorphism $\psi : Q^*/P^* \to T^*$ defined by $\psi(\epsilon P^*) = \beta(\epsilon) = \alpha(\epsilon)$ for all $\epsilon P^* \in Q^*/P^*$.
Suppose that $k \in T^*$, then

$$\left(\mu_{\psi(Q/P)}(k)\right)^q = \max\{(\mu_{Q/P}(\epsilon P^*))^q : \epsilon \in Q^*, \psi(\epsilon P^*) = k\}$$

$$= \max\{\max\{(\mu_Q(t))^q : t \in \epsilon P^*\} : \epsilon \in Q^*, \beta(\epsilon) = k\}$$

$$= \max\{(\mu_Q(t))^q : t \in Q^*, \beta(t) = k\}$$

$$= \max\{(\mu_Q(t))^q : t \in Q^*, \beta(t) = k\}$$

$$= \max\{(\mu_Q(t))^q : t \in V, \alpha(t) = k\}$$

$$= (\mu_{\alpha(Q)}(k))^q$$

$$= (\mu_T(k))^q$$

$$= (\mu_{T''}(k))^q \quad (Since\, k \in T^*)$$

Similarly,

$$\left(\nu_{\psi(Q/P)}(k)\right)^q = (\nu_{T''}(k))^q$$

Thus, $Q/P \cong T''$.

Definition 14. *Let Q and P be two q-ROFSGs of V, then the product of Q and P is denoted by $Q \circ P$ and is defined as;*

(i) $\mu_{Q\circ P}(\epsilon) = \max\{\min(\mu_Q(u), \mu_P(v)) : u, v \in V, uv = \epsilon\}$

(ii) $\nu_{Q\circ P}(\epsilon) = \min\{\max(\nu_Q(u), \nu_P(v)) : u, v \in V, uv = \epsilon\}$

for all $\epsilon \in V$.

Lemma 4. *Let Q and P be two q-ROFSGs of V. Then*

(i) $(Q \cap P)^* = Q^* \cap P^*$.

(ii) $(Q \circ P)^* = Q^* P^*$

Proof.

(i) Suppose that $\epsilon \in V$, then

$$\epsilon \in (Q_1 \cap Q_2)^* \Leftrightarrow (\mu_{Q_1\cap Q_2}(\epsilon))^q > 0 \textit{ and } (\nu_{Q_1\cap Q_2}(\epsilon))^q < 1$$

$$\Leftrightarrow \min\left((\mu_{Q_1}(\epsilon))^q, (\mu_(Q_2)(\epsilon))^q\right) > 0 \, and \, \min\left((\nu_(Q_1)(\epsilon))^q, (\nu_(Q_2)(\epsilon))^q\right) < 1$$

$$\Leftrightarrow \left(\mu_(Q_1)(\epsilon)\right)^q, (\mu_(Q_2)(\epsilon))^q > 0 \, and \, (\nu_(Q_1)(\epsilon))^q, (\nu_(Q_2)(\epsilon))^q < 1 \Leftrightarrow \epsilon \in Q_1^*, Q_2^* \Leftrightarrow \epsilon \in Q_1^* \cap Q_2^*.$$

Thus, $(Q_1 \cap Q_2)^* = Q_1^* \cap Q_2^*$.

(ii) Suppose that $\epsilon \in V$, then

$$\epsilon \in (Q \circ P)^* \Leftrightarrow (\mu_{Q\circ P}(\epsilon))^q > 0 \, and \, (\nu_{Q\circ P}(\epsilon))^q < 1$$

$$\Leftrightarrow \max\{\min(\mu_Q(u), \mu_P(v))^q : u, v \in V, uv = \epsilon\} > 0$$

and

$$\nu_{Q\circ P}(\epsilon) = \min\{\max(\nu_Q(u), \nu_P(v)) : u, v \in V, uv = \epsilon\} < 1$$

$$\Leftrightarrow \mu_Q(u) > 0, \mu_P(v) > 0 \, and \, \nu_Q(u), \nu_P(v) < 1, \, where \, u, v \in V \, such \, that \, uv = \epsilon$$

$$\Leftrightarrow u \in Q^* \, and \, v \in P^*, \, where \, u, v \in V \, such \, that \, uv = \epsilon \Leftrightarrow \epsilon = uv \in Q^* P^*.$$

Thus, $(Q \circ P)^* = Q^* P^*$.

Definition 15. *Assume that Q_1 and Q_2 are q-ROFSGs of V and W respectively. Then an isomorphism $\alpha : V \to W$ is called a weak q-rung isomorphism (Wq-ROFIso) from Q_1 to Q_2 if $\alpha(Q_1) \subseteq Q_2$. The existence of Wq-ROFIso between Q_1 and Q_2 is denoted by $Q_1 \simeq Q_2$.*

Theorem 21. *(Second Isomorphism Theorem in q-rung orthopair Fuzzy Settings) Let Q be a q-ROFNSG of V and P be a q-ROFSG of V. Then $P/Q \cap P \simeq (Q \circ P)/Q$.*

Proof. By Theorem 9, we have $Q^* \trianglelefteq V$. Also from Lemma 2, we obtain P^* is a subgroup of V. The second isomorphism theorem of crisp groups shows that there is an isomorphism ψ between $P^*/Q^* \cap P^*$ and Q^*P^*/Q^* that is defined by $\psi(k(Q^* \cap P^*)) = kQ^*$ for all $k \in P^*$. By Lemma 4; $(Q \cap P)^* = Q^* \cap P^*$ and $(QP)^* = Q^*P^*$, therefore $P^*/Q^* \cap P^*$ is isomorphic to Q^*P^*/Q^*.
Now

$$\begin{aligned}\left(\mu_{\psi(P/Q\cap P)}(kQ^*)\right)^q &= \left(\mu_{P/Q\cap P}(k(Q\cap P)^*)\right)^q \quad (\psi\, is\, one-one)\\ &= \max\{(\mu_P(t))^q : t \in k(Q\cap P)^*\}\\ &\leq \max\{((\mu_Q \circ \mu_P)(t))^q : t \in k(Q^* \cap P^*)\}\\ &\leq \max\{((\mu_Q \circ \mu_P)(t))^q : t \in kQ^*\}\\ &= (((\mu_Q \circ \mu_P)/\mu_Q)(kQ^*))^q\end{aligned}$$

The same reasoning leads us to

$$\left(\nu_{\psi(P/Q\cap P)}(kQ^*)\right)^q \geq (((\nu_Q \circ \nu_P)/\nu_Q)(kQ^*))^q\,.$$

Therefore, $\psi(P/Q \cap P) \subseteq (Q \circ P)/Q$ implying that $P/Q \cap P \simeq (Q \circ P)/Q$.

Theorem 22. *(Third Isomorphism Theorem in q-rung orthopair Fuzzy Settings) Let Q, P and T be q-ROFSG of V. If Q and P are q-ROFNSGs of T and $Q \subseteq P$. Then $(T/Q)/(P/Q) \cong T/P$.*

Proof. According to Theorem 9, $Q^* \trianglelefteq T^*$ and $P^* \trianglelefteq T^*$. Since $Q \subseteq P$ therefore $Q^* \trianglelefteq P^*$. The third isomorphism theorem associated to crisp groups theory reveals that there is an isomorphism ψ between $(T^*/Q^*)/(P^*/Q^*)$ and T^*/P^*, defined by $\psi(lQ^*(P^*/Q^*)) = lP^*$ for all $l \in T^*$.

$$\begin{aligned}\left(\mu_{\psi(((T/Q))/((P/Q)))}(lP^*)\right)^q &= \left(\mu_(((T/Q))/((P/Q)))(lQ^*(P^*/Q^*))\right)^q\\ &= \max\{(\mu_{T/Q}(l'Q^*))^q : l' \in T^*, l'Q^* \in lQ^*(P^*/Q^*)\}\\ &= \max\{\max\{(\mu_T(t))^q : t \in l'Q^*\} : l' \in T^*, l'Q^* \in lQ^*(P^*/Q^*)\}\\ &= \max\{(\mu_T(z))^q : z \in T^*, zQ^* \in lQ^*(P^*/Q^*)\}\end{aligned}$$

We know, $zQ^* \in lQ^*(P^*/Q^*)$, then $lQ^*(P^*/Q^*) = zQ^*(P^*/Q^*)$ (Since $\lambda \in gH \Rightarrow gH = \lambda H$). So, $\psi(lQ^*(P^*/Q^*)) = \psi(zQ^*(P^*/Q^*))$ implying that $lP^* = zP^*$. Since $z \in zP^*$, thus, $z \in lP^*$.
Therefore,

$$\begin{aligned}(\mu_\psi(((T/Q))/((P/Q)))(lP^*))^q &= \max\{(\mu_T(z))^q : z \in T^*, z \in lP^*\}\\ &= \left(\mu_{T/P}(lP^*)\right)^q\end{aligned}$$

By the same method, we obtain;

$$(\nu_\psi(((T/Q))/((P/Q)))(lP^*))^q = (\nu_(T/P)(lP^*))^q$$

Therefore, $\psi\left(((T/Q))/((P/Q))\right) = T/P$, thus, ψ is a q-ROFIso from $(T/Q)/(P/Q)$ to T/P.

6. Conclusion

The primary goal of this study is to analyze fundamental isomorphism theorems in the context of q-ROFSs. Many concepts from group theory, particularly cosets, normal subgroups, quotient groups, homomorphisms and isomorphisms are translated into the q-rung orthopair fuzzy structure and various theorems are proved in this context. As a consequence of this research, we want to investigate further topics of traditional group theory such as Abelian groups, commutator subgroups, Lagrange's and Caley's theorems, free group and p-groups etc in q-rung orthopair fuzzy environment. In addition, we will extend these results to other types (rings, vector spaces etc) of homomorphism and under different extensions of the FSs. Furthermore, we intend to employ the findings from this research in a variety of practical contexts, including cryptography, picture encryption, and other related areas.

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