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Multisorted Algebras of Trees of a Weakly Fixed Variable

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Abstract. For any algebra of type τ, this paper introduces a novel class of terms (or terms) of a weakly fixed variable of type τ. Algebraic structures in the sence of multisorted algebras are studied. In fact, it is shown that the set of such terms and the multisort operations forms a multisorted algebra that satisfies some axioms from the theory of clone. As a tool for classifying arbitrary algebras to subclasses called weakly fixed variable solid varieties, the seminearring of weakly fixed variable hypersubstitutions is proposed. Characterizations for any variety V of algebras of type τ to be a weakly fixed variable solid variety are explored.

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1. Introduction and background

The concept of multisorted algebras (also called many-sorted algebras or heterogeneous algebras) generalizes the concept of one-sorted algebras. Any module and vector space are basic examples of multisorted algebras. In general, the S-sorted sets $A = (A_s)_{s\in S}$ are essential. The set S is called a set of sorts. In addition, the sort mapping $\phi : A \to B$ from an S-sorted set $A = (A_s)_{s\in S}$ to an S-sorted set $B = (B_s)_{s\in S}$ is an S-sorted family $\phi : (\phi_s)_{s\in S}$ of mappings $\phi_s : A_s \to B_s$ where $s \in S$. The authors always refer to [1, 4, 19] for more details.

Terms or trees in a study of automata and logic can be applied to form multisorted algebras called the multisorted algebra of terms of type τ. To attain this, we recall some important definitions. Let I be a nonempty indexed set and $(f_i)_{i\in I}$ be a sequence of operation symbols. To every operation symbol f_i, we assign a natural number $n_i \in \mathbb{N} :=$

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$\{1, 2, \ldots\}$, called the arity of f_i. The type is a sequence $\tau := (n_i)_{i \in I}$. Let $n \geq 1$, we denote by $X_n := \{x_1, \ldots, x_n\}$ a finite set called an *alphabet* and each x_i in X_n is called a *variable*. The set of all *n-ary terms of type* τ is the smallest set which contains X_n denoted by $W_\tau(X_n)$ inductively defined by: (1) $X_n \subseteq W_\tau(X_n)$ and (2) If $t_1, \ldots, t_{n_i} \in W_\tau(X_n)$ and f_i is an operation symbol of the arity n_i, then $f_i(t_1, \ldots, t_{n_i}) \in W_\tau(X_n)$. For infinitely many variables X, we denote by

$$W_\tau(X) := (W_\tau(X_n))_{n \in \mathbb{N}},$$

i.e., for the infinite sequence

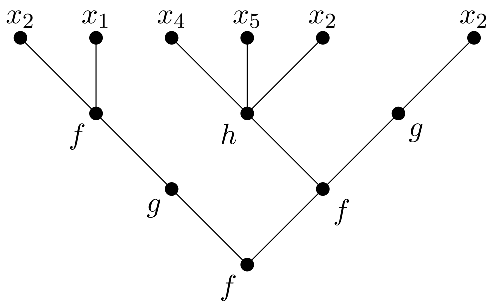
$$(W_\tau(X_1), W_\tau(X_2), W_\tau(X_3), \ldots)$$

the multisorted set of all terms of type τ. In this matter, the sorts are the sets of n-ary terms of type τ for all $n \in \mathbb{N}$. Recent developments of terms in various directions can be found, for instance, in [8, 10, 12, 15, 16, 18].

It is commonly seen that any term can be represented as a tree diagram. For this, we can visualize a term from the left to the right by treating each operation symbol as a vertex and each variable as a leaf of the tree. For example, the term

$$t = f(g(f(x_2, x_1)), f(h(x_4, x_5, x_2), g(x_2)))$$

can be visualized as the following figure.



For more backgrounds of terms representative by trees, we refer to [7].

The multisorted algebra of terms belongs to the variety of all abstract clones which is a family of $\mathbb{N}$-sorted algebras satisfying the following three identities:

(C1) $\tilde{S}^n_m(\tilde{S}^p_n(\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p), \tilde{X}_1, \ldots, \tilde{X}_n)$
$\approx \tilde{S}^p_m(\tilde{Z}, \tilde{S}^n_m(\tilde{Y}_1, \tilde{X}_1, \ldots, \tilde{X}_n), \ldots, \tilde{S}^n_m(\tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_n)), m, n, p \in \mathbb{N}$,

(C2) $\tilde{S}^n_m(\lambda_j, \tilde{X}_1, \ldots, \tilde{X}_n) \approx \tilde{X}_j, n, m \in \mathbb{N}, 1 \leq j \leq n$,

(C3) $\tilde{S}^n_n(\tilde{Y}, \lambda_1, \ldots, \lambda_n) \approx \tilde{Y}, n \in \mathbb{N}$,

where $\tilde{S}^n_m$, $\tilde{S}^p_n$, $\tilde{S}^p_m$, $\tilde{S}^n_n$ are operation symbols, $\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{Y}$ are variables for terms, and λ_j are symbols for variables. In general, (C1) is said to be the *superassociative law* since it generalizes the associative law. A class of algebras that satisfies (C1) is called a Menger algebra or a superassociative algebra. In (C1), if $n = m = p = 1$, then it reduces to the usual associative law. For an overview of clone theory we refer to [2, 3, 9]. The viewpoint taken in Menger algebras can be found in [5, 6, 11, 14].

The multisorted superposition operation defined on $(W_\tau(X_n))_{n\in\mathbb{N}}$ is a multisorted mapping

$$S^n_m : W_\tau(X_n) \times (W_\tau(X_m))^n \to W_\tau(X_m)$$

defined by:

(1) $S^n_m(x_i, t_1, \ldots, t_n) \; = \; t_i$ if $x_i \in X_n$,

(2) $S^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) \; = \; f_i(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n))$

where $n, m \in \mathbf{N}$ and $t_1, \ldots, t_n \in W_\tau(X_m)$.

Thus, the multisorted algebra of terms of type τ or the clone of all terms of type τ denoted by

$$clone(\tau) \; = \; ((W_\tau(X_n))_{n\in\mathbb{N}}, (S^n_m)_{m,n\in\mathbb{N}}, (x_i)_{i\leq n\in\mathbb{N}})$$

is formed. Actually, it also satisfies the axioms (C1), (C2) and (C3), thus it is an example of abstract clones. Particularly, if we let $\mathcal{A} := (A, (f_i^{\mathcal{A}})_{i\in I})$ be an algebra of type τ and let t be an n-ary term of type τ, then a term t induces an n-ary operation $t^{\mathcal{A}}$ on $\mathcal{A}$ as follows:

(1) If $t = x_j \in X_n$, then $t^{\mathcal{A}} = x_j^{\mathcal{A}} = \mathrm{pr}_j^{n,\mathcal{A}}$ where $\mathrm{pr}_j^{n,\mathcal{A}}$ is an n-ary projection mapping on A,

(2) if $t = f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ and $t_1^{\mathcal{A}}, \ldots, t_{n_i}^{\mathcal{A}}$ are the term operations which are induced by $t_1, \ldots, t_{n_i}$, then $t^{\mathcal{A}} = f_i^A(t_1^{\mathcal{A}}, \ldots, t_{n_i}^{\mathcal{A}})$.

Hence, $t^{\mathcal{A}}$ is called the term operation induced by the term t on the algebra $\mathcal{A}$. The set of all n-ary term operations on $\mathcal{A}$ will be denoted by $W_\tau(X_n)^{\mathcal{A}}$. Moreover, let

$$Id\mathcal{A} = \{s \approx t \in W_\tau(X) \times W_\tau(X) \mid s^{\mathcal{A}} = t^{\mathcal{A}}\}$$

be the set of all identities satsified in $\mathcal{A}$. The multisorted algebra

$$\mathrm{Clone}\mathcal{A} = ((W_\tau(X_n)^{\mathcal{A}})_{n\in\mathbb{N}}, (\mathcal{O}^{n,A}_m)_{n,m\in\mathbb{N}}, (\mathrm{pr}_i^{n,A})_{i\leq n, n\in\mathbb{N}})$$

is constructed. Another example is the quotient algebra

$$clone(V) \; = \; clone(\tau)/Id(V)$$

where $Id(V)$ is a congruence in the form of the multisorted set $(Id_n(V))_{n\in\mathbb{N}}$ of all n-ary identities $s \approx t$ satisfied in a variety V of algebras of type τ.

Recall from [3] that a mapping

$$\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$$

such that for each $i \in I$, $\sigma(f_i) \in W_\tau(X_{n_i})$ is called a hypersubstitution of type τ. Moreover, each $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ can be uniquely extended to the mapping

$$\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$$

defined by:

(1) $\widehat{\sigma}[x_i] = x_i$ for every $x_i \in X$,

(2) $\widehat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = S^{n_i}_m(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.

The set $Hyp(\tau)$ of all hypersubstitutions of type τ forms a monoid under the associative binary operation defined by:

$$\sigma \circ_h \alpha = \widehat{\sigma} \circ \alpha$$

for all $\sigma, \alpha \in Hyp(\tau)$ and the hypersubstitution $\sigma_{id} : \{f_i \mid i \in I\} \to W_\tau(X)$ defined by $\sigma_{id}(f_i) = f_i(x_1, \ldots, x_{n_i})$ for all $i \in I$ which acts as an identity element.

In fact, each hypersubstitution can be considered as a multisorted mapping. For $n \in \mathbb{N}$, let $I_n \subseteq I$ be the set of all indexes such that f_j with $j \in I_n$ is an n-ary. Let $F^n_\tau = \{f_j \mid j \in I_n\}$. Thus, a hypersubstitution is the sequence $(\sigma_n)_{n\in\mathbb{N}}$ where $\sigma : F^n_\tau \to W_\tau(X_n)$. Let $(Hyp_n(\tau))_{n\in\mathbb{N}}$ be the multisorted set of all hypersubstitutions of type τ. By the definition $\widehat{\sigma_n}[x_i] = x_i$ and $\widehat{\sigma_n}[f_i(t_1, \ldots, t_n)] = S^n_n(\sigma_n(f_i), \widehat{\sigma_n}[t_1], \ldots, \widehat{\sigma_n}[t_n])$ we obtain the extension of each σ_n in $(\sigma_n)_{n\in\mathbb{N}}$.

Recently, in [17], the multisorted set $(W^{fv}_\tau(X_n))_{n\in\mathbb{N}}$ of terms of a fixed variable is introduced. For instance, let us consider the type $\tau = (2)$ with a binary operation symbol f. Then,

$$x_1, x_2, f(x_1, x_1), f(x_2, x_2), f(f(x_1, x_1), x_1), f(x_2, f(x_2, x_2)) \in W^{fv}_\tau(X_2),$$

$$x_1, x_2, x_3, f(x_1, x_1), f(x_3, x_3), f(f(x_3, x_3), f(x_3, x_3)) \in W^{fv}_\tau(X_3).$$

Conversely,

$$f(x_1, x_2), f(x_3, x_1), f(x_2, f(x_1, x_2)) \in W_\tau(X_3) \setminus W^{fv}_\tau(X_3).$$

By the formal definition, if t is a term, then the set $\text{var}(t)$ consisting of all variables of X that appear in t is called the set of all variables for t. Thus, an n-ary term of a fixed variable of type τ is inductively defined by:

(1) Every $x_i \in X_n$ is an n-ary term of a fixed variable of type τ,

(2) if $t_1, \ldots, t_{n_i}$ are n-ary terms of a fixed variable of type τ, and if $\text{var}(t_j) = \text{var}(t_k)$ for all $1 \le j < k \le n_i$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of a fixed variable of type τ,

(3) the set $W^{fv}_\tau(X_n)$ is the smallest set which is closed under finite application of (2).

This concept always plays a key role in a study of the variety of bands, i.e. all algebras of type (2) satisfying $f(x, x) \approx x$. Closed identities of a fixed variable and closed varieties of a fixed variable are also investigated based on multisorted hypersubstitutions of a fixed variable.

The main purpose of this paper is to generalize the multisorted set of terms of a fixed variable by naturally reducing certain conditions and constructing the multisorted algebras under the superposition operation. The paper is organized as follows. Section 2 introduces the concept of terms of a weakly fixed variable of type τ and provides some algebraic properties. In Section 3, we present essential tools for defining a novel class of algebras that satisfy identities generated by terms of a weakly fixed variable. Additionally, the multisorted algebras consisting of the multisorted set of mapping whose images are terms of a weakly fixed variable and two associative binary operations are constructed. Applications of the multisorted algebras of terms of a weakly fixed variable are given in Section 4. Finally, we provide some concluding remarks in Section 5.

2. Terms of a weakly fixed variable

This section begins with the definition of terms of a weakly fixed variable. We also construct the multisorted algebra of such terms under the multisorted superposition operation and projections.

Definition 1. *For a natural number n, an n-ary term of a weakly fixed variable of type τ is inductively defined by the following:*

(1) Every variable x_i in an alphabet X_n is an n-ary term of a weakly fixed variable of type τ.

(2) If $t_1, \ldots, t_{n_i}$ are n-ary terms of a weakly fixed variable of type τ and $var(t_l) = var(t_p)$ for some $1 \leq l < p \leq n_i$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of a weakly fixed variable of type τ.

(3) The set $W_\tau^{wfv}(X_n)$ of all n-ary terms of a weakly fixed variable of type τ is the smallest set closed under finite application of (2).

Some concrete examples are given.

Example 1. *Consider a type $(3, 2)$ with a ternary operation symbol $\boxplus$ and a binary operation symbol $\boxminus$ and an alphabet X_4. Then quaternary terms of type $(3, 2)$ which are quaternary terms of a weakly fixed variable are listed, for example, as follows:*

$$x_1, x_2, x_3, x_4, \boxplus(x_1, x_1, x_3), \boxplus(x_2, x_2, x_4), \boxminus(x_4, x_4), \boxplus(\boxminus(x_3, x_3), x_3, \boxplus(x_1, x_2, x_1)).$$

On the other hand,

$$\boxplus(x_1, x_2, x_3), \boxplus(x_4, x_2, x_1), \boxminus(x_2, \boxminus(x_1, x_1)), \boxplus(x_3, \boxminus(x_1, x_1), x_4)$$

are not quaternary terms of a weakly fixed variable of type $(3, 2)$.

We note that the set of terms of a weakly fixed variable can be viewed as a generalization of the set of terms of a fixed variable as follows.

Remark 1. *For any $n \geq 1$, the connection between the set $W_\tau^{wfv}(X_n)$ and $W_\tau^{fv}(X_n)$ is described as follows:*

(1) If $\tau = (1, 1, \ldots,)$ or $\tau = (2, 2, \ldots)$, then $W_\tau^{wfv}(X_n) = W_\tau^{fv}(X_n)$.

(2) If $\tau = (n_i)_{i\in I}$ and $n_i > 2$ for some $i \in I$, then $W_\tau^{fv}(X_n) \subset W_\tau^{wfv}(X_n)$.

To guaruntee that the multisorted superposition can be applied to the family of the set of all terms of a weakly fixed variable of type τ, the following lemma is required.

Lemma 1. *For $m, n \in \mathbb{N}$, if s is an n-ary term of a weakly fixed variable of type τ and $t_1, \dots, t_n$ are m-ary terms of a weakly fixed variable of type τ, then*

$$S_m^n(s, t_1, \dots, t_n) \in W_\tau^{wfv}(X_m).$$

Proof. Suppose that $s \in W_\tau^{wfv}(X_n)$ and $t_1, \dots, t_n \in W_\tau^{wfv}(X_m)$. We prove on the complexity of a term s that $S_m^n(s, t_1, \dots, t_n) \in W_\tau^{wfv}(X_m)$. It is obvious that $S_m^n(s, t_1, \dots, t_n)$ is a term of a weakly fixed variable in $W_\tau^{wfv}(X_m)$ if s is a variable x_k for all $1 \leq i \leq n$. We now inductively assume that $s = f_i(s_1, \dots, s_{n_i})$ and $\text{var}(s_l) = \text{var}(s_k)$ for some fixed integers $1 \leq l < k \leq n_i$. From the definition of multisorted superposition, we have

$$S_m^n(f_i(s_1, \dots, s_{n_i}), t_1, \dots, t_n) = f_i(S_m^n(s_1, t_1, \dots, t_n), \dots, S_m^n(s_{n_i}, t_1, \dots, t_n)),$$

we prove that each $S_m^n(s_j, t_1, \dots, t_n)$ is an m-ary term of a weakly fixed variable of type τ for every $1 \leq j \leq n_i$ and $\text{var}(S_m^n(s_l, t_1, \dots, t_n)) = \text{var}(S_m^n(s_k, t_1, \dots, t_n))$ for some fixed integers $1 \leq l < k \leq n_i$.

Let $j \in \{1, \dots, n_i\}$. If $s_j = x_p$ for some $x_p \in X_n$, then

$$S_m^n(s_j, t_1, \dots, t_n) = S_m^n(x_p, t_1, \dots, t_n) = t_p \in W_\tau^{wfv}(X_m).$$

Assume that $s_j = f_i(s'_1, \dots, s'_{n_i})$ and $S_m^n(s'_p, t_1, \dots, t_n) \in W_\tau^{wfv}(X_m)$ for all $p = 1, \dots, n_i$. Without loss of generality, suppose that $\text{var}(s'_l) = \text{var}(s'_k)$ for some $1 \leq l < k \leq n_i$. Then we obtain that $S_m^n(f_i(s'_1, \dots, s'_{n_i}), t_1, \dots, t_n) \in W_\tau^{wfv}(X_m)$. Actually, since we know that $\text{var}(s_l) = \text{var}(s_k)$ for some fixed integers $1 \leq l < k \leq n_i$, then we have

$$\text{var}(S_m^n(s_l, t_1, \dots, t_n)) = \text{var}(S_m^n(s_k, t_1, \dots, t_n)),$$

which completes the proof.

To enhance understanding of the computation process for terms with a weakly fixed variable under the multisorted superposition, the following example is provided.

Example 2. *Consider $n = 3$ and $m = 4$ and the multisorted superposition S_4^3. On the sets $W_{(3,2)}^{wfv}(X_3)$ and $W_{(3,2)}^{wfv}(X_4)$, if we put*

$$a = \boxplus(x_1, x_3, x_1), \qquad b = \boxplus(x_4, \boxplus(x_2, x_1, x_2), x_4),$$
$$c = \boxminus(x_4, x_4), \qquad d = \boxminus(x_2, \boxplus(x_2, x_2, x_2)),$$
$$e = \boxplus(\boxminus(x_4, x_4), \boxminus(x_4, x_4), x_3), \qquad f = \boxminus(x_1, x_1),$$

then

$$\begin{aligned} S_4^3(a,b,c,c) &= S_4^3(\boxplus(x_1,x_3,x_1),\boxplus(x_4,\boxplus(x_2,x_1,x_2),x_4),\boxminus(x_4,x_4),\boxminus(x_4,x_4)) \\ &= \boxplus(\boxplus(x_4,\boxplus(x_2,x_1,x_2),x_4),\boxminus(x_4,x_4),\boxplus(x_4,\boxplus(x_2,x_1,x_2),x_4)) \end{aligned}$$

belongs to $W_{(3,2)}^{wfv}(X_4)$ *because the sets of variables in the first and the third positions are equal, i.e.,* $\{x_1,x_2,x_4\}$. *Moreover,* $S_4^3(y,z_1,z_2,z_3) \in W_{(3,2)}^{wfv}(X_4)$ *if* $y \in \{a,d,f\}$ *and* $z_1,z_2,z_3 \in \{a,b,c,d,e,f\}$ *and* $S_3^4(u,w_1,w_2,w_3,w_4) \in W_{(3,2)}^{wfv}(X_3)$ *if* $u \in \{a,b,c,d,e,f\}$ *and* $w_1,w_2,w_3,w_4 \in \{a,d,f\}$.

By Lemma 1, the multisorted superpositions S_m^n can be applied to the family $(W_\tau^{wfv}(X_n))_{n\in\mathbb{N}}$, which means that we have following multisorted mappings

$$S_m^n : W_\tau^{wfv}(X_n) \times (W_\tau^{wfv}(X_m))^n \to W_\tau^{wfv}(X_m)$$

for $m,n \in \mathbb{N}$.

As a consequence, the multisorted algebra

$$\overline{W_\tau^{wfv}(X)} := ((W_\tau^{wfv}(X_n))_{n\in\mathbb{N}},(S_m^n)_{n,m\in\mathbb{N}},(x_i)_{i\leq n,n\in\mathbb{N}})$$

is formed.

Since we know that $(W_\tau^{wfv}(X_n))_{n\in\mathbb{N}} \subseteq (W_\tau(X_n))_{n\in\mathbb{N}}$, then by Lemma 1 the following result is concluded.

Theorem 1. *The multisorted algebra* $\overline{W_\tau^{wfv}(X)}$ *satisfies the axioms (C1), (C2) and (C3).*

3. Weakly fixed variable hypersubstitutions

This section introduces a mapping whose images are terms of a weakly fixed variable and discusses the multisorted composition.

Definition 2. *A hypersubstitution* $\sigma \in Hyp(\tau)$ *is called a weakly fixed variable hypersubstitution of type* τ *if for all* $i \in I$, σ *maps each operation symbol* f_i *to an* n_i*-ary term of a weakly fixed variable of type* τ. *The set of all weakly fixed variable hypersubstitutions of type* τ *is denoted by* $Hyp^{wfv}(\tau)$, *i.e.,*

$$Hyp^{wfv}(\tau) = \{\sigma \mid \sigma : \{f_i \mid i \in I\} \to W_\tau^{wfv}(X)\}.$$

For instance, let $\tau = (3,2)$ be a type with a ternary operation symbol $\boxplus$ and a binary operation symbol $\boxminus$. A hypersubstitution $\sigma : \{\boxplus,\boxminus\} \to W_\tau(X)$ such that $\sigma(\boxplus) = \boxplus(x_3,\boxminus(x_2,x_2),x_3)$ and $\sigma(\boxminus) = \boxminus(x_1,x_1)$. Then we get that σ a weakly fixed variable hypersubstitution of type τ. On the other hand, we let $\beta : \{\boxplus,\boxminus\} \to W_\tau(X)$ such that $\beta(\boxplus) = \boxplus(x_1,\boxminus(x_2,x_2),x_3)$ and $\beta(\boxminus) = \boxminus(x_1,x_2)$. Then β is not a weakly fixed variable hypersubstitution of type τ since $\beta(\boxplus) \notin W_\tau^{wfv}(X)$ and $\beta(\boxminus) \notin W_\tau^{wfv}(X)$.

To construct the algebra of weakly fixed variable hypersubstitutions, we need the following result to confirm that each extension of σ takes from the set of terms of a weakly fixed variable into itself.

Lemma 2. *For any weakly fixed variable hypersubstitution σ, we have*

$$\widehat{\sigma} : W_\tau^{wfv}(X) \to W_\tau^{wfv}(X).$$

Proof. Let σ be a mapping on $Hyp^{wfv}(\tau)$ and t be an element in $W_\tau^{wfv}(X)$. We show that $\widehat{\sigma}[t] \in W_\tau^{wfv}(X)$. Clearly, $\widehat{\sigma}[x_i] \in W_\tau^{wfv}(X)$ for every variable x_i. Suppose now that $t = f_i(t_1, \ldots, t_{n_i})$ such that $\widehat{\sigma}[t_k] \in W_\tau^{wfv}(X)$ for all $k \in \{1, \ldots, n_i\}$. Without loss of generality, we may assume here that $\text{var}(t_l) = \text{var}(t_p)$ for some $l, p \in \{1, \ldots, n_i\}$ and $l \neq p$. Then we obtain $\text{var}(\widehat{\sigma}[t_l]) = \text{var}(\widehat{\sigma}[t_p])$. Since $\sigma(f_i)$ belongs to $W_\tau^{wfv}(X)$, then by the hypothesis, i.e., $\text{var}(t_l) = \text{var}(t_p)$, we have

$$\widehat{\sigma}[t] = S_m^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]) \in W_\tau^{wfv}(X).$$

This finishes the proof.

For $n \in \mathbb{N}$, the multisorted mapping $\widehat{\sigma_n} : W_\tau^{wfv}(X_n) \to W_\tau^{wfv}(X_n)$ can be defined by

(1) $\widehat{\sigma_n}[x_i] = x_i$ for any $x_i \in X_n$ and

(2) $\widehat{\sigma_n}[f_i(t_1, \ldots, t_{n_i})] = S_n(\sigma_{n_i}(f_i), \widehat{\sigma_n}[t_1], \ldots, \widehat{\sigma_n}[t_{n_i}])$ if each $\widehat{\sigma_n}[t_j]$ is already known for $1 \leq j \leq n_i$.

Then we prove:

Theorem 2. *The extension $\widehat{\sigma}$ of each σ in $Hyp^{wfv}(\tau)$ is an endomorphism on the superassociative system $\overline{W_\tau^{wfv}(X)}$.*

Proof. Let $\sigma \in Hyp^{wfv}(\tau)$. To prove that $\widehat{\sigma}$ is an endomorphism on $\overline{W_\tau^{wfv}(X)}$, we aim to show that the equation

$$\widehat{\sigma_m}[S_m^n(s, t_1, \ldots, t_n)] = S_m^n(\widehat{\sigma_n}[s], \widehat{\sigma_m}[t_1], \ldots, \widehat{\sigma_m}[t_n]) \tag{1}$$

holds for any $s \in W_\tau^{wfv}(X_n)$, $t_1, \ldots, t_n \in W_\tau^{wfv}(X_m)$. To do this, we give a proof on the complexity of a term s. If s is a variable x_j in X_n, then

$$\widehat{\sigma_m}[S_m^n(x_j, t_1, \ldots, t_n)] = \widehat{\sigma_m}[t_j] = S_m^n(x_j, \widehat{\sigma_m}[t_1], \ldots, \widehat{\sigma_m}[t_n]) = S_m^n(\widehat{\sigma_n}[x_j], \widehat{\sigma_m}[t_1], \ldots, \widehat{\sigma_m}[t_n]).$$

Suppose that $s = f_i(s_1, \ldots, s_{n_i})$ and inductively assume that the equation (1) is satisfied for $s_1, \ldots, s_{n_i}$. Without loss of generality we may assume that $\text{var}(s_l) = \text{var}(s_k)$ for some $1 \leq l < k \leq n_i$. Then by Theorem 1, we obtain

$$\begin{aligned}
&\widehat{\sigma_m}[S_m^n(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n)] \\
&= \widehat{\sigma_m}[f_i(S_m^n(s_1, t_1, \ldots, t_n), \ldots, S_m^n(s_{n_i}, t_1, \ldots, t_n))] \\
&= S_m^{n_i}(\sigma_{n_i}(f_i), \widehat{\sigma_m}[S_m^n(s_1, t_1, \ldots, t_n)], \ldots, \widehat{\sigma_m}[S_m^n(s_{n_i}, t_1, \ldots, t_n)]) \\
&= S_m^{n_i}(\sigma_{n_i}(f_i), S_m^n(\widehat{\sigma_n}[s_1], \widehat{\sigma_m}[t_1], \ldots, \widehat{\sigma_m}[t_n]), \ldots, S^{n_m}(\widehat{\sigma_n}[s_{n_i}], \widehat{\sigma_m}[t_1], \ldots, \widehat{\sigma_m}[t_n])) \\
&= S_m^n(S_n^{n_i}(\sigma_{n_i}(f_i), \widehat{\sigma_n}[s_1], \ldots, \widehat{\sigma_n}[s_{n_i}]), \widehat{\sigma_m}[t_1], \ldots, \widehat{\sigma_m}[t_n]) \\
&= S_m^n(\widehat{\sigma_n}[f_i(s_1, \ldots, s_{n_i})], \widehat{\sigma_m}[t_1], \ldots, \widehat{\sigma_m}[t_n]),
\end{aligned}$$

which shows that (1) holds for $s = f_i(s_1, \ldots, s_{n_i})$.

Acutally, weakly fixed variable hypersubstitutions of type τ can be considered as multisorted mappings $(\sigma_n)_{n\in\mathbb{N}}$. Let $Hyp_n(\tau)$ be the set of all σ_n and let $(Hyp_n(\tau))_{n\in\mathbb{N}}$ be the multisorted sets of all weakly fixed variable hypersubstitutions.

We see that the composition mapping

$$(\overline{F}_\tau^n)_{n\in\mathbb{N}} \xrightarrow{(\alpha_n)_{n\in\mathbb{N}}} (W_\tau^{wfv}(X_n))_{n\in\mathbb{N}} \xrightarrow{(\widehat{\sigma_n})_{n\in\mathbb{N}}} (W_\tau^{wfv}(X_n))_{n\in\mathbb{N}}$$

is an element in the multisorted set $(Hyp_n^{wfv}(\tau))_{n\in\mathbb{N}}$. Consequently, as in the one-sorted case, the composition between any two mappings $Hyp_n^{wfv}(\tau)$ is defined by

$$\sigma_n \circ_n^h \alpha_n = \widehat{\sigma_n} \circ_n \alpha_n$$

where $\circ_n$ is a usual composition on the nth sort.

Then we prove:

Theorem 3. *$(Hyp_n^{wfv}(\tau))_{n\in\mathbb{N}}$ is a subsemigroup of $(Hyp_n(\tau))_{n\in\mathbb{N}}$ with respect to the multisorted operation $(o_n^h)_{n\in\mathbb{N}}$.*

Proof. The proof follows from Lemma 2.

Note that there is no an identity element in $(Hyp_n^{wfv}(\tau))_{n\in\mathbb{N}}$ bacause the image of the mapping σ_{id} given by $\sigma_{id}(f_i) = f_i(x_1, x_2, \dots, x_{n_i})$ is not a term of a weakly fixed variable. Consequently, $(Hyp_n^{wfv}(\tau))_{n\in\mathbb{N}}$ does not form a monoid.

For any σ_n and α_n on $Hyp_n^{wfv}(\tau)$, we define the binary operation $+_n$ on $Hyp_n^{wfv}(\tau)$ by

$$(\sigma_n +_n \alpha_n)(f_i) = S_n^{n_i}(\sigma_n(f_i), \alpha_n(f_i), \dots, \alpha_n(f_i)).$$

It is obvious that $\sigma_n +_n \alpha_n$ is again a weakly fixed variable hypersubstitution of type τ. From this, we have the following result.

Theorem 4. *$((Hyp_n^{wfv}(\tau))_{n\in\mathbb{N}}, (\circ_n^h)_{n\in\mathbb{N}}, (+_n)_{n\in\mathbb{N}})$ forms a left-seminearring.*

Proof. For each $n \in \mathbb{N}$, we first show that the operation $+_n$ is associative, i.e.,

$$((\sigma_n +_n \alpha_n) + \beta_n) = (\sigma_1 + (\sigma_2 + \sigma_3))$$

for all $\sigma_n, \alpha_n, \beta_n \in Hyp^{wfv}(\tau)$. For this, let f_i be an operation symbol. Then by Theorem 1, we obtain

$$\begin{aligned}
&((\sigma_n +_n \alpha_n) + \beta_n)(f_i) \\
&= S_n^n((\sigma_n +_n \alpha_n)(f_i), \beta_n(f_i), \dots, \beta_n(f_i)) \\
&= S_n^n(S_n^n(\sigma_n(f_i), \alpha_n(f_i), \dots, \alpha_n(f_i)), \beta_n(f_i), \dots, \beta_n(f_i)) \\
&= S_n^n(\sigma_n(f_i), S_n^n(\alpha_n(f_i), \beta_n(f_i), \dots, \beta_n(f_i)), \dots, S_n^n(\alpha_n(f_i), \beta_n(f_i), \dots, \beta_n(f_i))) \\
&= S_n^n(\sigma_n(f_i), (\alpha_n +_n \beta_n)(f_i), \dots, (\alpha_n +_n \beta_n)(f_i)) \\
&= (\sigma_n +_n (\alpha_n + \beta_n))(f_i).
\end{aligned}$$

Furthermore, the left distributive law, i.e.,

$$\sigma_n \circ_n^h (\alpha_n +_n \beta_n) = (\sigma_n \circ_n^h \alpha_n) +_n (\sigma_n \circ_n^h \beta_n)$$

is also obtained. Indeed, from the fact that the extension of each mapping on $Hyp^{wfv}(\tau)$ preserves the operations on the multisorted algebra $\overline{W_\tau^{wfv}(X)}$ proved in Theorem 2, we have

$$\begin{aligned}
(\sigma_n \circ_n^h (\alpha_n +_n \beta_n))(f_i) &= \widehat{\sigma}_n[(\alpha_n +_n \beta_n)(f_i)] \\
&= \widehat{\sigma}_n[S_n^n(\alpha_n(f_i), \beta_n(f_i), \ldots, \beta_n(f_i))] \\
&= S_n^n(\widehat{\sigma}_n[\alpha_n(f_i)], \widehat{\sigma}_n[\beta_n(f_i)], \ldots, \widehat{\sigma}_n[\beta_n(f_i)]) \\
&= S_n^n((\sigma_n \circ_n^h \alpha_n)(f_i), (\sigma_n \circ_n^h \beta_n)(f_i), \ldots, (\sigma_n \circ_n^h \beta_n)(f_i)) \\
&= ((\sigma_n \circ_n^h \alpha_n) +_n (\sigma_n \circ_n^h \beta_n))(f_i).
\end{aligned}$$

Therefore, $((Hyp_n^{wfv}(\tau))_{n\in\mathbb{N}}, (\circ_n^h)_{n\in\mathbb{N}}, (+_n)_{n\in\mathbb{N}})$ is a left-seminearring.

Generally, the right distributivity does not hold, as demonstrated by the following counterexample.

Example 3. *Let $\tau = (3)$ be a type with a ternary operation symbol $\boxplus$. Assume that σ_3, α_3 and β_3 be weakly fixed variable hypersubstitutions of type (3) which are defined by*

$$\begin{aligned}
\sigma_3(\boxplus) &= \boxplus(x_1, x_3, x_1), \\
\alpha_3(\boxplus) &= \boxplus(x_3, x_1, x_1), \\
\beta_3(\boxplus) &= \boxplus(x_2, x_3, x_3).
\end{aligned}$$

Consider

$$\begin{aligned}
((\sigma_3 +_3 \alpha_3) \circ_3^h \beta_3)(\boxplus) &= \widehat{(\sigma_3 +_3 \alpha_3)}[\boxplus(x_2, x_3, x_3)] \\
&= S_3^3((\sigma_3 +_3 \alpha_3)(\boxplus), x_2, x_3, x_3) \\
&= S_3^3(\boxplus(\boxplus(x_3, x_1, x_1), \boxplus(x_3, x_1, x_1), \boxplus(x_3, x_1, x_1)), x_2, x_3, x_3) \\
&= \boxplus(\boxplus(x_3, x_2, x_2), \boxplus(x_3, x_2, x_2), \boxplus(x_3, x_2, x_2))
\end{aligned}$$

and $((\sigma_3 \circ_3^h \beta_3) +_3 (\alpha_3 \circ_3^h \beta_3))(\boxplus)$

$$= S_3^3((\sigma_3 \circ_3^h \beta_3)(\boxplus), (\alpha_3 \circ_3^h \beta_3)(\boxplus), (\alpha_3 \circ_3^h \beta_3)(\boxplus), (\alpha_3 \circ_3^h \beta_3)(\boxplus)).$$

Since

$$(\sigma_3 \circ_3^h \beta_3)(\boxplus) = (\alpha_3 \circ_3^h \beta_3)(\boxplus) = \boxplus(\boxplus(x_2, x_3, x_3), \boxplus(x_2, x_3, x_3), \boxplus(x_2, x_3, x_3)),$$

we conclude that

$$((\sigma_3 +_3 \alpha_3) \circ_3^h \beta_3)(\boxplus) \neq ((\sigma_3 \circ_3^h \beta_3) +_3 (\alpha_3 \circ_3^h \beta_3))(\boxplus),$$

which means that the right distributivity on $((Hyp_n^{wfv}(\tau))_{n\in\mathbb{N}}, (\circ_n^h)_{n\in\mathbb{N}}, (+_n)_{n\in\mathbb{N}})$ does not hold.

4. Weakly fixed variable hyperidentities

In this section, we apply the concepts of terms of a weakly fixed variable and wfv-hypersubstitutions to describe classes of algebras. We start with the following definition.

Definition 3. *Let V be a variety of algebras of type τ. An identity $s \approx t \in Id(V)$ is said to be a weakly fixed variable identity of V, also called wfv-identity of V, if both s and t come from the set $W_\tau^{wfv}(X_n)$ for some $n \in \mathbb{N}$.*

For example, the identity

$$f(a, b, a) = a$$

in the variety *Reg* of regular semigroups is a weakly fixed variable identity of *Reg*.

For the set $Id_n(V)$ of all n-ary identities of the variety V, we let

$$Id_n^{wfv}(V) := \{s \approx t \mid s \approx t \in Id(V), s, t \in W_\tau^{wfv}(X_n)\}.$$

Alternatively, we say that

$$Id_n^{wfv}(V) = (W_\tau^{wfv}(X_n))^2 \cap Id_n(V).$$

Moreover, we consider

$$Id^{wfv}(V) := (Id_n^{wfv}(V))_{n\in\mathbb{N}}.$$

Then we prove:

Theorem 5. *Let V be a variety of algebras of type τ. Then $Id^{wfv}(V)$ is a congruence on the multisorted algebra $\underline{W_\tau^{wfv}(X)}$.*

Proof. It is clear that $(Id_n^{fv}(V))_{n\in\mathbb{N}^+}$ is preserved by the constant fundamental operations, i.e., projections, of $\underline{W_\tau^{wfv}(X)}$.

Assume now that $p \approx q \in Id_n^{wfv}(V)$ and $p_1 \approx q_1, \ldots, p_n \approx q_n \in Id_m^{wfv}(V)$. We aim to show that

$$S_m^n(p, p_1, \ldots, p_n) \approx S_m^n(q, q_1, \ldots, q_n) \in Id_m^{wfv}(V).$$

From Lemma 1, we have that the terms $S_m^n(p, p_1, \ldots, p_n)$ and $S_m^n(q, q_1, \ldots, q_n)$ contain in the set $W_\tau^{wfv}(X_m)$. Thus

$$S_m^n(p, p_1, \ldots, p_n) \approx S_m^n(q, q_1, \ldots, q_n) \in Id_m(V).$$

As a result,

$$S_m^n(p, p_1, \ldots, p_n) \approx S_m^n(q, q_1, \ldots, q_n) \in Id_m^{wfv}(V).$$

Theorem 5 allows us to consider the quotient systems of the following form:

$$(W_\tau^{wfv}(X_n))_{n\in\mathbb{N}}/(\mathrm{Id}_n^{wfv}(V))_{n\in\mathbb{N}},$$

which we will call the quotient algebra of terms of a weakly fixed variable of a variety V.

Moreover, the multisorted operations on $(W_\tau^{wfv}(X_n))_{n\in\mathbb{N}}/(\mathrm{Id}_n^{wfv}(V))_{n\in\mathbb{N}}$ denoted by

$$\overline{\overline{S}}_m^n : W_\tau^{wfv}(X_n)/\mathrm{Id}_n^{wfv}(V) \times (W_\tau^{wfv}(X_m)/\mathrm{Id}_m^{wfv}(V))^n \to W_\tau^{wfv}(X_m)/\mathrm{Id}_m^{wfv}(V)$$

can be naturally defined by

$$\overline{\overline{S}}_m^n([t]_{\mathrm{Id}_n^{wfv}(V)}, [t_1]_{\mathrm{Id}_m^{wfv}}(V), \ldots, [t_n]_{\mathrm{Id}_m^{wfv}}(V)) = [s]_{\mathrm{Id}_m^{wfv}(V)}.$$

Thus, we denote

$$Q_\tau^{wfv}(V) := ((W_\tau^{wfv}(X_n))_{n\in\mathbb{N}}/(\mathrm{Id}_n^{wfv}(V))_{n\in\mathbb{N}}, (\overline{\overline{S}}_m^n)_{n,m\in\mathbb{N}}).$$

Normally, the natural homomorphism is the multisorted mapping

$$(\mathrm{nat}^{wfv}\mathrm{Id}^{wfv}V_n)_{n\in\mathbb{N}} : (W_\tau^{wfv}(X_n))_{n\in\mathbb{N}} \to (W_\tau^{wfv}(X_n))_{n\in\mathbb{N}}/(\mathrm{Id}_n^{wfv}(V))_{n\in\mathbb{N}}$$

defined by

$$\mathrm{nat}^{wfv}\mathrm{Id}^{wfv}V_n(t) = [t]_{\mathrm{Id}_n^{wfv}(V)}$$

for all $t \in W_\tau^{wfv}(X_n)$. Clearly, $(\mathrm{nat}^{wfv}\mathrm{Id}^{wfv}V_n)_{n\in\mathbb{N}}$ is a homomorphism from $\overline{W_\tau^{wfv}(X)}$ to $Q_\tau^{wfv}(V)$.

In the study of algebra, the concept of hyperidentities represents an extension of identities to a higher level, see [3, 13]. We now discuss identities that involve terms of a weakly fixed variable.

Definition 4. *Let V be a variety of algebras of type τ and let $(Hyp_n^{wfv}(\tau))_{n\in\mathbb{N}}$ be the multisorted semigroup of weakly fixed variable hypersubstitutions of type τ. A weakly fixed variable identity $s \approx t$ in V is said to be a weakly fixed variable hyperidentity in V if*

$$\widehat{\sigma_n}[s] \approx \widehat{\sigma_n}[t] \in Id_n(V)$$

for $s, t \in W_\tau^{wfv}(X_n)$, $\sigma_n \in Hyp_n^{wfv}(\tau)$ and $n \in \mathbb{N}$.

Furthermore, we call a variety V a weakly fixed variable solid variety if

$$\widehat{\sigma_n}[s] \approx \widehat{\sigma_n}[t] \in Id_n(V)$$

for $s, t \in W_\tau^{wfv}(X_n)$, $\sigma_n \in Hyp_n^{wfv}(\tau)$ and $n \in \mathbb{N}$.

From Definition 4, we define

$$HId_n^{wfv}(V) := \{s \approx t \mid s, t \in W_\tau^{wfv}(X_n), \widehat{\sigma_n}[s] \approx \widehat{\sigma_n}[t] \in Id_n(V), \sigma_n \in Hyp_n^{wfv}(\tau)\}.$$

Then $(HId_n^{wfv}(V))_{n\in\mathbb{N}}$ is a multisorted equivalence on $(W_\tau^{wfv}(X_n))_{n\in\mathbb{N}}$.

Theorem 6. *Let V be a variety of type τ. Then $(HId_n^{wfv}(V))_{n\in\mathbb{N}}$ is a congruence on $\overline{W_\tau^{wfv}(X)}$.*

Proof. Let $p \approx q \in HId_n^{wfv}(V)$ and let $p_j \approx q_j \in HId_m^{wfv}(V)$ for $j = 1, \dots, n$. According to the definition of weakly fixed variable hyperidentities in a variety V, we have

$$\widehat{\sigma_n}[p] \approx \widehat{\sigma_n}[q] \in Id_n^{wfv}(V) \text{ and } \widehat{\sigma_m}[p_j] \approx \widehat{\sigma_m}[q_j] \in Id_m^{wfv}(V)$$

for all $j = 1, \dots, n$ and $\sigma \in (Hyp_n^{wfv}(\tau))_{n\in\mathbb{N}}$. Our aim is to show that

$$\widehat{\sigma_m}[S_m^n(p, p_1, \dots, p_n)] \approx \widehat{\sigma_m}[S_m^n(q, q_1, \dots, q_n)] \in Id_m^{wfv}(V)$$

for all $\sigma \in (Hyp_n^{wfv}(\tau))_{n\in\mathbb{N}}$.

For this, we let σ be a weakly fixed variable hypersubstitution on the multisorted semigroup $(Hyp_n^{wfv}(\tau))_{n\in\mathbb{N}}$. From $\hat{\sigma}_n[p] \approx \hat{\sigma}_n[q] \in Id_n^{wfv}(V)$, we have that $\widehat{\sigma_n}[p]$ and $\widehat{\sigma_n}[q]$ are m-ary terms of a weakly fixed variable of type τ and thus $\widehat{\sigma_n}[p] \approx \widehat{\sigma_n}[q]$ is an identity in V, i.e.,

$$\widehat{\sigma_n}[p] \approx \widehat{\sigma_n}[q] \in Id_n(V).$$

Similarly, because for every $j = 1, \dots, n$,

$$\widehat{\sigma_m}[p_j] \approx \widehat{\sigma_m}[q_j] \in Id_m^{wfv}(V),$$

then

$$\widehat{\sigma_m}[p_j] \approx \widehat{\sigma_m}[q_j] \in Id_m(V)$$

for all $j = 1, \dots, n$, and

$$\widehat{\sigma_m}[p_1], \dots, \widehat{\sigma_m}[p_n], \widehat{\sigma_m}[q_1], \dots, \widehat{\sigma_m}[q_n] \in W_\tau^{wfv}(X_m).$$

By the fact that $\widehat{\sigma_n}[p] \approx \widehat{\sigma_n}[q] \in Id_n(V)$ and $\widehat{\sigma_m}[p_j] \approx \widehat{\sigma_m}[q_j] \in Id_m(V)$, we obtain

$$S_m^n(\widehat{\sigma_n}[p], \widehat{\sigma_m}[p_1], \dots, \widehat{\sigma_m}[p_n]) \approx S_m^n(\widehat{\sigma_n}[q], \widehat{\sigma_m}[q_1], \dots, \widehat{\sigma_m}[q_n]) \in Id_m(V)$$

because $(Id_n(V))_{n\in\mathbb{N}}$ is a congruence on the multisorted algebra of terms.

From $\widehat{\sigma_n}[p], \widehat{\sigma_n}[q] \in W_\tau^{wfv}(X_n)$ and $\widehat{\sigma_m}[p_1], \dots, \widehat{\sigma_m}[p_n], \widehat{\sigma_m}[q_1], \dots, \widehat{\sigma_m}[q_n] \in W_\tau^{wfv}(X_m)$, we also conclude that

$$S_m^n(\widehat{\sigma_n}[p], \widehat{\sigma_m}[p_1], \dots, \widehat{\sigma_m}[p_n]) \approx S_m^n(\widehat{\sigma_n}[q], \widehat{\sigma_m}[q_1], \dots, \widehat{\sigma_m}[q_n]) \in W_\tau^{wfv}(X_m).$$

This implies that

$$S_m^n(\widehat{\sigma_n}[p], \widehat{\sigma_m}[p_1], \dots, \widehat{\sigma_m}[p_n]) \approx S_m^n(\widehat{\sigma_n}[q], \widehat{\sigma_m}[q_1], \dots, \widehat{\sigma_m}[q_n]) \in Id_m^{wfv}(V).$$

From the fact that $(\widehat{\sigma_n})_{n\in\mathbb{N}}$ is an endomorphism on the multisorted algebra $\overline{W_\tau^{wfv}(X)}$ proved in Theorem 2, we have

$$\widehat{\sigma_m}[S_m^n(p, p_1, \dots, p_n)] \approx \widehat{\sigma_m}[S_m^n(q, q_1, \dots, q_n)] \in Id_m^{wfv}(V).$$

Therefore, $(HId_n^{wfv}(V))_{n\in\mathbb{N}}$ is a congruence on $\overline{W_\tau^{wfv}(X)}$.

A congruence $(HId_n^{wfv}(V))_{n\in\mathbb{N}}$ on $\overline{W_\tau^{wfv}(X)}$ is said to be fully invariant if it is compatible with all endomorphism $\widehat{\sigma}$ on $\overline{W_\tau^{wfv}(X)}$.

Then we prove the following theorem which gives a necessary condition for any variety V to be weakly fixed variable.

Theorem 7. *Let V be a variety of type τ. If a congruence $(HId_n^{wfv}(V))_{n\in\mathbb{N}}$ is fully invariant, then V is a weakly fixed variable solid variety.*

Proof. Suppose first that a congruence $(HId_n^{wfv}(V))_{n\in\mathbb{N}}$ is fully invariant. On a variety V of type τ, we let $s \approx t \in HId_n^{wfv}(V)$ and $(\sigma_n)_{n\in\mathbb{N}} \in (Hyp_n^{wfv}(\tau))_{n\in\mathbb{N}}$. Applying Theorem 2, we have

$$\widehat{\sigma_n}[s] \approx \widehat{\sigma_n}[t] \in Id_n(V)$$

for all $n \in \mathbb{N}$. Hence, $s \approx t$ is a weakly fixed variable hyperidentity in V, whence, a variety V is weakly fixed variable.

We close this section with the following theorem which gives connection between weakly fixed variable identities and weakly fixed variable hyperidentities in a variety V of algebras of type τ.

Theorem 8. *Every weakly fixed variable identity in a variety V is a weakly fixed variable hyperidentity in V.*

Proof. Suppose first that $s \approx t \in Id^{wfv}(V)$ is an identity in the multisorted quotient algebra $Q_\tau^{wfv}(V)$. Let $\sigma \in Hyp^{wfv}(\tau)$. By Theorem 2 and the property of a natural homomorphism, thus the composition mapping

$$nat_{n,Id^{wfv}(V)} \circ \widehat{\sigma_n} : \overline{W_\tau^{wfv}(X)} \to Q_\tau^{wfv}(V)$$

is a homomorphism. By the hypothesis, we obtain

$$nat_{n,Id^{vf}(V)} \circ \hat{\sigma}_n(s) \ = \ nat_{n,Id^{vf}(V)} \circ \hat{\sigma}_n(t).$$

That is,

$$nat_{n,Id^{wfv}(V)}(\widehat{\sigma_n}[s]) \ = \ nat_{n,Id^{wfv}(V)}(\widehat{\sigma_n}[t]).$$

Again by a natural homomorphism $nat_{n,Id^{wfv}(V)}$, we also get

$$[\widehat{\sigma_n}[s]]_{Id^{wfv}(V)} \ = \ [\widehat{\sigma_n}[t]]_{Id^{wfv}(V)},$$

which means that

$$\widehat{\sigma_n}[s] \approx \widehat{\sigma_n}[t] \in Id^{wfv}(V).$$

Thus, $s \approx t$ is a weakly fixed variable hyperidentity in V.

5. Concluding remarks

As a generalization of terms of fixed variable introduced in [17], this work presents an extension called terms of a weakly fixed variable by relaxing the conditions on inductive construction of any term t from an alphabet X_n. Several multibased structures for such terms are developed, including the superassociative system with respect to the multisorted superposition operation, the seminearring of mappings which images are terms of a weakly fixed variable, the quotient multisorted set obtained by dividing the set of all identities induced by terms of a weakly fixed variable, which acts as a congruence. Furthermore, applications of terms of a weakly fixed variable for classifying a variety V of algebras of type τ are discussed.

Another direction of the future research in this domain should be devoted to characterize the set of idempotent and regular elements on the semigroup $(Hyp_n^{wfv}(\tau))_{n\in\mathbb{N}}$.

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