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$\mathcal{C}_\alpha$-Rectifying Curves in a New Conformable Differential Geometry

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Abstract. In this study, we reintroduce the theory of curves by incorporating local fractional calculus. We elucidate the condition for a naturally parametrized curve to be conformable, and we define the orthonormal conformable frame of such a curve at any given point. Then, we provide a comprehensive explanation of how these newly derived conformable geometric concepts are related to their classical counterparts. Furthermore, we introduce the concept of a conformable rectifying curve and provide its characterizations in terms of this differentiation with respect to arbitrary order. Some illustrative graphs are provided.

2020 Mathematics Subject Classifications: 53A04, 26A33

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1. Introduction

One of the most interesting topics in differential geometry is the theory of curves. The reason for this is that curves are used in modeling many problems that we encounter in real life. For example, curves are used when observing the motion of a charged particle in a magnetic field [1, 2]. In addition, many operations are performed using curves in computer-aided geometric designs [3, 4]. The first thing to do when designing these geometric models is to characterize the curve. Because curves are concepts that are characterized and studied. There are some methods used when characterizing curves. The most important of these are that the curve is characterized by its curvatures and Frenet vectors. The rectifying curves that are the subject of the article are characterized by both Frenet vectors and curvatures by B.Y. Chen [5]. A naturally parametrized curve is called a rectifying curve if it lies in the plane formed by its tangent and binormal at a point. Also, the ratio of

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torsion to curvature of rectifying curves results in a linear equation. Moreover, it is known that centrodes (i.e. angular velocity vectors) play some important roles in mechanics and joint kinematics [6, 7]. In this direction B.Y. Chen and F. Dillen observed that rectifying curves can be viewed as centrodes and extremal curves in $\mathbb{E}^3$ [8]. In addition, rectifying curves are studied by many researchers in different spaces and different dimensions [9–13].

Fractional calculus represents a generalization of classical derivative and integral concepts, extensively explored by contemporary researchers. The notion of fractional derivative essentially refers to derivatives of non-integer order. Remarkably, the concepts of fractional derivatives and integrals have an age parallel to that of integer derivatives and integrals, with the term "fractional derivative" first mentioned in Leibniz's 1695 letter to L'Hospital, as documented in various sources. In this letter, Leibniz posed the query, "Can the notion of integer derivatives be extended to fractional derivatives?" The concept of fractional calculus has captivated the interest of numerous mathematicians, becoming a broad area of study. It has emerged as an indispensable cornerstone across various domains within basic sciences and engineering, purportedly offering more nuanced numerical results, particularly in the realm of differential equation solutions. Fractional calculus has gained widespread popularity, leading to diverse definitions and features proposed by numerous researchers. Notable among these are the Riemann-Liouville (R-L), Caputo, Grünwald-Letnikov, Wely, and Riesz fractional derivatives [14–16]. While they share common features, each fractional calculus possesses unique rules. For instance, non-local fractional derivative types diverge from satisfying the classical Leibniz and chain rules. Notably, except for the Caputo fractional derivative, the derivative of a constant is non-zero in non-local fractional derivatives [17]. On the other hand, local fractional derivatives such as Conformable, Alternative, M-fractional, and V-fractional adhere to satisfying Leibniz's and the chain rule. Hence, local fractional derivatives hold an advantageous position in algebraically constructed subjects due to their adherence to these classical rules [18–22]. The author in [23] studied more features of conformable fractional derivatives where the endpoints are allowed to appear in the weight of the conformable integral to define the concepts of left and right conformable derivatives. Recalling that, if a function f is differentiable then its conformable derivative $D_\alpha f(t)$ of order $\alpha \in (0, 1]$ will equal to $t^{1-\alpha} f'(t)$, we can relate conformable derivatives to a fractal type. Indeed, we have

$$\lim_{t \to s} \frac{f(t) - f(s)}{t^\alpha - s^\alpha} = \lim_{t \to s} \frac{f(t) - f(s)}{t^\alpha - s^\alpha} . \frac{t - s}{t - s}. \tag{1}$$

Then, we have

$$\lim_{t \to s} \frac{f(t) - f(s)}{t^\alpha - s^\alpha} = \lim_{t \to s} \frac{t - s}{t^\alpha - s^\alpha} f'(t) = \frac{1}{\alpha} t^{1-\alpha} f'(t). \tag{2}$$

That is

$$\lim_{t \to s} \frac{f(t) - f(s)}{t^\alpha - s^\alpha} = \frac{1}{\alpha} D_\alpha f(t). \tag{3}$$

The theory of curves involves examining the movement of a point within a plane or space using tools from linear algebra and calculus. In recent research, there's a notable trend where fractional calculus has begun to be applied to the study of curves and surfaces within differential geometry. This trend was initiated with the pioneering work of T. Yajima and K. Kamasaki, who conducted the first study employing fractional calculus to analyze surfaces [24]. Subsequently, T. Yajima et al. extended this exploration by deriving Frenet formulas using fractional derivatives [25]. Another significant contribution came from K.A. Lazopoulos and A.K. Lazopoulos, who delved into the realm of fractional differentiable manifolds [26]. Additionally, M.E. Aydın et al. investigated plane curves within the context of fractional order equiaffine geometry [27]. Exploring the foundational concepts of curves and the Frenet frame within the domain of fractional order, U. Gozutok et al. conducted an analysis utilizing conformable local fractional derivatives [28]. Furthermore, A. Has and B. Yılmaz investigated specific curves and curve pairs within the context of fractional order, employing conformable Frenet frames [29, 30]. Moreover, the exploration of electromagnetic fields and magnetic curves under fractional derivatives has been undertaken by A. Has and B. Yılmaz [31–33]. These studies collectively showcase the burgeoning interest and application of fractional calculus in diverse aspects of curve theory, bringing forth new insights and methodologies within the field of differential geometry.

In this study, algebraic and calculus-based properties of curves are reconstructed with the help of conformable local fractional derivatives. First of all, line, plane and sphere, which are the most basic concepts of geometry, are redefined in fractional order. Afterward, the concepts of unit and orthogonality, which are the algebraic basis of curves, are defined in accordance with the fractional order. Then, the conformable frame of the conformable naturally parameterized curve is defined. Throughout this study, definitions based on conformable analysis are denoted by $\mathcal{C}_\alpha$. For example $\mathcal{C}_\alpha-$frame, $\mathcal{C}_\alpha-$naturally parameterized curve etc. It should be noted here that the conformable frame defined in this study is different from the frame discussed in the study [28]. The conformable frame mentioned in this article is completely defined by the vectors' conformable local fractional derivative and gives different results from the classical Frenet frame. In addition, the rectifying curves defined by Chen and also called Chen curves in the article are examined with a conformable local fractional derivative and their fractional order characterizations are obtained. Finally, in the study, examples of the concepts obtained from fractional order are given and their graphs are drawn.

2. Preliminaries

2.1. Basics parametrized curves

A regular natural parametrization of class C^k, with $k \geq 1$ of a curve in $\mathbb{R}^3$ is a vector valued function $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$, $s \mapsto \mathbf{x}(s) = (\mathbf{x}_1(s), \mathbf{x}_2(s), \mathbf{x}_3(s))$ defined on an interval I which satisfies x is of class C^k and $\mathbf{x}'(s) \neq 0$ for all $s \in I$ where $\mathbb{E}$ denotes Euclidean space. A curve $\mathbf{x}$ is continuously differentiable if $\mathbf{x}'(s)$ exists for all $s \in I$ and the derivative $\mathbf{x}'(s)$ is a continuous function; thinking dynamically, the vector $\mathbf{x}'(s)$ is the velocity of the curve at time s. We call $\mathbf{x}(s)$ *a naturally parametrized curve* if $\mathbf{x}_i(s)$ $(i = 1, 2, 3)$ is of class C^k and $\|\mathbf{x}'(s)\| = 1$, for each $s \in I$ [34].

Let $\mathbf{x}(s)$ be *biregular*, that is, $\mathbf{x}'(s) \times \mathbf{x}''(s) \neq 0$, for each $s \in I$. We consider a trihedron $\{T(s), N(s), B(s)\}$ along $\mathbf{x}(s)$, so-called *Frenet frame*, where [34]

$$T(s) = \mathbf{x}'(s), \quad N(s) = \frac{T'(s)}{\|T'(s)\|}, \quad B(s) = T(s) \times N(s).$$

The curvature κ, a non-negative scalar field, is defined by setting $\kappa(s) = \|T'(s)\|$ and torsion is defined by setting $\tau(s) = \langle N'(s), B(s)\rangle$. The naturally parametrized curve $\mathbf{x}$ has unit speed and strictly positive curvature then the following equations hold [34]

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{4}$$

2.2. Basics in conformable fractional calculus

Given $s \mapsto x(s) \in \mathbb{E}^3$, $s \in I \subset \mathbb{R}$, the *conformable derivative* of x at s is defined by [19]

$$D_\alpha(x)(s) = \lim_{\varepsilon \to 0} \frac{x(s + \varepsilon s^{1-\alpha}) - x(s)}{\varepsilon}.$$

Let $Dx(s) = dx(s)/ds$. We then notice

$$D_\alpha x(s) = s^{1-\alpha} dx(s)/ds.$$

Denote by $D_\alpha x(s)$ the α-th order conformable derivative of $x(s)$ for each $s > 0$, $0 < \alpha < 1$.

It can be said that the conformable derivative provides some properties such as linearity, Leibniz rule and chain rule as in the classical derivative as follows

(i) $D_\alpha(ax + by)(s) = aD_\alpha(x)(s) + bD_\alpha(y)(s)$, for all $a, b \in \mathbb{R}$,

(ii) $D_\alpha(s^p) = ps^{p-\alpha}$ for all $p \in \mathbb{R}$,

(iii) $D_\alpha(\lambda) = 0$, for all constant functions $x(s) = \lambda$,

(iv) $D_\alpha(xy)(s) = x(s)D_\alpha y(s) + y(s)D_\alpha x(s)$,

(v) $D_\alpha(\frac{x}{y})(s) = \frac{x(s)D_\alpha y(s) - y(s)D_\alpha x(s)}{y^2(s)}$,

(vi) $D_\alpha(y \circ x)(s) = x(s)^{\alpha-1} D_\alpha x(s) D_\alpha y(x(s))$

where x, y be conformable differentiable for each $s > 0$ and $0 < \alpha < 1$ [19].

The conformable integral is defined as the inverse operator to the conformable derivative. Specifically, the conformable integral of a function x(s) is formally expressed as [19]

$$I_\alpha^a f(t) = I_1^a(t^{\alpha-1} f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx.$$

The impact of conformable analysis on vector-valued functions is a subject of investigation, exploring both the limits and derivatives of these functions within this framework. The subsequent theorem delineates the formulation of the conformable derivative applied to vector-valued functions.

Theorem 1. *[35] Let $x = (x_1(s), x_2(s), x_3(s), ..., x_n(s))$ be a vector-valued function with n variables. So x is $\alpha-$differentiable at $s \in \mathbb{R}$, as follows*

$$D_\alpha x(t) = (D_\alpha x_1(t), ..., D_\alpha x_m(t)).$$

3. Conformable parametrized curves and their conformable frame

In this section, basic vector operations and parameterized curves will be reconstructed with conformable calculus. First of all, let's define the concepts of conformable angle and conformable orthogonality, which are the most important concepts of geometry, with the help of conformable calculus as follows.

The geometric interpretation of the conformable derivative is based on the notion of fractal geometry. In fractal geometry, objects exhibit self-similarity at different scales. The conformable derivative captures this self-similar behavior of a function by considering its local fractional variations. Geometrically, it can be understood as analyzing the "zooming in" behavior of the function at that point, similar to the classical derivative capturing the local linear behavior. Overall, the geometric interpretation of the conformable derivative relates to the self-similarity and scaling properties of functions, enabling us to understand their behavior at different levels of detail and resolution. *More specifically, the conformable derivative can be explained as a measure of how much a straight line and plane bends to form a curve and a surface.* Figure 1 shows how a line is curved with the conformable calculus effect.

Example 1. *Let consider the $s \mapsto \mathbf{x}(s) = (s, \int s^{1-\alpha} ds)$, $\mathcal{C}_\alpha-$line passing through the point $P = (0, 0)$ and whose direction is $v = (s^{1-\alpha}, s^{1-\alpha})$.*
In Figure 1 we present the graph of the conformable line for different α values.

As seen in Figure 1, there is no classical line in the $\mathcal{C}_\alpha-$ (fractional) system. This is only achieved when $\alpha \to 1$. Accordingly, it requires a new concept of angle in $\mathcal{C}_\alpha-$space. This angle is called the $\mathcal{C}_\alpha-$ angle, which gives the angle between two $\mathcal{C}_\alpha-$ lines. In addition, the concept of orthogonality in this $\mathcal{C}_\alpha-$ space is different from the classical one. Because we cannot talk about classical directness, we cannot talk about steepness in the classical sense. We will explain this below.

Notation: Along the study, expressions that are equal to 1 when $\alpha \to 1$ will be denoted as $\mathbf{1}_\alpha$, and expressions that are equal to 0 when $\alpha \to 1$ will be denoted as $\mathbf{0}_\alpha$.

Suppose that $\mathbf{x}$ and $\mathbf{y}$ are $\mathcal{C}_\alpha-$unit vector that is, they are vectors of the form $\|\mathbf{x}\| = \mathbf{1}_\alpha$ and $\|\mathbf{y}\| = \mathbf{1}_\alpha$. Then, the *$\alpha-$conformable radian measure* of *$\mathcal{C}_\alpha-$angle* between $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\theta_\alpha = \arccos\left(\frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|\|\mathbf{y}\|}\right).$$

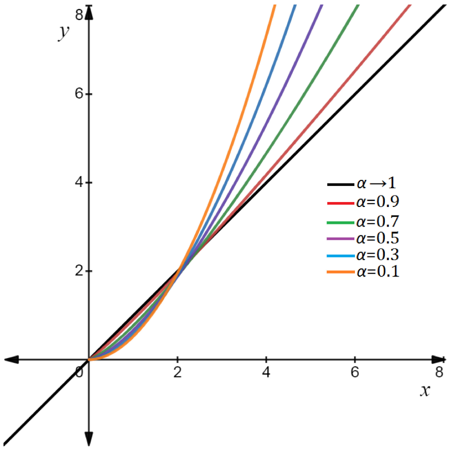

Figure 1: Transformation from line to curve.

In this sense,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{0}_\alpha$$

when $\mathbf{x}$ and $\mathbf{y}$ are $\mathcal{C}_\alpha-$orthogonal. For example, the $\mathcal{C}_\alpha-$ vectors $u = (s^{1-\alpha}, 1-\alpha, \frac{1}{s^{1-\alpha}})$ and $v = (\frac{1-\alpha}{s^\alpha}, s^\alpha, 2-2\alpha)$ are orthogonal to each other in the $\mathcal{C}_\alpha-$ sense, and we present this in Figure 2.

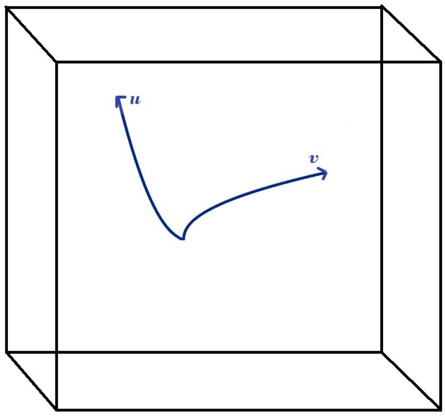

Figure 2: $\mathcal{C}_\alpha-$orthogonal vectors.

In addition, vectors u, v and $u \times v$ form the fractional orthogonal system. For example, if $u = (s^{1-\alpha}, 1-\alpha, \frac{1}{s^{1-\alpha}})$ and $v = (\frac{1-\alpha}{s^\alpha}, s^\alpha, 2-2\alpha)$, it becomes $u \times v = (2\alpha^2 - 4\alpha - s^{2\alpha-1} + 2, 2\alpha s^{1-\alpha} - 2s^{1-\alpha} - \frac{\alpha}{s} + \frac{1}{s}, -\alpha^2 s^{-\alpha} + 2\alpha s^{-\alpha} - s^{-\alpha} + s)$. The fractional orthogonal system is shown in Figure 3.

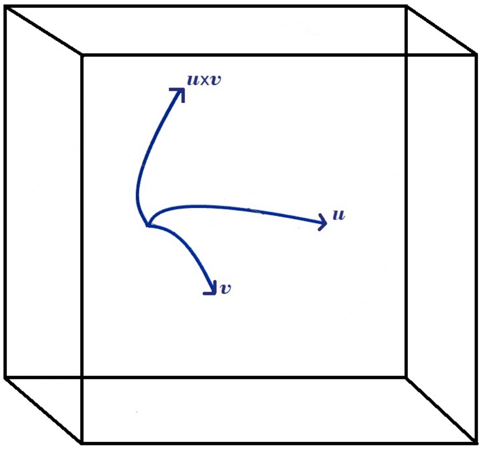

Figure 3: $\mathcal{C}_\alpha-$orthogonal system.

Let $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$ be a vector-valued function where $s \mapsto \mathbf{x}(s) = (\mathbf{x}_1(s), \mathbf{x}_2(s), \mathbf{x}_3(s))$. Then, $D_\alpha \mathbf{x}(s) = (D_\alpha \mathbf{x}_1(s), D_\alpha \mathbf{x}_2(s), D_\alpha \mathbf{x}_3(s))$. We call $\mathbf{x}(s)$ $\mathcal{C}_\alpha-$*naturally parametrized curve* if $\mathbf{x}_i(s)$ $(i = 1, 2, 3)$ is of class C^α and $\|D_\alpha \mathbf{x}(s)\| = s^{1-\alpha}$, for each $s \in I$. Here α is the maximum order that we will need.

In the remaining part, unless otherwise specified, we will assume that $\mathbf{x}(s)$ in $\mathbb{E}^3$ is a $\mathcal{C}_\alpha-$naturally parametrized curve.

Let $\mathbf{x}(s)$ be $\mathcal{C}_\alpha-$*biregular*, that is, $D_\alpha \mathbf{x}(s) \times D^2_\alpha \mathbf{x}(s) \neq 0_\alpha$, for each $s \in I$. We consider a trihedron $\{E_1(s), E_2(s), E_3(s)\}$ along $\mathbf{x}(s)$, so-called $\mathcal{C}_\alpha-$*frame*, where

$$E_1(s) = D_\alpha \mathbf{x}(s), \quad E_2(s) = \frac{D_\alpha E_1(s)}{\|D_\alpha E_1(s)\|}, \quad E_3(s) = E_1(s) \times E_2(s). \tag{5}$$

where $\{E_1(s), E_2(s), E_3(s)\}$ trihedron is called $\mathcal{C}_\alpha-$tangent, $\mathcal{C}_\alpha-$principal normal and the $\mathcal{C}_\alpha-$binormal of the $\mathcal{C}_\alpha-$curve $\mathbf{x}$, respectively. Also, considering Eq. (5) $\mathcal{C}_\alpha-$tangent, $\mathcal{C}_\alpha-$principal normal and $\mathcal{C}_\alpha-$binormal of the $\mathcal{C}_\alpha-$curve $\mathbf{x}$, they different from the Frenet vectors by the effect of the conformable calculus. However, these vectors turn into Frenet vectors, respectively, in case $\alpha \to 1$. In addition, the set $\{E_1(s), E_2(s), E_3(s)\}$ is mutually $\mathcal{C}_\alpha-$orthogonal and $\mathcal{C}_\alpha-$unit speed vectors.

We call $\kappa_\alpha(s) = \|D_\alpha E_1(s)\|$ $\mathcal{C}_\alpha-$*curvature* and $\tau_\alpha(s) = \langle D_\alpha E_2(s),\ E_3(s)\rangle$ $\mathcal{C}_\alpha-$*torsion*. The $\mathcal{C}_\alpha-$*frame formulae* are now [36]

$$\begin{bmatrix} D_\alpha E_1 \\ D_\alpha E_2 \\ D_\alpha E_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ -\kappa_\alpha & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}. \tag{6}$$

Conclusion 1. ***(What is the advantage of $\mathcal{C}_\alpha-$frame?)***
The concept of α-differentiability offers a distinctive perspective where functions exhibit α-differentiability at points where classical differentiability fails. For instance, consider

the function $f(t) = 2\sqrt{t}$. *At* $t = 0$, *the classical derivative* $f'(0)$ *doesn't exist. However, employing the conformable fractional derivative yields the result* $D_{\frac{1}{2}} f(0) = 1$ *quite straightforwardly. In this example, function* f *lacks a classical tangent at the point* $t = 0$, *but an approximation to this tangent is achievable through the conformable fractional derivative. The Frenet frame of a curve heavily relies on the existence of the curve's tangent at a given point. Yet, the* $\mathcal{C}\alpha$*-frame resolves this issue. In points within the* $\mathcal{C}\alpha$*-frame where the curve's tangent is absent, fractional values are assigned to approximate the tangent at that specific point. Moreover, when* $\alpha \to 1$, *the* $\mathcal{C}\alpha$*-frame aligns with the Frenet frame. In such instances, the* $\mathcal{C}\alpha$*-frame encompasses the classical Frenet frame while extending advantages to researchers dealing with points where the Frenet frame lacks definition.*

Theorem 2. *[36] Let* $\mathbf{x} = \mathbf{x}(s)$ *be* $\mathcal{C}_\alpha-$*naturally parametrized curve in the Euclidean* $3-$*space where* s *measures its* $\mathcal{C}_\alpha-$*arc length. When* $\alpha \to 1$, *as follows*

$$\kappa_\alpha = s^{1-\alpha}\sqrt{(1-\alpha)^2 s^{-2\alpha} + s^{2-2\alpha}\kappa^2}. \tag{7}$$

and

$$\tau_\alpha = \frac{s^{5-5\alpha}\kappa^2}{\kappa_\alpha^2}\tau. \tag{8}$$

Proposition 1. *Let the* $\alpha-$*conforamable frame of a naturally parameterized* $\alpha-$*conformable* $\mathbf{x}$ *curve be* $\{E_1, E_2, E_3\}$ *and the* $\alpha-$*conformable curvature and torsion be* κ_α *and* τ_α, *respectively. If the* $\alpha-$*conformable curve* $\mathbf{x}$ *lies on the* $\alpha-$*conformable sphere* $\mathbb{S}^2_\alpha(C_\alpha, r_\alpha)$, *the curve* $\mathbf{x}$ *is called* $\alpha-$*conformable spherical curve. Then* $\langle \mathbf{x}, E_1\rangle = 0_\alpha$ *and as follows*

$$\mathbf{x}(s) = C_\alpha + 0_\alpha\lambda_1 - \frac{1_\alpha}{\kappa_\alpha}E_2 - \frac{1_\alpha}{\tau_\alpha}\left(\frac{1_\alpha}{\kappa_\alpha}\right)' E_3.$$

Example 2. *Let* $\alpha-$*conformable spherical curve be* $\mathbf{y}$ *in* $\mathbb{S}^2_\alpha$ *for* $r_\alpha = u^{\alpha-1}$, $C_\alpha = (0,0,0)$ *given by the parametrization*

$$\mathbf{y}(s) = \left(\frac{1}{2}u^{\alpha-1}\cos 2s, \frac{1}{2}u^{\alpha-1}\sin 2s, \frac{\sqrt{3}}{2}u^{\alpha-1}\right). \tag{9}$$

In Figure 4, we present a figures of the $\alpha-$*conformable spherical curve according to different values of* α.

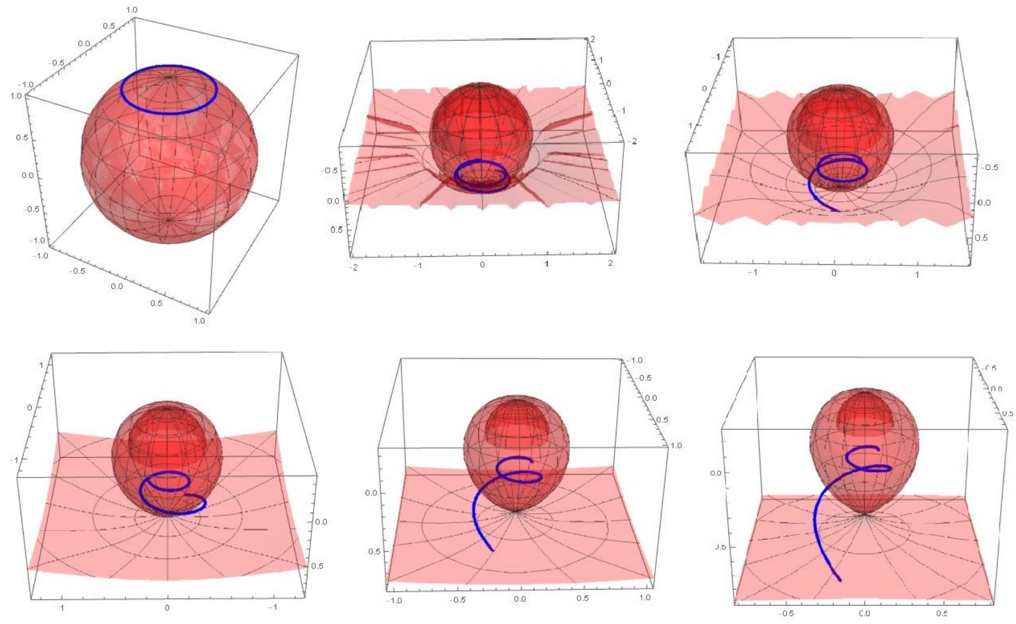

Figure 4: $\alpha-$Conformable spherical curve $\mathbf{y}(s)$ for for $\alpha \to 1,\ \alpha = 0.9,\ \alpha = 0.7,\ \alpha = 0.5,\ \alpha = 0.3$ *and* $\alpha = 0.1$, respectively.

4. $\mathcal{C}_\alpha-$rectifying curves

Let $\mathbf{x}(s)$ be a $\mathcal{C}_\alpha-$naturally parametrized curve in $\mathbb{E}^3$, $s \in I \subset \mathbb{R}$ and $\{E_1(s), E_2(s), E_3(s)\}$ denotes the $\mathcal{C}_\alpha-$frame. Suppose that $\mathcal{C}_\alpha-$biregular. We call $\mathbf{x}(s)$ $\mathcal{C}_\alpha-$*rectifying curve* if there is a linear relation as

$$\mathbf{x}(s) = \lambda(s)E_1(s) + \mu(s)E_3(s). \tag{10}$$

Here the functions $\lambda(s)$ and $\mu(s)$ are of class C^n on I called $\mathcal{C}_\alpha-$*tangential* and $\mathcal{C}_\alpha-$*binormal components* of $\mathbf{x}(s)$, respectively. Hence, $\lambda(s) = \langle \mathbf{x}(s), E_1(s)\rangle$ and $\mu(s) = \langle \mathbf{x}(s), E_3(s)\rangle$. Remark also that $\mathcal{C}_\alpha-$rectifying curve $\mathbf{x}(s)$ holds $\langle \mathbf{x}(s), E_2(s)\rangle = 0$ for each $s \in I$.

Taking a conformable differentiation in Eq. (10) together with considering $\mathcal{C}_\alpha-$frame formulae, we have

$$(D_\alpha\lambda - 1)E_1 + (\lambda\kappa_\alpha - \mu\tau_\alpha)E_2 + D_\alpha\mu E_3 = 0. \tag{11}$$

According to this equation, the following results are given

$$D_\alpha\lambda = 1, \tag{12}$$

$$\frac{\tau_\alpha}{\kappa_\alpha} = \frac{\lambda}{\mu}, \tag{13}$$

$$D_\alpha\mu = 0. \tag{14}$$

Conclusion 2. *From Eqs. (12) and (14) one concludes that* $\lambda(s) = \frac{s^\alpha}{\alpha} + c_1$ *and* $\mu(s) = c_2$ *for* $c_1, c_2 \in \mathbb{R}$. *Notice here that* $c_2 \neq 0$ *because otherwise one derives from in Eq.* (13) *that* $\kappa_\alpha(s)$ *is* 0. *This contradicts with biregularity of* $\mathbf{x}(s)$. *Analogously,* $\tau_\alpha(s)$ *is nowhere* 0. *In conclusion, we deduce that a* $\mathcal{C}_\alpha-$*rectifying curve has to be* $\mathcal{C}_\alpha-$*twisted.*

Theorem 3. *Let* $\mathbf{x}(s) \subset \mathbb{E}^3$ *be a* $\mathcal{C}_\alpha-$*rectifying curve and* $\kappa_\alpha \neq 0$ *and* τ_α *is nowhere* 0_α. *Then*

$$\rho^2(s) = 1_\alpha(s^{2\alpha} + cs^\alpha + d) \tag{15}$$

where $\rho(s) = \|\mathbf{x}(s)\|$ *is the distance function,* c *and* d *real numbers.*

Proof. Consider the a $\mathcal{C}_\alpha-$rectifying curve $\mathbf{x}(s)$ in Eq. (10), so the following equation exists

$$\rho^2(s) = \langle \mathbf{x}(s), \mathbf{x}(s)\rangle = \|E_1\|\lambda^2(s) + \|E_3\|\mu^2(s)$$

where E_1 and E_3 are $\mathcal{C}_\alpha-$unit speed vectors. So this equation is edited

$$\rho^2(s) = 1_\alpha(\frac{s^{2\alpha}}{\alpha^2} + 2c_1\frac{s^\alpha}{\alpha} + c_2^2) \tag{16}$$

or

$$\rho^2(s) = \frac{1_\alpha}{\alpha^2}(s^{2\alpha} + 2c_1\alpha s^\alpha + c_2^2\alpha^2). \tag{17}$$

Here, when $\alpha \to 1$, since $\frac{1_\alpha}{\alpha^2} = 1$ is $\frac{1_\alpha}{\alpha^2} = 1_\alpha$ can be written. Also, since α is a real number, if $2c_1\alpha = c$ and $c_2^2\alpha^2 = d$ are selected, we get the following

$$\rho^2(s) = 1_\alpha(s^{2\alpha} + cs^\alpha + d). \tag{18}$$

Theorem 4. *Let* $\mathbf{x}(s) \subset \mathbb{E}^3$ *be a* $\mathcal{C}_\alpha-$*rectifying curve. Then the ratio of the conformable curvatures is for* $a, b \in \mathbb{R}$

$$\frac{\tau_\alpha}{\kappa_\alpha} = as^\alpha + b.$$

Proof. From Eq. (13), we know the following equation

$$\frac{\tau_\alpha}{\kappa_\alpha} = \frac{\lambda}{\mu}.$$

Now, if we use the results of the λ and μ conformable differentiable equations available in Conclusion 2 in the above equation, we get

$$\frac{\tau_\alpha}{\kappa_\alpha} = \frac{\frac{s^\alpha}{\alpha} + c_1}{c_2}.$$

or

$$\frac{\tau_\alpha}{\kappa_\alpha} = \frac{s^\alpha}{\alpha c_2} + \frac{c_1}{c_2}$$

Since c_1, c_2 and α are real numbers, if we select new real numbers as $\frac{1}{\alpha c_2} = a$ and $\frac{c_1}{c_2} = b$, it can be easily seen that the following equation is achieved

$$\frac{\tau_\alpha}{\kappa_\alpha} = as^\alpha + b.$$

Theorem 5. *Let* $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$ *be a* $\mathcal{C}_\alpha-$*rectifying curve* E^3 *with* $\kappa_\alpha \neq 0_\alpha$. *In this case, the following result applies*

$$\mathbf{x}(s) = 1_\alpha \sqrt{f(t)^{2\alpha} + n^{2\alpha}}\mathbf{y}(s), \tag{19}$$

where $f(t)$ *is a conformable differentiable function and* n *is a positive number and* $\mathbf{y} = \mathbf{y}(s)$ *is a* $\mathcal{C}_\alpha-$*unit speed spherical curve in* S^2_α.

Proof. Let $\mathbf{x}(s) \subset \mathbb{E}^3$ be a $\mathcal{C}_\alpha-$rectifying curve with $\kappa_\alpha \neq 0_\alpha$. Suppose that 0_α lies in I and $\mathbf{x} = \mathbf{x}(s)$ is $\mathcal{C}_\alpha-$unit speed curve. By the Eq. (18), the distance function $\rho = \|\mathbf{x}\|$ of the curve satisfies $\rho^2(s) = 1_\alpha(s^{2\alpha} + cs^\alpha + d)$ for some constant c and d. After a conformable translation in s, we may take $\rho^2(s) = 1_\alpha(s^{2\alpha} + m)$, for some constant m. Since $0_\alpha \in I$, $m > 0_\alpha$. Therefore, we may put $m = n^{2\alpha}$ for $n \in \mathbb{R}^+$. Introduce a spherical curve $\mathbf{y}(s)$ as

$$\mathbf{x}(s) = 1_\alpha \sqrt{s^{2\alpha} + n^{2\alpha}}\mathbf{y}(s). \tag{20}$$

By taking the conformable derivative of this equation according to s, we get

$$D_\alpha \mathbf{x} = \left(0_\alpha \sqrt{s^{2\alpha} + n^{2\alpha}} + \frac{1_\alpha \alpha s^\alpha}{\sqrt{s^{2\alpha} + n^{2\alpha}}} \right) \mathbf{y} + 1_\alpha \sqrt{s^{2\alpha} + n^{2\alpha}} D_\alpha \mathbf{y} \tag{21}$$

Note that the conformable derivative of $D_\alpha 1_\alpha = 0_\alpha$. Also, since $\mathbf{y}$ is $\mathcal{C}_\alpha-$unit and $\mathbf{y}$ and $D_\alpha \mathbf{y}$ are $\mathcal{C}_\alpha-$orthogonal, from Eq. (21) the following equation is obtained

$$\|D_\alpha \mathbf{y}\| = \sqrt{\frac{1_\alpha s^{2\alpha}(1 - \alpha^2) + n^{2\alpha}}{(s^{2\alpha} + n^{2\alpha})^2}}$$

where it is clear that $\|D_\alpha \mathbf{y}\|$ is the $\mathcal{C}_\alpha-$velocity of the $\mathcal{C}_\alpha-$spherical curve $\mathbf{y}$. Then

$$t = I_0^s \sqrt{\frac{1_\alpha s^{2\alpha}(1 - \alpha^2) + n^{2\alpha}}{(s^{2\alpha} + n^{2\alpha})^2}} = f^{-1}(s).$$

So, $s = f(t)$ is obtained. Here $f(t)$ is a function that gives the result $n.tant$ when $\alpha \to 1$. If this result is written in Eq (20), we get

$$\mathbf{x}(s) = 1_\alpha \sqrt{f(t)^{2\alpha} + n^{2\alpha}}\mathbf{y}(s). \tag{22}$$

5. Conclusion

While ordinary analysis and the differential geometry are related to ordinary derivatives, fractional calculus provides us with the more fractional analysis which depends on differentiation and integration with respect to arbitrary order. In the last few decades, fractional analysis has been used extensively in almost all basic sciences, especially in Physics, Chemistry and Engineering. It is claimed that fractional analysis gives more numerical results than classical analysis. This makes fractional analysis more advantageous.

Since conformable derivatives have several well-behaved properties like ordinary derivatives, in this work the basic concepts of geometry have been re-examined and studied in the frame of conformable fractional analysis. Our new differential geometry concepts we have studied give the readers a more general approach to deal with. The limiting case $\alpha \to 1$ sends us back to the classical geometry.

- The $\mathcal{C}_\alpha-$ frame has been constructed differently from previous similar studies and the classical Frenet frame. The advantage of this frame is that when $\alpha \to 1$ it gives the classical Frenet frame, it also gives the opportunity to examine the frame of the curve for all cases in the range of $0 < \alpha < 1$. In other words, it exhibits a more general situation compared to the classical Frenet frame.

- Curves defined according to the $\mathcal{C}_\alpha-$frame take on a different variation of the curve for each α value.

- An example of this can be seen very well in Eq. (22). For each α value in this equation, the function $f(t)$ will give a different result. Thus, the $\mathbf{x}$ curve will turn into a different a $\mathcal{C}_\alpha-$rectifying curve in a conformable sense according to each α value, and when $\alpha \to 1$ it will turn into a classical rectifying curve.

- The variation of the $\mathcal{C}_\alpha-$frame at any point of the $\mathcal{C}_\alpha-$curve and of each defined curve depending on this frame within the range of $0 < \alpha < 1$ can be examined. In addition, a curve can be generated for the resulting $\mathcal{C}_\alpha-$frame for each α value.

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