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Supra Soft Somewhat Open Sets: Characterizations and Continuity

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Abstract. In this manuscript, we used the supra soft interior operator to define a new approach of generalized sets named, supra soft somewhat (briefly, SS-sw-) open sets. We discuss its relationships with the other generalizations and provide the necessary examples and counterexamples. After that, we define new continuity inspired by this new approach, named SS-sw-continuous function. We characterize several of its essential properties. We use the SS-sw-closure (interior) operators to present several equivalent conditions for the new approach. Furthermore, we define a new type of functions related to SS-sw-open sets, named SS-sw-open functions.

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1. Introduction

In light of the broadest crisp (fuzzy) sets, Molodtsov [1] in 1999, outlined the concept of soft sets. Maji et al. [2], introduced more operations to soft theory. In 2001, Ahmad and Kharal [3], defined the concept of soft continuity.

Shabir and Naz [4] defined the notions of soft topological space (STS, for short), which investigated by Aygunoüglu and Aygün in [5]. In 2012, Zorlutuna et al. [6] presented

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more properties to STS. Later, several types of broader soft open sets and generalized soft continuity are explored in [7–11].

It was first explained in [12] what soft ideal and soft local functions are. After then, the soft semi-local functions approach was defined by several authors [13, 14] by using the definition of soft semi-open sets. To define new soft ideal rough topological spaces, Abd El-latif [15] employed the soft ideal for this purpose.

By utilizing the soft ideal notion, numerous soft open weaker classes have been expanded in [16–21]. Subsequently, new approaches based on soft ideals were presented for the soft separation axioms [22, 23], soft connectedness [24], and soft semi-compactness [25].

El-Sheikh et al. [26] presented the definition of supra soft topological space (SSTS, for short) in 2014. They also introduced many types of wider soft sets and soft continuity in SSTS. Later, several valuable papers have been presented related to SS-locally closed sets [27], SS-b-open sets [28], SS-δ_i-open sets [29, 30], SS-(strongly) generalized closed sets [31, 32], SS-separation axioms [33, 34], SS-regular open sets [35], the Baire categories of soft sets [36] and SS-sd-sets [37]. Recently, several soft topological spaces are introduced to SSTS in [39–43].

Ameen et al. [44], defined the notion of soft somewhat open (sw-open) sets. In this regards, Al-shami [45] applied this class to medical application. Also, he and others [46] used this notion to defined new categories of connectedness and compactness.

The supra soft interior operator was utilized in this manuscript to construct a novel generalized set approach known as SS-sw-open sets. We examined the key features of this new approach. The relevant examples and counterexamples are given, and its connections with the other generalizations are discussed. In addition, we applied this novel concept for soft continuity. Furthermore, we presented a number of analogous conditions for our novel methods using the SS-sw-closure (interior) operators.

2. Preliminaries

Definition 1. *[1] Let χ be the initial universe set and η be the set of parameters. Then, a pair (K, η) is called a soft set, which is defined by*

$$K_\eta = \{K(\vartheta) : \vartheta \in \eta,\ K : \eta \to P(\chi)\}.$$

the category of all soft sets will be represented by $S(\chi)_\eta$. Also, the absolute (null) soft set will represented by $\tilde{\chi}$ ($\tilde{\varphi}$), where $\tilde{\chi}(\vartheta) = \chi$ and $\tilde{\varphi}(\vartheta) = \varphi$, for all $\vartheta \in \eta$.

Definition 2. *[4] The class $\sigma \subseteq S(\chi)_\eta$ is called a soft topology on χ if σ contains $\tilde{\chi}, \tilde{\varphi}$ and closed under finite soft intersection and arbitrary soft union.*

The triplet (χ, σ, η) is referred to as an STS over χ. Also, for any soft set (G, η), if $(G, \eta) \in \sigma$, then (G, η) is called soft open set and its soft complements $(G^{\tilde{c}}, \eta)$ is called soft closed set.

Definition 3. *[4, 6] Let (χ, σ, η) be an STS and $(K, \eta) \in S(\chi)_\eta$, then*

(1) $int(K,\eta) = \sqcup\{(O,\eta) : (O,\eta) \in \sigma$ *and* $(O,\eta)\tilde{\subseteq}(K,\eta)\}$.

(2) $cl(K,\eta) = \sqcap\{(H,\eta) : (H,\eta) \in \sigma^c$ *and* $(K,\eta)\tilde{\subseteq}(H,\eta)\}$.

Definition 4. *[3] Let* $\psi_{sw} : (\chi_1, \sigma_1, \eta_1) \rightarrow (\chi_2, \sigma_2, \eta_2)$ *be a function, where* $s : \chi_1 \rightarrow \chi_2$ *and* $w : \eta_1 \rightarrow \eta_2$*. Then*

(1) *The image of* (K, η_1) *under* ψ_{sw}*, represented by* $\psi_{sw}(K, \eta_1) = (\psi_{sw}(K), s(\eta_1))$*, is a soft set in* $S(\chi_2)_{\eta_2}$ *such that*
$$\psi_{sw}(K)(\theta) = \begin{cases} \sqcup_{\theta \in s^{-1}(\theta)\sqcap\eta_1} \; w(K(\vartheta)), & s^{-1}(\theta) \sqcap K \neq \varphi, \\ \varphi, & otherwise. \end{cases}$$
for all $\theta \in \eta_2$.

(2) *The pre-image of* (H, η_2) *under* ψ_{sw}*, represented by* $\psi_{sw}^{-1}(H, \eta_2) = (\psi_{sw}^{-1}(H), s^{-1}(\eta_2))$*, is a soft set in* $S(\chi_1)_{\eta_1}$ *such that*
$$\psi_{sw}^{-1}(H)(\vartheta) = \begin{cases} w^{-1}(H(s(\vartheta))), & s(\vartheta) \in \eta_2, \\ \varphi, & otherwise. \end{cases}$$
for all $\vartheta \in \eta_1$.

If s *and* w *are surjective (injective) together, then* ψ_{sw} *is surjective (injective).*

Theorem 5. *[3] For the soft function* $\psi_{sw} : (\chi_1, \sigma_1, \eta_1) \rightarrow (\chi_2, \sigma_2, \eta_2)$*, the following statements hold.*

(1) $\psi_{sw}^{-1}((N^{\tilde{c}}, \eta_2)) = (\psi_{sw}^{-1}(N, \eta_2))^{\tilde{c}}\ \forall\ (N, \eta_2) \in S(\chi_2)_{\eta_2}$.

(2) $\psi_{sw}(\psi_{sw}^{-1}((N, \eta_2)))\tilde{\subseteq}(N, \eta_2)\ \forall\ (N, \eta_2) \in S(\chi_2)_{\eta_2}$.

(3) $(M, \eta_1)\tilde{\subseteq}\psi_{sw}^{-1}(\psi_{sw}((M, \eta_1)))\ \forall\ (M, \eta_1) \in S(\chi_1)_{\eta_1}$.

(4) $\psi_{sw}(\tilde{\chi_1})\tilde{\subseteq}\tilde{\chi_2}$.

Definition 6. *[26] The collection* $\rho \subseteq S(\chi)_\eta$ *is called a supra soft topology (or SSTS) on* χ *if it contains* $\tilde{\chi}, \tilde{\varphi}$ *and closed under arbitrary soft union.*
For any soft set (G, η)*, if* $(G, \eta) \in \rho$*, then* (G, η) *is called supra soft open (shortly, SS-open) set or and its soft complements* $(G^{\tilde{c}}, \eta)$ *is called SS-closed. Also, if* $\sigma \subset \rho$*, then* ρ *is called an SSTS associated with* σ*.*

Definition 7. *[26]* (χ, ρ, η) *be an SSTS and* $(K, \eta) \in S(\chi)_\eta$*, then the SS-interior (closure), denoted by* $int^s(K, \eta)$ $(cl^s(K, \eta))$ *where:*

(1) $int^s(K, \eta) = \sqcup\{(O, \eta) : (O, \eta) \in \rho$ *and* $(O, \eta)\tilde{\subseteq}(K, \eta)\}$.

(2) $cl^s(K, \eta) = \sqcap\{(H, \eta) : (H, \eta) \in \rho^c$ *and* $(K, \eta)\tilde{\subseteq}(H, \eta)\}$.

Definition 8. *[26] If* $\psi_{sw}^{-1}(G, \eta_2) \in \rho_1\ \forall\ (G, \eta_2) \in \sigma_2$*, then the soft function* $\psi_{sw} : (\chi_1, \sigma_1, \eta_1) \rightarrow (\chi_2, \sigma_2, \eta_2)$ *with* ρ_1 *as an associated SSTS with* σ_1 *will called SS-continuous.*

Definition 9. *[26, 37, 38] A soft subset* (G, η) *of an SSTS* (χ, ρ, η) *is called*

(1) *SS-semi-open set if* $(G,\eta)\tilde{\subseteq}cl^s(int^s(G,\eta))$.

(2) *SS-*β*-open set if* $(G,\eta)\tilde{\subseteq}cl^s(int^s(cl^s(G,\eta)))$.

(3) *SS-*α*-open set if* $(G,\eta)\tilde{\subseteq}int^s(cl^s(int^s(G,\eta)))$.

(4) *SS-dense set if* $cl^s(G,\eta)=\tilde{\chi}$.

(5) *SS-co-dense set if* $int^s(G,\eta)=\tilde{\varphi}$.

(6) *SS-regular open set if* $int^s(cl^s(G,\eta))=(G,\eta)$.

(7) *SS-sd-set if there is* $\tilde{\varphi}\neq(O,\eta)\in\rho$ *such that* $(O,\eta)\tilde{\subseteq}cl^s[(O,\eta)\tilde{\sqcap}(K,\eta)]$.

The categories of SS-semi-open (respectively, β*-open,* α*-open, regular-open, sd-) sets shall be indicated by* $SOS^s(\chi)_\eta$ *(respectively,* $\beta OS^s(\chi)_\eta$, $\alpha OS^s(\chi)_\eta$, $ROS^s(\chi)_\eta$, $SD^s(\chi)_\eta$*).*

Definition 10. *[26, 37, 38] A soft function* $\psi_{sw}:(\chi_1,\sigma_1,\eta_1)\rightarrow(\chi_2,\sigma_2,\eta_2)$ *with* ρ_1 *as an associated SSTS with* σ_1 *is referred to as*

(1) *SS-semi-cts if* $\psi_{sw}^{-1}(G,\eta_2)\in SOS^s(\chi_1)_{\eta_1}\ \forall\ (G,\eta_2)\in\sigma_2$.

(2) *SS-*β*-cts if* $\psi_{sw}^{-1}(G,\eta_2)\in\beta OS^s(\chi_1)_{\eta_1}\ \forall\ (G,\eta_2)\in\sigma_2$.

(3) *SS-*α*-cts* $\psi_{sw}^{-1}(G,\eta_2)\in\alpha OS^s(\chi_1)_{\eta_1}\ \forall\ (G,\eta_2)\in\sigma_2$.

(4) *SS-regular cts if* $\psi_{sw}^{-1}(G,\eta_2)\in ROS^s(\chi_1)_{\eta_1}\ \forall\ (G,\eta_2)\in\sigma_2$.

(5) *SS-sd-cts if* $\psi_{sw}^{-1}(G,\eta_2)\in SD^s(\chi_1)_{\eta_1}\ \forall\ (G,\eta_2)\in\sigma_2$.

Theorem 11. *[26] A soft subset* (G,η) *of an SSTS* (χ,ρ,η) *is SS-semi-open set if and only if* $cl^s(G,\eta)=cl^s(int^s(G,\eta))$.

3. Supra soft sw-open sets and relationships

In this section, we present a new generalization of soft open sets in SSTS named SS-sw-open sets. The relationships with other different types of SS-open sets are discussed. With the confirmations of the counterexamples, we show that, this new class forms an SSTS and fail to form an STS.

Definition 12. *A soft subset* (K,η) *of an SSTS* (χ,ρ,η) *is said to be SS-sw-open set if it is null or its SS-interior points is non-null.*
The soft complement of an SS-sw-open set is called SS-sw-closed. The class of all SS-sw-open (respectively, SS-sw-closed) sets will denoted by $SWO^s(\chi)_\eta$ *(respectively,* $SWC^s(\chi)_\eta$*).*

Proposition 13. *For an SSTS* (χ,ρ,η) *we have that:*

(1) $(K,\eta)\in S(\chi)_\eta$ *is SS-sw-closed if it is the absolute soft set or it is not SS-dense set.*

(2) *A non-null soft set (K,η) is SS-sw-open if and only if there is $\tilde{\varphi} \neq (O,\eta) \in \rho$ such that $(O,\eta)\tilde{\sqsubseteq}(K,\eta)$.*

(3) *A proper soft set (K,η) is SS-sw-closed if and only if there is $\tilde{\chi} \neq (C,\eta) \in \rho^c$ such that $(K,\eta)\tilde{\sqsubseteq}(C,\eta)$.*

Proof. Obvious from Definition 12.

Corollary 14. *A non-null soft subset (H,η) of an SSTS (χ,ρ,η) is SS-sw-open if and only if it is a neighborhood for each soft point in $\tilde{\chi}$.*

Proof. It is follows from Proposition 13.

Proposition 15. *Every soft superset (subset) of an SS-sw-open (SS-sw-closed) set is SS-sw-open (SS-sw-closed).*

Proof. It is immediately from Definition 12.

Remark 16. *The next example will confirm that, in general the above proposition is not conversely.*

Example 17. *Assume that $\chi = \{x_1,x_2,x_3,x_4\}$. Let $\eta = \{\vartheta_1,\vartheta_2\}$ be the set of parameters. Let $(J_i,\eta), i = 1,2,...,5$, be soft sets over χ, where*
$J_1(\vartheta_1) = \{x_1,x_2\}, \quad J_1(\vartheta_2) = \{x_1,x_3,x_4\},$
$J_2(\vartheta_1) = \{x_1\}, \quad J_2(\vartheta_2) = \varphi,$
$J_3(\vartheta_1) = \{x_1,x_2\}, \quad J_3(\vartheta_2) = \{x_3,x_4\},$
$J_4(\vartheta_1) = \{x_3,x_4\}, \quad J_4(\vartheta_2) = \{x_1,x_2\},$
$J_5(\vartheta_1) = \chi, \quad J_5(\vartheta_2) = \{x_1,x_2\}.$
Then, $\rho = \{\tilde{\chi},\tilde{\varphi},(J_i,\eta), i = 1,2,...,5\}$ defines an SSTS on U. Hence, the soft set (J_4,η), is an SS-sw-open set, since
$int^s(J_4,\eta)) \neq \tilde{\varphi}$. However, we have that $(B,\eta)\tilde{\subseteq}(J_4,\eta)$, where
$B(\vartheta_1) = \{x_3,x_4\}, \quad B(\vartheta_2) = \varphi,$
is not SS-sw-open set, since $int^s(B,\eta)) = \tilde{\varphi}$.
Also, for the soft sets (J_3,η), (T,η), where
$T(\eta_1) = \{x_1,x_2,x_3\}, \quad T(\eta_2) = \{x_1,x_3,x_4\}.$
We have that $(J_3,\eta)\tilde{\subseteq}(T,\eta)$, and (J_3,η) is an SS-sw-closed set whereas (T,η) is not SS-sw-closed.

Lemma 18. *A soft subset (J,η) of an SSTS (χ,ρ,η) is SS-sw-open set if and only if $int^s(J,\eta)$ is SS-sw-open set.*

Proof. It is immediately from Definition 12.

Lemma 19. *If $(T,\eta)\tilde{\cap}(S,\eta) = \tilde{\varphi}$ for some $(T,\eta) \in SWO^s(\chi)_\eta$ and $(S,\eta) \in S(\chi)_\eta$, then $(S,\eta) \in SWC^s(\chi)_\eta$.*

Proof. Assume that, $(T,\eta)\tilde{\cap}(S,\eta) = \tilde{\varphi}$ such that $(T,\eta) \in SWO^s(\chi)_\eta$ and $(S,\eta) \in S(\chi)_\eta$. Then,

$$(S,\eta)\tilde{\subseteq}(T^{\tilde{c}},\eta),\ (T^{\tilde{c}},\eta)\in SWC^{s}(\chi)_{\eta}.$$

Given Proposition 15, $(S,\eta)\in SWC^{s}(\chi)_{\eta}$.

Theorem 20. *If $\{(G_\imath,\eta),\imath\in \mathrm{I}\}$ is a family of SS-sw-open subsets of an SSTS (χ,ρ,η), then*

(1) $\tilde{\bigcup}_{\imath\in \mathrm{I}}(G_\imath,\eta)\in SWO^{s}(\chi)_{\eta}$.

(2) $\tilde{\bigcap}_{\imath\in \mathrm{I}}(G_\imath^{\tilde{c}},\eta)\in SWC^{s}(\chi)_{\eta}$.

Proof.

(1) Let $\{(G_\imath,\eta),\imath\in I\}$ is a family of SS-sw-open sets. Then, $int^{s}(G_\imath,\eta)\neq\tilde{\varphi}$ for each $\imath\in I$. Hence, $\tilde{\varphi}\neq\tilde{\bigcup}_{\imath\in I}int^{s}(G_\imath,\eta)\tilde{\subseteq}int^{s}[\tilde{\bigcup}_{\imath\in I}(G_\imath,\eta)]$. Therefore, $\tilde{\bigcup}_{\imath\in I}(G_\imath,\eta)\in SWO^{s}(\chi)_{\eta}$.

(2) It is clear from (1).

Remark 21. *If $\{(W_\imath,\eta),\imath=1,2,....,n\}$ is a finite family of SS-sw-open subsets of an SSTS (χ,ρ,η), then $\tilde{\bigcap}_{\imath=1}^{n}(W_\imath,\eta)\notin SWO^{s}(\chi)_{\eta}$ generally, as the example that follows illustrates.*

Example 22. *Suppose that $\rho=\{\tilde{R},\tilde{\varphi},(J_1,\eta),(J_2,\eta),(J_3,\eta)\}$ is an SSTS defined on the set of real numbers R and the set of parameters $\eta=\{\vartheta_1,\vartheta_2\}$ where*
$J_1(\vartheta_1)=[5,6]$, $\quad J_1(\vartheta_2)=[7,8]$,
$J_2(\vartheta_1)=[6,7]$, $\quad J_2(\vartheta_2)=[8,9]$,
$J_3(\vartheta_1)=[5,7]$, $\quad J_3(\vartheta_2)=[7,9]$.
Hence, the soft sets (J_1,η) and (J_2,η) are SS-sw-open sets, but their soft intersection $(J_1,\eta)\tilde{\cap}(J_2,\eta)=\{(\eta_1,\{6\}),(\eta_2,\{8\})\}$ is not SS-sw-open set.

Corollary 23. *If a non-null soft subset (G,η) of an SSTS (χ,ρ,η) is SS-semi-open set, then $int^{s}(G,\eta)\neq\tilde{\varphi}$.*

Proof. Suppose contrary that, $int^{s}(G,\eta)=\tilde{\varphi}$ for an SS-semi-open set (G,η). According to Theorem 11, $cl^{s}(G,\eta)=cl^{s}(int^{s}(G,\eta))=\tilde{\varphi}$. If follows that, $(G,\eta)=\tilde{\varphi}$, which is a contradiction.

Note 24. *According to Corollary 23, if a soft subset (G,η) of an SSTS (χ,ρ,η) is SS-semi-open set, then it is an SS-sw-open, but not conversely. In Example 17, the soft set (N,η) where :*
$N(\eta_1)=\{x_1,x_3,x_4\}$, $\quad N(\eta_2)=\{x_1,x_4\}$ *is an SS-sw-open set but not SS-semi-open.*

Proposition 25. *If soft subset (G,η) of an SSTS (χ,ρ,η) is SS-sw-open, then it is an SS-sd-set.*

Proof. Suppose contrary that, (G,η) is not SS-sd-set, then $int^{s}(cl^{s}(G,\eta))=\tilde{\varphi}$. Since $int^{s}(G,\eta)\tilde{\subseteq}int^{s}(cl^{s}(G,\eta))=\tilde{\varphi}$, $int^{s}(G,\eta)=\tilde{\varphi}$, which is a contradiction. The converse of this result is not generally accurate, refer to Example 22, the soft set $\{(\eta_1,\{6\}),(\eta_2,\{8\})\}$ is an SS-sd-set but not SS-sw-open.

Remark 26. *The classes of SS-β-open sets and SS-sw-open sets are independent, as shall demonstrated in the upcoming examples.*

Examples 27. (1) *Let R be the set of real numbers, $\eta = \{\vartheta_1, \vartheta_2\}$ and let*

$$\rho = \{\tilde{R}, \tilde{\varphi}, (A,\eta), (B,\eta), (C,\eta)\}, where:$$

$$A(\vartheta_1) = [3,5], \quad A(\vartheta_2) = [5,7].$$

$$B(\vartheta_1) = [4,5], \quad B(\vartheta_2) = [6,7].$$

$$C(\vartheta_1) = [3,4], \quad C(\vartheta_2) = [5,6].$$

$$Then,\ (T,\eta) = \{(\vartheta_1, \{4\}), (\vartheta_2, \{6\})\}$$

is an SS-β-subset of $\tilde{R}$ but not SS-sw-open.

(2) *Let $\chi = \{r_1, r_2, r_3, r_4\}$, $\eta = \{\vartheta_1, \vartheta_2\}$ and let*

$$\rho = \{\tilde{\chi}, \tilde{\varphi}, (I_1,\eta), (I_2,\eta), (I_3,\eta), (I_4,\eta), (I_5,\eta), (I_6,\eta), (I_7,\eta)\},\ where:$$

$$I_1(\vartheta_1) = \{r_1\}, \quad I_1(\vartheta_2) = \varphi.$$

$$I_2(\vartheta_1) = \{r_1, r_2\}, \quad I_2(\vartheta_2) = \{r_1\}.$$

$$I_3(\vartheta_1) = \{r_1, r_2\}, \quad I_3(\vartheta_2) = \{r_3, r_4\}.$$

$$I_4(\vartheta_1) = \{r_3, r_4\}, \quad I_4(\vartheta_2) = \{r_1, r_2\}.$$

$$I_5(\vartheta_1) = \{r_1, r_3, r_4\}, \quad I_5(\vartheta_2) = \{r_1, r_2\}.$$

$$I_6(\vartheta_1) = U, \quad I_6(\vartheta_2) = \{r_1, r_2\}.$$

$$I_7(\vartheta_1) = \{r_1, r_2\}, \quad I_7(\vartheta_2) = \{r_1, r_3, r_4\}.$$

$$Then,\ (Z,\eta) = \{(\vartheta_1, \{r_2, r_3, r_4\}), (\vartheta_2, \chi)\}$$

is an SS-sw-open set, but it is not SS-β-open.

Corollary 28. *We can summarize the above relationships with the help of [Corollary 3.19, [37]], in the subsequent ramifications for an SSTS (χ, ρ, η), which cannot be reversed .*

$$\begin{tikzcd}
ROS^s(\chi)_\eta \arrow[r] & OS^s(\chi)_\eta \arrow[r] & \alpha OS^s(\chi)_\eta \arrow[r] & SOS^s(\chi)_\eta \arrow[r] \arrow[dr] & \beta OS^s(\chi)_\eta \arrow[r] \arrow[d, leftrightarrow, "/" marking] & SD^s(\chi)_\eta \\
 & & & & SWO^s(\chi)_\eta \arrow[ur] &
\end{tikzcd}$$

Figure 1. The relationships among SS-sw-open sets and other generalizations.

4. Soft continuity (openness) inspired by supra soft sw-open sets

In this section, we introduce new types of soft continuity related to SS-sw-open sets, named SS-sw-cts functions. We characterize many of its essential properties. Also, we have studied its relationships with previous similar types of generalizations. We used SS-sw-closure (interior) operators to present several equivalent conditions of our new approach. Furthermore, we define a new type of functions inspired by SS-sw-open sets, named SS-sw-open functions.

Definition 29. *A soft function $\psi_{sw} : (\chi_1, \sigma_1, \eta_1) \to (\chi_2, \sigma_2, \eta_2)$ with ρ_1 as an associated SSTS with σ_1 is said to be an SS-sw-cts if $\psi_{sw}^{-1}(G, \eta_2) \in SWO^s(\chi_1)_{\eta_1}$ $\forall$ $(G, \eta_2) \in \sigma_2$.*

Note 30. *According to Figure 1, we have the following diagram.*

$$\begin{tikzcd}
\text{SS-regular-cts} \arrow[r] & \text{SS-cts} \arrow[r] & \text{SS-}\alpha\text{-cts} \arrow[r] & \text{SS-semi-cts} \arrow[r] \arrow[dr] & \text{SS-}\beta\text{-cts} \arrow[r] \arrow[d, phantom, "\not\Updownarrow"] & \text{SS-sd-cts} \\
 & & & & \text{SS-sw-cts} \arrow[ur] &
\end{tikzcd}$$

Figure 2. *The relationships between some generalizations of SS-continuity*

The next examples show that, the implications in Figure 2 are not reversible.

Examples 31. **(1)** *Let $\chi_1 = \{r_1, r_2, r_3, r_4\}$, $\chi_2 = \{t_1, t_2, t_3, t_4\}$, $\eta_1 = \{\vartheta_1, \vartheta_2\}$ and $\eta_2 = \{\theta_1, \theta_2\}$.*

$$Define\ s : \chi_1 \to \chi_2 \ and\ w : \eta_1 \to \eta_2\ as\ follows:$$

$$s(r_1) = t_1,\ s(r_2) = t_4,\ s(r_3) = t_2,\ s(r_4) = t_3,\ w(\vartheta_1) = \theta_1,\ w(\vartheta_2) = \theta_2.$$

$$Let\ \sigma_1 = \{\tilde{\chi_1}, \tilde{\varphi}, (A, \eta_1)\}\ be\ an\ STS\ over\ \chi_1,\ where$$

$$A(\vartheta_1) = \{r_1\}, \quad A(\vartheta_2) = \varphi.$$

$$Let\ \rho_1 = \{\tilde{\chi_1}, \tilde{\varphi}, (Q_i, \eta_1), i = 1, 2, .., 5\}$$

$$is\ an\ associated\ SSTS\ with\ \sigma_1, where:$$

$$Q_1(\vartheta_1) = \chi_1, \quad Q_1(\vartheta_2) = \{r_1, r_2\}.$$

$$Q_2(\vartheta_1) = \{r_3, r_4\}, \quad Q_2(\vartheta_2) = \{r_1, r_2\}.$$

$$Q_3(\vartheta_1) = \{r_1, r_2\}, \quad Q_3(\vartheta_2) = \{r_3, r_4\}.$$

$$Q_4(\vartheta_1) = \{r_1\}, \quad Q_4(\vartheta_2) = \varphi.$$

$$Q_5(\vartheta_1) = \{r_1, r_2\}, \quad Q_5(\vartheta_2) = \{r_1, r_3, r_4\}.$$

$$Let\ \sigma_2 = \{\tilde{\chi_2}, \tilde{\varphi}, (P, \Theta_2)\}\ be\ an\ STS\ over\ \chi_2\ , where:$$

$$P(\theta_1) = \{t_1, t_2, t_3\}, \quad P(\theta_2) = \{t_1, t_3\}.$$

Then,

$$\psi_{sw}^{-1}(P,\Theta_2)=\{(\vartheta_1,\{r_1,r_3,r_4\}),(\vartheta_2,\{r_1,r_4\})\}$$

is an SS-sw-subset of $\tilde{\chi_1}$*, but not SS-semi-open. Therefore,* ψ_{sw} *is an SS-sw-cts, but not SS-semi-cts.*

(2) *Let* R *be the set of real numbers,* $\eta_1=\{\vartheta_1,\vartheta_2\}$ *and* $\eta_2=\{\theta_1,\theta_2\}$.

$$Let\ s:R\to R\ and\ w:\eta_1\to\eta_2\ be\ the\ identity\ functions.$$

$$Let\ \sigma_1=\{\tilde{R},\tilde{\varphi},(A,\eta_1)\}\ be\ an\ STS\ over\ R,\ and\ let$$
$$\rho_1\ in\ Examples\ 27\ (1)\ be\ an\ associated\ SSTS\ with\ \sigma_1.$$
$$Let\ \sigma_2=\{\tilde{R},\tilde{\varphi},(H,\eta_2)\}\ be\ an\ STS\ over\ R\ ,where:$$
$$H(\theta_1)=\{4\},\quad H(\theta_2)=\{6\}.$$

Then,

$$\psi_{sw}^{-1}(H,\eta_2)=\{(\vartheta_1,\{4\}),(\vartheta_2,\{6\})\}$$

is an SS-sd-subset of $\tilde{R}$ *but not SS-sw-open. Therefore,* ψ_{sw} *is an SS-sd-cts but not SS-sw-cts.*

(3) *In (2), we have* $\psi_{sw}^{-1}(H,\eta_2)=\{(\vartheta_1,\{4\}),(\vartheta_2,\{6\})\}$ *is an SS-*β*-subset of* $\tilde{R}$ *but not SS-sw-open. Therefore,* ψ_{sw} *is an SS-*β*-cts but not SS-sw-cts.*

(4) *Let* $\chi_1=\{r_1,r_2,r_3,r_4\}$, $\chi_2=\{t_1,t_2,t_3,t_4\}$, $\eta_1=\{\vartheta_1,\vartheta_2\}$ *and* $\eta_2=\{\theta_1,\theta_2\}$.

$$Define\ s:\chi_1\to\chi_2\ and\ w:\eta_1\to\eta_2\ as\ follows:$$

$$s(r_1)=t_4,\ s(r_2)=t_3,\ s(r_3)=t_1,\ s(r_4)=t_2,\ w(\vartheta_1)=\theta_1,\ w(\vartheta_2)=\theta_2.$$
$$Let\ \sigma_1=\{\tilde{\chi_1},\tilde{\varphi},(I_2,\eta_1)\}\ be\ an\ STS\ over\ \chi_1\ and$$
$$\rho_1\ in\ Examples\ 27\ (2)\ be\ an\ associated\ SSTS\ with\ \sigma_1.$$
$$Let\ \sigma_2=\{\tilde{\chi_2},\tilde{\varphi},(S,\eta_2)\}\ be\ a\ STS\ over\ \chi_2\ where,$$
$$S(\theta_1)=\{t_1,t_2,t_3\},\quad S(\theta_2)=\chi_2,$$

Then,

$$\psi_{sd}^{-1}((S,\eta_2))=\{(\vartheta_1,\{r_2,r_3,r_4\}),(\vartheta_2,\chi_1)\}$$

is an SS-sw-open set, but it is not SS-β*-open. Hence,* ψ_{sd} *is an SS-sw-cts, but it is not SS-*β*-cts .*

Definition 32. *Let* (G,η) *be a soft subset of an SSTS* (χ,ρ,η)*, then*

(1) $int^s_{sw}(G,\eta)=\sqcup\{(O,\eta):(O,\eta)\in SWO^s(\chi)_\eta\ and\ (O,\eta)\tilde{\subseteq}(G,\eta)\}$.

(2) $cl^s_{sw}(G,\eta)=\sqcap\{(H,\eta):(H,\eta)\in SWC^s(\chi)_\eta\ and\ (G,\eta)\tilde{\subseteq}(H,\eta)\}$.

(3) *(G,η) is called SS-sw-co-dense if $int_{sw}(G,\eta)=\tilde{\varphi}$.*

(4) *(G,η) is called SS-sw-dense if $cl_{sw}(G,\eta)=\tilde{\chi}$.*

Theorem 33. *Let $\psi_{sw}:(\chi_1,\sigma_1,\eta_1)\rightarrow(\chi_2,\sigma_2,\eta_2)$ be a soft function with ρ_1 as an associated SSTS with σ_1; then, the subsequent statements are equivalent:*

(1) *ψ_{sw} is an SS-sw-cts.*

(2) *For each $(E,\eta_2)\in\sigma_2^c$, $\psi_{sw}^{-1}(E,\eta_2)\in SWC^s(\chi_1)_{\eta_1}$.*

(3) *$cl_{sw}^s(\psi_{sw}^{-1}(E,\eta_2))\tilde{\subseteq}\psi_{sw}^{-1}(cl(E,\eta_2))\ \forall\ (E,\eta_2)\tilde{\subseteq}\tilde{\chi}_2$.*

(4) *$\psi_{sw}(cl_{sw}^s(G,\eta_1))\tilde{\subseteq}cl(\psi_{sw}(G,\eta_1))\ \forall\ (G,\eta_1)\tilde{\subseteq}\tilde{\chi}_1$.*

(5) *$\psi_{sw}^{-1}(int(E,\eta_2))\tilde{\subseteq}int_{sw}^s(\psi_{sw}^{-1}(E,\eta_2))\ \forall\ (E,\eta_2)\tilde{\subseteq}\tilde{\chi}_2$.*

Proof.

(1) $\Rightarrow$ (2) Let $(E,\eta_2)\in\sigma_2^c$; then, $(E^{\tilde{c}},\eta_2)\in\sigma_2$. Given (1), $\psi_{sw}^{-1}(E^{\tilde{c}},\eta_2)=[\psi_{sw}^{-1}(E,\eta_2)]^{\tilde{c}}\in SWO^s(\chi_1)_{\eta_1}$. Hence,

$$\psi_{sw}^{-1}(E,\eta_2)\in SWC^s(\chi_1)_{\eta_1}.$$

(2) $\Rightarrow$ (3) Since $cl(E,\eta_2)\in\sigma_2^c$ for each $(E,\eta_2)\tilde{\subseteq}\tilde{\chi}_2$,

$$\psi_{sw}^{-1}(cl(E,\eta_2))\in SWC^s(\chi_1)_{\eta_1},\ given\ (2), which\ implies$$

$$cl_{sw}^s(\psi_{sw}^{-1}(E,\eta_2))\tilde{\subseteq}cl_{sw}^s(\psi_{sw}^{-1}(cl(E,\eta_2)))=\psi_{sw}^{-1}(cl(E,\eta_2)).$$

(3) $\Rightarrow$ (4) Given that, $\psi_{sw}(G,\eta_1)\tilde{\subseteq}\tilde{\chi}_2$ for each $(G,\eta_1)\tilde{\subseteq}\tilde{\chi}_1$, and applying (3), we have

$$cl_{sw}^s(\psi_{sw}^{-1}(\psi_{sw}(G,\eta_1)))\tilde{\subseteq}\psi_{sw}^{-1}(cl(\psi_{sw}(G,\eta_1))).$$

Hence,

$$\psi_{sw}[cl_{sw}^s(\psi_{sw}^{-1}(\psi_{sw}(G,\eta_1)))]\tilde{\subseteq}\psi_{sw}[\psi_{sw}^{-1}(cl(\psi_{sw}(G,\eta_1)))]\tilde{\subseteq}cl(\psi_{sw}(G,\eta_1)),\ from\ Theorem\ 5\ (2).$$

Therefore,

$$\psi_{sw}(cl_{sw}^s(G,\eta_1))\tilde{\subseteq}cl(\psi_{sw}(G,\eta_1)),\ from\ Theorem\ 5\ (3).$$

(4) $\Rightarrow$ (5) Since $\psi_{sw}^{-1}(E^{\tilde{c}},\eta_2)\tilde{\subseteq}\tilde{\chi}_1$ for each $(E^{\tilde{c}},\eta_2)\tilde{\subseteq}\tilde{\chi}_2$. Applying (4),

$$\psi_{sw}[cl_{sw}^s[\psi_{sw}^{-1}(E^{\tilde{c}},\eta_2)]]\tilde{\subseteq}cl(\psi_{sw}[\psi_{sw}^{-1}(E^{\tilde{c}},\eta_2)])\tilde{\subseteq}cl(E^{\tilde{c}},\eta_2)=[int(E,\eta_2)]^{\tilde{c}}.$$

It follows that,

$$\psi_{sw}^{-1}[\psi_{sw}(cl_{sw}^s[\psi_{sw}^{-1}(E^{\tilde{c}},\eta_2)])]\tilde{\subseteq}\psi_{sw}^{-1}[[int(E,\eta_2)]^{\tilde{c}}]=[\psi_{sw}^{-1}(int(E,\eta_2))]^{\tilde{c}}.$$

Hence,

$$cl^s_{sw}[(\psi^{-1}_{sw}(E,\eta_2))]^{\tilde{c}}\tilde{\subseteq}[\psi^{-1}_{sw}(int(E,\eta_2))]^{\tilde{c}},\ from\ Theorem\ 5\ (3).$$

Therefore,

$$\psi^{-1}_{sw}(int(E,\eta_2))\tilde{\subseteq}[cl^s_{sw}[(\psi^{-1}_{sw}(E,\eta_2))]^{\tilde{c}}]^{\tilde{c}} = int^s_{sw}(\psi^{-1}_{sw}(E,\eta_2)).$$

(5) $\Rightarrow$ (1) Since $(E,\eta_2) = int(E,\eta_2)$ for each $(E,\eta_2) \in \sigma_2$. Then,

$$\psi^{-1}_{sw}(E,\eta_2)\tilde{\subseteq}int^s_{sw}(\psi^{-1}_{sw}(E,\eta_2)),\ from\ (5),$$

and so ,

$$int^s_{sw}(\psi^{-1}_{sw}(E,\eta_2)) = \psi^{-1}_{sw}(E,\eta_2) \neq \tilde{\varphi}.$$

It follow that,

$$\psi^{-1}_{sw}(E,\eta_2) \in SWO^s(\chi_1)_{\eta_1}.$$

Thus, ψ_{sw} is an SS-sw-cts.

Theorem 34. *Let $\psi_{sw} : (\chi_1,\sigma_1,\eta_1) \rightarrow (\chi_2,\sigma_2,\eta_2)$ be a soft function with ρ_1 as an associated SSTS with σ_1; then the subsequent statements are equivalent:*

(1) *ψ_{sw} is an SS-sw-cts.*

(2) *There exists $\tilde{\varphi} \neq (O,\eta_1) \in \rho_1$ such that $(O,\eta_1)\tilde{\sqsubseteq}\psi^{-1}_{sw}(E,\eta_2)$, for each $(E,\eta_2) \in \sigma_2$ with $\psi^{-1}_{sw}(E,\eta_2) \neq \tilde{\varphi}$.*

(3) *There exists $\tilde{\chi_1} \neq (C,\eta_1) \in \rho_1^c$ such that $\psi^{-1}_{sw}(E,\eta_2)\tilde{\sqsubseteq}(C,\eta_1)$, for each $(E,\eta_2) \in \sigma_2^c$ with $\psi^{-1}_{sw}(E,\eta_2) \neq \tilde{\chi_1}$.*

(4) *$\psi_{sw}(G,\eta_1)$ is SS-dense over $\psi_{sw}(\chi_1)$, for each (G,η_1) SS-dense over χ_1.*

Proof.

(1) $\Rightarrow$ (2) Immediate from Proposition 13 (2).

(2) $\Rightarrow$ (3) Let $(E,\eta_2) \in \sigma_2^c$ with $\psi^{-1}_{sw}(E,\eta_2) \neq \tilde{\chi_1}$. It follows that, $(E^{\tilde{c}},\eta_2) \in \sigma_2$ with $\psi^{-1}_{sw}(E^{\tilde{c}},\eta_2) \neq \tilde{\varphi}$. Given (2), there exists $\tilde{\varphi} \neq (O,\eta_1) \in \rho_1$ such that $(O,\eta_1)\tilde{\sqsubseteq}\psi^{-1}_{sw}(E^{\tilde{c}},\eta_2)$. Hence, $\psi^{-1}_{sw}(E,\eta_2)\tilde{\sqsubseteq}(O^{\tilde{c}},\eta_1)$, $\tilde{\chi_1} \neq (O^{\tilde{c}},\eta_1) \in \rho_1^c$.

(3) $\Rightarrow$ (4) Assume conversely, $\psi_{sw}(G,\eta_1)$ is not SS-dense over $\psi_{sw}(\chi_1)$, for some (G,η_1) SS-dense over χ_1. Then, there exists $\tilde{\chi_2} \neq (E,\eta_2) \in \sigma_2^c$ such that $\psi_{sw}(G,\eta_1)\tilde{\sqsubseteq}(E,\eta_2)\tilde{\sqsubseteq}\psi_{sw}(\chi_1)$, and so $(G,\eta_1)\tilde{\sqsubseteq}\psi^{-1}_{sw}(E,\eta_2)$. Given (3), there exists $\tilde{\chi_1} \neq (C,\eta_1) \in \rho_1^c$ such that $(G,\eta_1)\tilde{\sqsubseteq}\psi^{-1}_{sw}(E,\eta_2)\tilde{\sqsubseteq}(C,\eta_1) \neq \tilde{\chi_1}$, which contradicts our assumption.

(4) $\Rightarrow$ (1) Let $(E, \eta_2) \in \sigma_2$ with $\psi_{sw}^{-1}(E, \eta_2) \neq \tilde{\varphi}$. Assume contrary that $\psi_{sw}^{-1}(E, \eta_2)$ is not SS-sw-cts. Then, $int(\psi_{sw}^{-1}(E, \eta_2)) = \tilde{\varphi}$, which follows $cl(\psi_{sw}^{-1}(E^{\tilde{c}}, \eta_2)) = \tilde{\chi_1}$. This means that $\psi_{sw}^{-1}(E^{\tilde{c}}, \eta_2)$ is SS-dense over χ_1. Given (4), $\psi_{sw}[\psi_{sw}^{-1}(E^{\tilde{c}}, \eta_2)]$ is an SS-dense over $\psi_{sw}(\chi_1)$, and so $(E, \eta_2) = \tilde{\varphi}$, which contradicts our assumption. Thus, ψ_{sw} is an SS-sw-cts.

Proposition 35. *Let $\psi_{sw} : (\chi_1, \sigma_1, \eta_1) \to (\chi_2, \sigma_2, \eta_2)$ be an one to one soft function with ρ_1 as an associated SSTS with σ_1; then the subsequent statements are equivalent:*

(1) *ψ_{sw} is an SS-sw-cts.*

(2) *$\psi_{sw}(G, \eta_1)$ is soft co-dense over χ_2, for each SS-sw-co-dense subset (G, η_1) of $\tilde{\chi_1}$.*

Proof.

(1) $\Rightarrow$ (2) Assume conversely that, $\psi_{sw}(G, \eta_1)$ is not soft co-dense set over χ_2 for any SS-sw-co-dense subset (G, η_1) of $\tilde{\chi_1}$. It follows that, $int[\psi_{sw}(G, \eta_1)] \neq \tilde{\varphi}$. Given (1), $\psi_{sw}^{-1}[int[\psi_{sw}(G, \eta_1)]] \in SWO^s(\chi_1)_{\eta_1}$. Since ψ_{sw} is one to one,

$$\tilde{\varphi} \neq int_{sw}[\psi_{sw}^{-1}[int[\psi_{sw}(G, \eta_1)]]] \tilde{\sqsubseteq} int_{sw}[\psi_{sw}^{-1}[\psi_{sw}(G, \eta_1)]] = int_{sw}(G, \eta_1).$$

Hence, (G, η_1) is not SS-sw-co-dense set, which is a contradiction.

(2) $\Rightarrow$ (1) Let $\tilde{\varphi} \neq (G, \eta_1) \in \sigma_2$. Assume conversely that, $\psi_{sw}^{-1}(G, \eta_1) \notin SWO^s(\chi_1)_{\eta_1}$, then $int_{sw}[\psi_{sw}^{-1}(G, \eta_1)] = \tilde{\varphi}$. By condition and given ψ_{sw} is one to one, we get $\tilde{\varphi} = int(\psi_{sw}[int_{sw}[\psi_{sw}^{-1}(G, \eta_1)]]) \tilde{\sqsubseteq} int(\psi_{sw}[\psi_{sw}^{-1}(G, \eta_1)]) = int(G, \eta_1)$, which is a contradiction.

Definition 36. *A soft function $\psi_{sw} : (\chi_1, \sigma_1, \eta_1) \to (\chi_2, \sigma_2, \eta_2)$ with ρ_2 as an associated SSTS with σ_2 is said to be an SS-sw-open if $\psi_{sw}(G, \eta_1) \in SWO^s(\chi_2)_{\eta_2}\ \forall\ (G, \eta_1) \in \sigma_1$.*

Proposition 37. *A soft function $\psi_{sw} : (\chi_1, \sigma_1, \eta_1) \to (\chi_2, \sigma_2, \eta_2)$ with ρ_2 as an associated SSTS with σ_2 is SS-sw-open if and only if for each $\tilde{\varphi} \neq (G, \eta_1) \in \sigma_1$, there exists $\tilde{\varphi} \neq (H, \eta_2) \in SWO^s(\chi_2)_{\eta_2}$ such that $(H, \eta_2) \tilde{\sqsubseteq} \psi_{sw}(G, \eta_1)$.*

Proof. Immediate from Definition 36.

Theorem 38. *Let $\psi_{sw} : (\chi_1, \sigma_1, \eta_1) \to (\chi_2, \sigma_2, \eta_2)$ be a soft function with ρ_2 as an associated SSTS with σ_2; then the subsequent statements are equivalent:*

(1) *ψ_{sw} is an SS-sw-open.*

(2) *$\psi_{sw}(int(W, \eta_1)) \tilde{\sqsubseteq} int^s_{sw}(\psi_{sw}(W, \eta_1))$, for each $(W, \eta_1) \tilde{\sqsubseteq} \tilde{\chi_1}$.*

(3) *$\psi_{sw}^{-1}(cl^s_{sw}(Z, \eta_2)) \tilde{\sqsubseteq} cl(\psi_{sw}^{-1}(Z, \eta_2))$, for each $(Z, \eta_2) \tilde{\sqsubseteq} \tilde{\chi_2}$.*

Proof.

(1) $\Rightarrow$ (2) Since $int(W, \eta_1) \tilde{\sqsubseteq} (W, \eta_1)$, $\psi_{sw}(int(W, \eta_1)) \tilde{\sqsubseteq} \psi_{sw}((W, \eta_1))$. Given (1),

$$int^s_{sw}[\psi_{sw}(int(W,\eta_1))] = \psi_{sw}(int(W,\eta_1))\tilde{\subseteq} int^s_{sw}[\psi_{sw}((W,\eta_1))].$$

(2) $\Rightarrow$ (1) Assume that $\tilde{\varphi} \neq (W,\eta_1) \in \sigma_1$. Then,

$$\psi_{sw}(int(W,\eta_1)) = \psi_{sw}(W,\eta_1)\tilde{\subseteq} int^s_{sw}[\psi_{sw}(W,\eta_1)].$$

However,

$$int^s_{sw}[\psi_{sw}(W,\eta_1)]\tilde{\subseteq}\psi_{sw}(W,\eta_1), \text{ and therefore}$$

$$int^s_{sw}[\psi_{sw}(W,\eta_1)] = \psi_{sw}(W,\eta_1).$$

Thus,

$$(W,\eta_1) \in SWO^s(\chi_1)_{\Theta_1}; \text{ hence, } \psi_{sw} \text{ is SS-sw-open.}$$

(2) $\Rightarrow$ (3) Since $\psi^{-1}_{sw}(Z^{\tilde{c}},\eta_2)\tilde{\subseteq}\tilde{\chi_1}$ for each $(Z,\eta_2)\tilde{\subseteq}\tilde{\chi_2}$. Applying (2),

$$\psi_{sw}(int(\psi^{-1}_{sw}(Z^{\tilde{c}},\eta_2)))\tilde{\subseteq} int^s_{sw}(\psi_{sw}(\psi^{-1}_{sw}(Z^{\tilde{c}},\eta_2)))\tilde{\subseteq} int^s_{sw}(Z^{\tilde{c}},\eta_2) = [cl^s_{sw}(Z,\eta_2)]^{\tilde{c}},$$
from Theorem 5 (3).

So,

$$cl[(\psi^{-1}_{sw}(Z,\eta_2))]^{\tilde{c}} = int(\psi^{-1}_{sw}(Z^{\tilde{c}},\eta_2))\tilde{\subseteq}\psi^{-1}_{sw}[\psi_{sw}(int(\psi^{-1}_{sw}(Z^{\tilde{c}},\eta_2)))]\tilde{\subseteq}\psi^{-1}_{sw}[[cl^s_{sw}(Z,\eta_2)]^{\tilde{c}}].$$

Hence,

$$\psi^{-1}_{sw}(cl^s_{sw}(Z,\eta_2))\tilde{\sqsubseteq} cl(\psi^{-1}_{sw}(Z,\eta_2)).$$

(3) $\Rightarrow$ (2) By a similar technique.

Theorem 39. *Let $\psi_{sw} : (\chi_1,\sigma_1,\eta_1) \to (\chi_2,\sigma_2,\eta_2)$ be an one to one soft function with ρ_2 as an associated SSTS with σ_2; then the subsequent statements are equivalent:*

(1) *ψ_{sw} is an SS-sw-open.*

(2) *There exists $\tilde{\chi_2} \neq (B,\eta_2) \in \rho^c_2$ such that $\psi_{sw}(A,\eta_1)\tilde{\sqsubseteq}(B,\eta_2)$, for each $(A,\eta_1) \in \sigma^c_1$ with $\psi_{sw}(A,\eta_1) \neq \tilde{\chi_2}$.*

Proof.

(1) $\Rightarrow$ (2) Let $(A,\eta_1) \in \sigma^c_1$ with $\psi_{sw}(A,\eta_1) \neq \tilde{\chi_2}$. It follows that, $(A^{\tilde{c}},\eta_1) \in \sigma_1$. Given (1), there is $\tilde{\varphi} \neq (B,\eta_2) \in \rho_2$ such that $(B,\eta_2)\tilde{\sqsubseteq}\psi_{sw}(A^{\tilde{c}},\eta_1)$. That is, $\psi_{sw}(A,\eta_1)\tilde{\sqsubseteq}(B^{\tilde{c}},\eta_2)$, $\psi_{sw}(\tilde{\chi_1}) \neq (B^{\tilde{c}},\eta_2) \in \rho^c_2$.

(2) $\Rightarrow$ (1) Let $\tilde{\varphi} \neq (G, \eta_1) \in \sigma_1$. It follows, $\tilde{\chi_1} \neq (G^{\tilde{c}}, \eta_1) \in \sigma_1^c$. Applying the condition, there exists $\tilde{\chi_2} \neq (H, \eta_2) \in \rho_2^c$ such that $\psi_{sw}(G^{\tilde{c}}, \eta_1)\tilde{\sqsubseteq}(H, \eta_2)$. This implies, $(H^{\tilde{c}}, \eta_2)\tilde{\sqsubseteq}\psi_{sw}(G, \eta_1)$, $\tilde{\varphi} \neq (H^{\tilde{c}}, \eta_2) \in \rho_2$. Hence, $\psi_{sw}(G, \eta_1) \in SWO^s(\chi_2)_{\eta_2}$. Therefore, ψ_{sw} is an SS-sw-open.

Theorem 40. *Let $\psi_{sw} : (\chi_1, \sigma_1, \eta_1) \rightarrow (\chi_2, \sigma_2, \eta_2)$ be a soft function with ρ_2 as an associated SSTS with σ_2; then the subsequent statements are equivalent:*

(1) *ψ_{sw} is an SS-sw-open.*

(2) *$\psi_{sw}^{-1}(K, \eta_2)$ is soft dense over χ_1, for each (K, η_2) SS-sw-dense set over χ_2.*

Proof.

(1) $\Rightarrow$ (2) Assume conversely that, $\psi_{sw}^{-1}(K, \eta_2)$ is not soft dense set over χ_1 for arbitrary SS-sw-dense subset (K, η_2) of $\tilde{\chi_2}$. It follows that, there is $\tilde{\chi_1} \neq (V, \eta_1) \in \sigma_1^c$ such that $\psi_{sw}^{-1}(K, \eta_2)\tilde{\sqsubseteq}(V, \eta_1)$. It follows that,

$$\psi_{sw}(V^{\tilde{c}}, \eta_1)\tilde{\sqsubseteq}(K^{\tilde{c}}, \eta_2) \tag{1}$$

Since $\tilde{\varphi} \neq (V^{\tilde{c}}, \eta_1) \in \sigma_1$, given(1) there exists $\tilde{\varphi} \neq (H, \eta_2) \in SWO^s(\chi_2)_{\eta_2}$ such that

$$(H, \eta_2)\tilde{\sqsubseteq}\psi_{sw}(V^{\tilde{c}}, \eta_1), from\ Proposition\ 37. \tag{2}$$

From Eqs (1) and (2), $(K, \eta_2)\tilde{\sqsubseteq}(H^{\tilde{c}}, \eta_2)$, $(H^{\tilde{c}}, \eta_2) \in SWC^s(\chi_2)_{\eta_2}$. That is, (K, η_2) is not SS-sw-dense set, which contradicts our assumption. Therefore, $\psi_{sw}^{-1}(K, \eta_2)$ is soft dense set over χ_1.

(2) $\Rightarrow$ (1) Let $\tilde{\varphi} \neq (G, \eta_1) \in \sigma_1$. If $\psi_{sw}(G, \eta_1) \notin SWO^s(\chi_2)_{\eta_2}$, then $int[\psi_{sw}(G, \eta_1)] = \tilde{\varphi}$, and hence $cl[\psi_{sw}(G^{\tilde{c}}, \eta_1)] = \tilde{\chi_2}$. It follows that, $\psi_{sw}(G^{\tilde{c}}, \eta_1)$ is an SS-sw-dense set over χ_2. By assumption,

$$cl[\psi_{sw}^{-1}[\psi_{sw}(G^{\tilde{c}}, \eta_1)]] = \tilde{\chi_1} \tag{3}$$

However, we have $(G, \eta_1)\tilde{\sqsubseteq}\psi_{sw}^{-1}[\psi_{sw}(G, \eta_1)$ from Theorem 5. Hence, $\psi_{sw}^{-1}[\psi_{sw}(G^{\tilde{c}}, \eta_1)\tilde{\sqsubseteq}(G^{\tilde{c}}, \eta_1)$. Since $(G,^{\tilde{c}} \eta_1) \in \sigma_1^c$,

$$cl[\psi_{sw}^{-1}[\psi_{sw}(G^{\tilde{c}}, \eta_1)]\tilde{\sqsubseteq}cl[(G^{\tilde{c}}, \eta_1)] = (G^{\tilde{c}}, \eta_1) \tag{4}$$

From Eqs (3) and (4), $\tilde{\chi_1}\tilde{\sqsubseteq}(G^{\tilde{c}}, \eta_1)$, which follows $(G, \eta_1) = \tilde{\varphi}$, which is a contradiction. Hence, $\psi_{sw}(G, \eta_1) \in SWO^s(\chi_2)_{\eta_2}$, and therefore ψ_{sw} is an SS-sw-open.

5. Conclusion

In this paper, we used the supra soft interior operator to define a new approach of generalized sets named, SS-sw-open sets. We studied the essential characterizations of this new approach. We discuss its relationships with the other generalizations and provide the necessary examples and counterexamples. Furthermore, we applied this new notion to soft

continuity. Especially, we presented the notions of SS-sw-cts and Ss-sw-open functions. Moreover, we used the SS-sw-closure (interior) operators to present several equivalent conditions for our new approaches.
We plan to extend the previously mentioned concepts by basing them on the soft ideal [32]. Furthermore, by employing the aforementioned methods, additional topological characteristics like separation axioms, compactness and connectedness will be presented, and this will be the focus of our upcoming work. Lastly, using the presented generalizations, the enhancement of the accuracy measures for subsets in information systems will be taken into consideration.

6. Conflict of Interest

The author declares no conflicts of interest.

7. Acknowledgments

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