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On Filters of Implicative Negatively Partially Ordered Ternary Semigroups

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Abstract. In this paper, we study a special set in an implicative n.p.o.(negatively partially ordered) ternary semigroup, and prove that a filter can be represented by the union of such sets. Indeed, let $(T, [\ ], \leq, [\ ]^*)$ be an implicative n.p.o. ternary semigroup. For any $a, b \in T$, we define

$$S(a, b) := \{c \in T\ :\ [aa[bbc]^*]^* = 1\}.$$

We have the following:

(1) A non-empty subset F of T is a filter if and only if it satisfies the following conditions:

- (F3) $1 \in F$;
- (F4) for any $a, b, c \in T$, if $[abc]^* \in F$ and $a, b \in F$, then $c \in F$.

(2) If T is commutative and F is a filter of T, then

$$F = \bigcup_{a,b\in F} S(a, b).$$

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Key Words and Phrases: Implicative negatively partially ordered ternary semigroup (INPOTS), filter, left self-distributive

1. Introduction

Implicative negatively partially ordered semigroups and filters were introduced and studied in [3] by Chan and Shum. The implicative negatively partially ordered semigroup is a generalization of the implicative semilattice (cf. [2], [8]), it is closed to implications in mathematical logic (cf. [1], [4]). As demonstrated in [8], filters play a crucial role in implicative semilattice theory. Quotient structures of implicative negatively partially

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ordered semigroups through filters were constructed in [3]. Additionally, in [6], filters within commutative implicative negatively partially ordered semigroups were examined. In [5], the author introduced a set in an implicative negatively partially ordered semigroup, and gave an equivalent condition of a filter. Moreover, it is obtained that a filter is the union of that special sets.

In this paper, we follow these concepts to derive a special set in an implicative negatively partially ordered ternary semigroup, and prove that a filter can be represented by the union of such sets. Also, some important results are investigated.

2. Preliminaries

We collects results obtained in negatively partially ordered ternary semigroups.

Definition 1. *[7] A system $(T, [\ ], \leq)$ is called a NPOTS (negatively partially ordered ternary semigroup) if*

(1) $(T, [\ ])$ is a ternary semigroup;

(2) a partially order $\leq$ on T is compatible with $[\ ]$;

(3) $\forall a, b, c \in T, [abc] \leq a,\ [abc] \leq b,\ [abc] \leq c$.

Definition 2. *[7] A NPOTS $(T, [\ ], \leq)$ is called an INPOTS (implicative negatively partially ordered ternary semigroup) if there is an additional ternary multiplication $[\ ]^*$ on T such that for all $a, b, c, u \in T$,*

$$u \leq [cbc]^* \Leftrightarrow [uab] \leq c.$$

Here, $[\ ]^*$ is a ternary implication.

A multiplicative identity of a ternary semigroup $(T, [\ ])$ is an element 1 of T satisfying the condition $[1a1] = [11a] = [a11] = a$ for any $a \in T$.

Example 1. *Consider the INPOTS $(T, [\ ], \leq, [\ ]^*)$ defined as follows:*

$[\ ]$	1	a	0
11	1	0	0
$1a$	0	0	0
10	0	0	0

$[\ ]$	1	a	0
aa	0	0	0
$a1$	0	0	0
$a0$	0	0	0

$[\ ]$	1	a	0
00	0	0	0
01	0	0	0
$0a$	0	0	0

$[\ ]^*$	1	a	0
11	1	a	a
$1a$	1	1	1
10	1	1	1

$[\ ]^*$	1	a	0
aa	1	1	1
$a1$	1	1	1
$a0$	1	1	1

$[\ ]^*$	1	a	0
00	1	1	1
01	1	1	1
$0a$	1	1	1

and

$$\leq = \{(0,0),(1,1),(a,a),(a,1),(0,a),(0,1)\}.$$

We place x_1x_2 in the first column and x_3 in the first row to express the calculation $[x_1x_2x_3]$ using a multiplication table. Observed that the greatest element 1 is not identity since $[1a1] = 0 \neq a$.

The following shows that not every NPOTS with identity admits the INPOTS.

Example 2. *Let us consider a NPOTS $(T, [\ ], \leq)$ defined as follows:*

[ ]	1	2	3	4	6
11	1	2	3	4	6
12	2	2	6	4	6
13	3	3	3	6	6
14	4	4	6	4	6
16	6	6	6	6	6

[ ]	1	2	3	4	6
21	2	2	6	4	6
22	2	2	6	4	6
23	6	6	6	6	6
24	4	4	6	4	6
26	6	6	6	6	6

[ ]	1	2	3	4	6
31	3	6	3	6	6
32	6	6	6	6	6
33	3	6	3	6	6
34	6	6	6	6	6
36	6	6	6	6	6

[ ]	1	2	3	4	6
41	4	4	6	4	6
42	4	4	6	4	6
43	6	6	6	6	6
44	4	4	6	4	6
46	6	6	6	6	6

[ ]	1	2	3	4	6
61	6	6	6	6	6
62	6	6	6	6	6
63	6	6	6	6	6
64	6	6	6	6	6
66	6	6	6	6	6

and

$$\leq = \{(1,1),(2,2),(3,3),(4,4),(6,6),(3,1),(2,1),$$
$$(4,a),(4,1),(6,4),(6,2),(6,3),(6,1)\}.$$

The element 1 is the greatest element. Suppose T is an INPOTS with ternary implication $[\ ]^*$. Clearly, $[322] = 6 \leq 4$ and $[422] = 2 \leq 2$. Then $3 \leq [224]^*$ and $4 \leq [224]^*$, so $[224]^* = 1$. As $1 \leq [224]^*$, we have $2 = [122] \leq 4$. This is a contradiction. Hence, T is not an INPOTS.

Definition 3. *[9] An INPOTS $(T, [\ ], \leq, [\ ]^*)$ is called commutative if*

$$[a_1a_2a_3] = [a_{\alpha(1)}a_{\alpha(2)}a_{\alpha(3)}]$$

for any permutation $\alpha \in S_3$.

The example shows an infinite commutative INPOTS.

Example 3. *[7] Let $\mathbb{Z}^+$ be a TS such that $[abc] = abc$ for all $a, b, c \in \mathbb{Z}^+$. Consider*

$$\leq = \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : b \mid a\}.$$

Here, $b \mid a$ means b divides a. Then $(\mathbb{Z}^+, [\ ], \leq)$ is a commutative NPOTS; 1 is the greatest element. Define

$$[abc]^* = \frac{c}{\gcd(ab, c)}$$

for all $a, b, c \in \mathbb{Z}^+$. Then $(\mathbb{Z}^+, [\ ], \leq, [\ ]^*)$ is a commutative INPOTS.

Theorem 1. *[7] Let $(T, [\ ], \leq, [\ ]^*)$ be an INPOTS. Then, for $a, b \in T$,*

(1) $a \leq [aaa]^*$;

(2) $[aaa]^* = [bbb]^*$;

(3) $[aaa]^*$ *is the greatest element of T; then an INPOTS always contains the greatest element.*

Let 1 be the greatest element of a NPOTS $(T, [\ ], \leq)$ if exists. Assume 1 is the multiplicative identity. Then, for any $u, v, w \in T$, $[uvw] = 1 \Leftrightarrow u = 1,\ v = 1,\ w = 1$.

Throughout this paper, we assume 1 is both the multiplicative identity and the greatest element of an INPOTS.

Theorem 2. *[7] Let $(T, [\ ], \leq, [\ ]^*)$ be an INPOTS. Then, for $a, b, c, u, v \in T$,*

(1) $a \leq 1$, $[aaa]^* = 1$, $a = [11a]^*$;

(2) $a \leq [bc[abc]]^*$;

(3) $a \leq [aa[aaa]]^*$;

(4) $a \leq [bca]^*$;

(5) *if* $a \leq b$, *then* $[buv]^* \leq [auv]^*$ *and* $[uva]^* \leq [uvb]^*$;

(6) $a \leq b \Leftrightarrow [a1b]^* = 1 \Leftrightarrow [1ab]^* = 1$;

(7) $[ab[cuv]^*]^* = [[abc]uv]^* = [a[bcu]v]^*$.

3. Filters of implicative negatively partially ordered ternary semigroups

We begin with filters of an INPOTS.

Definition 4. *[7] Let $(T, [\ ], \leq, [\ ]^*)$ be an INPOTS. Then $\emptyset \neq F \subseteq T$ is called a filter of T if*

(F1) $[abc] \in F$ *for any* $a, b, c \in F$*;*

(F2) for any $x, y \in T$*,* $a \leq b$ *and* $a \in F$ *imply* $b \in F$*.*

Proposition 1. *Let* $(T, [\ ], \leq, [\ ]^*)$ *be an INPOTS. Then* $\emptyset \neq F \subseteq T$ *is a filter if and only if it holds the conditions:*

(F3) $1 \in F$*;*

(F4) for any $a, b, c \in T$*, if* $[abc]^* \in F$ *and* $a, b \in F$*, then* $c \in F$*.*

Proof. Assume that F is a filter of T. Since 1 is the greatest element of T, $1 \in F$. It is observed that for any $a, b, c \in T$, from $[abc]^* \leq [abc]^*$, we have

$$[[abc]^*ab] \leq c. \tag{2.1}$$

Let $a, b, c \in T$ be such that $[abc]^* \in F$ and $a, b \in F$. By assumption, we have $[[abc]^*ab] \in F$. Using (2.1), we get $[[abc]^*ab] \leq c$. This implies that $c \in F$. Hence, F satisfies (F3) and (F4).

Conversely, assume that F satisfies (F3) and (F4). If $a, b \in T$ such that $a \leq b$ and $a \in F$, then by Theorem 2 (6) we have $[1ab]^* = 1 \in F$. By (F4), $b \in F$. Thus, F satisfies (F2). Let $a, b, c \in F$. By Theorem 2 (2), $a \leq [bc[abc]]^*$, and so by (F2) we get $[bc[abc]]^* \in F$. From (F4), $[abc] \in F$. Hence, F satisfies (F1). Consequently, F is a filter of T.

Definition 5. *Let* $(T, [\ ], \leq, [\ ]^*)$ *be an INPOTS. For any* $a, b \in T$*, define*

$$S(a, b) := \{c \in T \,:\, [aa[bbc]^*]^* = 1\}.$$

Observe that $1, b \in S(a, b)$ for any $a, b \in T$.

Proposition 2. *For a commutative INPOTS* $(T, [\ ], \leq, [\ ]^*)$*,* $a \in S(a, b)$ *for all* $a, b \in T$*.*

Proof. Let $a, b \in T$. By Theorem 2 (7),

$$\begin{aligned}
[aa[bba]^*]^* &= [a[abb]a]^* \\
&= [a[bba]a]^* \\
&= [ab[baa]^*]^* \\
&= [[abb]aa]^* \\
&= [[bba]aa]^* \\
&= [bb[aaa]^*]^* \\
&= [bb1]^*.
\end{aligned}$$

From Theorem 2 (4), $1 \leq [bb1]^* \leq 1$. This implies that $[aa[bba]^*]^* = 1$, and so $a \in S(a, b)$.

Proposition 3. *Let $(T, [\ ], \leq, [\ ]^*)$ be an INPOTS, and $b \in T$. If $[buv]^* = 1$ for all $u, v \in T$, then $S(a, b) = T = S(b, a)$ for all $a \in T$.*

Proof. Assume that $[buv]^* = 1$ for all $u, v \in T$ and $a \in T$. Clearly, $S(a, b) \subseteq T$ and $S(b, a) \subseteq T$. By assumption, we have $[aa[bba]^*]^* = [aa1]^*$. Since $1 \leq [aa1]^* \leq 1$, we have $[aa[bba]^*]^* = 1$, and then $a \in S(a, b)$. Thus, $T \subseteq S(a, b)$. By assumption, $[bb[aaa]^*]^* = [bb1]^* = 1$. This shows that $a \in S(b, a)$, and so $T \subseteq S(b, a)$.

Example 4. *Let us consider the INPOTS $(T, [\ ], \leq, [\ ]^*)$ defined as follows:*

[ ]	1	2	3	4	5	7
11	1	2	3	4	5	7
12	2	3	3	5	7	7
13	3	3	3	7	7	7
14	4	5	7	4	5	7
15	5	7	7	5	7	7
17	7	7	7	7	7	7

[ ]	1	2	3	4	5	7
21	2	3	3	5	7	7
22	3	3	3	7	7	7
23	3	3	3	7	7	7
24	5	7	7	5	7	7
25	7	7	7	7	7	7
27	7	7	7	7	7	7

[ ]	1	2	3	4	5	7
31	3	3	3	7	7	7
$3a$	3	3	3	7	7	7
33	3	3	3	7	7	7
34	7	7	7	7	7	7
35	7	7	7	7	7	7
37	7	7	7	7	7	7

[ ]	1	2	3	4	5	7
41	4	5	7	4	5	7
42	5	7	7	5	7	7
43	7	7	7	7	7	7
44	4	5	7	4	5	7
45	5	7	7	5	7	7
47	7	7	7	7	7	7

[ ]	1	2	3	4	5	7
51	5	7	7	5	7	7
52	7	7	7	7	7	7
53	7	7	7	7	7	7
54	5	7	7	5	7	7
55	7	7	7	7	7	7
57	7	7	7	7	7	7

[ ]	1	2	3	4	5	7
71	7	7	7	7	7	7
72	7	7	7	7	7	7
73	7	7	7	7	7	7
74	7	7	7	7	7	7
75	7	7	7	7	7	7
77	7	7	7	7	7	7

$[\ ]^*$	1	2	3	4	5	7
11	1	2	3	4	5	7
12	1	1	2	4	4	5
13	1	1	1	4	4	4
14	1	2	3	1	2	3
15	1	1	2	1	1	2
17	1	1	1	1	1	1

$[\ ]^*$	1	2	3	4	5	7
21	1	1	2	4	4	5
22	1	1	1	4	4	4
23	1	1	1	4	4	4
24	1	1	3	1	1	2
25	1	1	1	1	1	1
27	1	1	1	1	1	1

$[\ ]^*$	1	2	3	4	5	7
31	1	1	1	4	4	4
$3a$	1	1	1	4	4	4
33	1	1	1	4	4	4
34	1	1	1	1	1	1
35	1	1	1	1	1	1
37	1	1	1	1	1	1

$[\ ]^*$	1	2	3	4	5	7
41	1	2	3	1	2	3
42	1	1	2	1	1	2
43	1	1	1	1	1	1
44	1	2	3	1	2	3
45	1	1	2	1	1	2
47	1	1	1	1	1	1

$[\ ]^*$	1	2	3	4	5	7
51	1	1	2	1	1	2
52	1	1	1	1	1	1
53	1	1	1	1	1	1
54	1	1	2	1	1	2
55	1	1	1	1	1	1
57	1	1	1	1	1	1

$[\ ]^*$	1	2	3	4	5	7
71	1	1	1	1	1	1
72	1	1	1	1	1	1
73	1	1	1	1	1	1
74	1	1	1	1	1	1
75	1	1	1	1	1	1
77	1	1	1	1	1	1

and

$$\begin{aligned} \leq = \{&(1,1),(2,2),(3,3),(4,4),(5,5),(7,7),(3,1),(3,2),(2,1),\\ &(4,1),(5,4),(5,1),(5,2)(7,1),(7,2),(7,3),(7,4),(7,5)\}. \end{aligned}$$

By Proposition 3, we have $S(a,7)=S(7,a)=T$ for all $a\in T$. Furthermore, we have $S(1,1)=\{1\}$, $S(1,2)=S(2,1)=S(1,3)=S(3,1)=S(2,3)=S(3,2)=S(2,2)=S(3,3)=\{1,2,3\}$, $S(1,4)=S(4,1)=S(4,4)=\{1,4\}$, and $S(2,4)=S(4,2)=S(3,4)=S(4,3)=S(1,5)=S(2,5)=S(3,5)=S(4,5)=S(5,5)=S(5,1)=S(5,2)=S(5,3)=S(5,4)=T$. We observe that for any $a,b\in T$, $S(a,b)$ is a filter of T.

Theorem 3. *Let $(T,[\ ],\leq,[\ ]^*)$ be a commutative INPOTS. If F is a filter, then $S(a,b)\subseteq F$ for all $a,b\in F$.*

Proof. Let F be a filter of T and let $a,b\in F$. If $c\in S(a,b)$, then $[aa[bbc]^*]^*=1\in F$, and by (F4) we have $c\in F$.

Theorem 4. *Let $(T,[\ ],\leq,[\ ]^*)$ be a commutative INPOTS. If F is a filter of T, then*

$$F=\bigcup_{a,b\in F}S(a,b).$$

Proof. Let F be a filter of T. By Proposition 2, $c\in S(c,1)$ for any $c\in F$. Then

$$F\subseteq\bigcup_{c\in F}S(c,1)\subseteq\bigcup_{a,b\in F}S(a,b).$$

For the reverse inclusion, let $c'\in\bigcup_{a,b\in F}S(a,b)$. Then there exist $x,y\in F$ such that $c'\in S(x,y)$. By Theorem 3, $c'\in F$. This shows that $\bigcup_{a,b\in F}S(a,b)\subseteq F$.

Corollary 1. *Let* $(T, [\ ], \leq, [\ ]^*)$ *be a commutative INPOTS. If* F *is a filter of* T*, then*

$$F = \bigcup_{a \in F} S(a, 1).$$

4. Conclusions

In this paper, we introduce the concept of filters in implicative negatively partially ordered ternary semigroups (Definition 4) and give a characterization of filters (Proposition 1). Then we consider the set $S(a, b) := \{c \in T : [aa[bbc]^*]^* = 1\}$ where a, b are elements of an implicative negatively partially ordered ternary semigroup $(T, [\ ], \leq, [\ ]^*)$. The main result obtained is that any filter can be represented by the union of such sets (Theorem 4), if $(T, [\ ], \leq, [\ ]^*)$ is commutative.

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