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Signed Fuzzy Graphs Domination in Environmental Monitoring Wireless Sensor Networks

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Abstract. Signed Fuzzy Graphs(Signed-FG), a mathematical structure having nodes associated with membership values and edges consisting of positive and negative signs, present a multi-faceted mathematical model to address uncertainty and polarity in graph-based networks. Exploring the theories of Signed-FG and its applications helps us understand complex networks and their characteristics. This study intends to examine the key theorems pertaining to vertex and edge degrees and their membership values by highlighting their real-time applications in Wireless Sensor Networks (WSNs) and Environmental Monitoring Systems (EMSs), where sensor nodes must maintain reliable communication. The main focus of this paper shifts from the exploration of theorems about vertex and edge degree and their membership values to the introduction of new domination metrics tailored for Signed-FGs, the domination sets and minimum domination sets, and their relevance to efficient placement and the operation of WSNs in EMSs. Signed-FGs facilitate effective data collection by ensuring network coverage. This study makes a major contribution to the field of Fuzzy Mathematics with a comprehensive understanding of complex networks and their characteristics.

2020 Mathematics Subject Classifications: 05C72, 05C22, 05C69, 05C12

Key Words and Phrases: Signed fuzzy graphs (Signed-FG), wireless sensor networks (WSN), environmental monitoring systems (EMS)

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1. Introduction

Graphs have long been recognized as powerful tools for modeling relationships, providing a simple way to represent connections between objects. In this model, objects are

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represented as vertices, and the relationships between them are defined as edges. Although they are helpful in many situations, traditional graph models are based on binary relationships with edges that may or may not exist. However, ambiguity or confusion in the definition of items, their relationships, or both are common in the real world. Binary edges can be used to make things like the quality of sensor data in environmental monitoring, the strength of friendships in social networks, and the dependability of transportation routes sufficiently variable and unexpected. By giving vertices and edges membership values, fuzzy graphs enable a more accurate description of relationships, offering a more reliable framework in this situation. They do not, however, adequately capture the dichotomous character of relationships, which can be either positive (supporting) or negative (antagonistic). Relationships are not just ambiguous but also divisive in many applications, including wireless sensor networks, ecosystems, and conflict and cooperation studies. Examples of connections and relationships include support and influence in social communication, relationships and relationships in ecosystems, and collaboration and competition in social or political interactions.

By adding the idea of polarity, signed-FGs expand on the notion of fuzzy graphs. A relationship value, which indicates the strength of the link, and a sign, which indicates whether the relationship is positive or negative, are characteristics of each edge in a Signed-FG. Signature fuzzy images can capture small changes in systems where the nature of the interaction and uncertainty are both significant thanks to this binary representation. By altering the polarity of the interactions to produce a framework, signed-FGs further enhance these properties. When combined, these models offer a cohesive framework for examining intricate communication systems across a range of fields, such as wireless communications, social networks, and environmental monitoring.

A more accurate framework for simulating environmental monitoring networks is offered by Signed-FGs, which depict both the type of interaction (positive or negative signs) and the degree of connectivity (fuzzy weights) between sensor nodes. They outperform traditional crisp graph models for real-world monitoring applications because they can capture changes in link quality, take interference or malfunctioning nodes into account, adjust to shifting environmental conditions, and support richer network analysis.

Graph[1] can be viewed as a binary relation between nonempty V. The basis of fuzzy graphs in the theory of fuzzy relations and fuzzy sets mentioned by Zadeh in 1965 [2]. This concept was introduced by Rosenfeld [3] in 1975, who discovered the relationship between fuzzy sets and paved the way for the research of fuzzy trees, blocks, bridges and cutting nodes. In [3], the ideas of blocks, cut nodes, fuzzy trees and bridges, in fuzzy graphs were examined. Yeh [4] established several concepts of connectedness in fuzzy graphs around the same period. Numerous academics have explored this area further since the seminal publications of Rosenfeld [3]. Yeh [4] in 1975, which described fundamental theoretic fuzzy graph notions and applications. They have discovered fuzzy analogues of many theoretic-graph concepts. These consist of fuzzy interval graphs [5, 6], fuzzy trees [4, 7], fuzzy line graphs [8], automorphisms of fuzzy graphs [9], and cycles and cocycles of fuzzy graphs [10].

In [4], many definitions of fuzzy trees that are compatible with cut-level representations were presented, along with the ideas of connectedness and acyclicity degrees for fuzzy graphs. The notions of dominance and absolute domination in fuzzy graphs were developed by Somasundaram [11, 12]. They also established boundaries and calculated the domination number for different sets of fuzzy graphs. Union, join, composition, and Cartesian product are among the operations on fuzzy graphs that Somasundaram [13] further examined and determined their dominating parameters. Moderson[7] examined fuzzy graphs with varying degrees of connectivity.

It was demonstrated how the structural characteristics of finite FGs can be used to solve operational research issues. The authors of the same work examined the properties of cut nodes, bridges, trees, and fuzzy cycles in fuzzy graphs. In order to demonstrate the interconnectedness of FGs, Nagoor Gani [14] looked at the interactions between degree, order, and size. Bhutani [15] created fuzzy end nodes in fuzzy graphs, showing that no node could be both a cut node and a fuzzy end node simultaneously. Additionally, they analyzed fuzzy end node characteristics in fuzzy trees and described fuzzy cycles that lacked fuzzy end nodes or cut nodes.

Strong arcs and strong pathways in fuzzy graphs were first proposed in [16], where it was shown that while a strong arc is not always a fuzzy bridge, a fuzzy bridge is always strong. Furthermore, the authors demonstrated the characteristics of strong arcs in fuzzy trees and characterized fuzzy trees using strong pathways.

Assia Alaoui [17] expanded the concepts of external stability, external stability, external dominance, and their combinations to fuzzy graphs. Similarly, Sameena and Sunitha researched the concept of strong arcs in maximum spanning trees [18] and its applications in neural networks [19, 20]. fuzzy graphs are also used in the literature for database theory, group structure analysis, and chemical systems [21, 22].

Signed-FG was introduced by [23], its integrity was discussed by [24], the domination parameter was examined by [25], and the operations of Signed-FGs were explored by [26].

1.1. Motivation

Conventional graph theory often limits itself to binary relationships, which may be insufficient to represent real-world scenarios where connections are ambiguous or fuzzy. This research aims to address such complexities by applying fuzzy graph principles. The paper emphasizes the importance of reliable communication between sensors used in environmental monitoring and explores the practical need to maintain network coverage. By leveraging the principles of Signed-FGs, it ensures efficient data collection despite potential connectivity issues.

The study of domination sets, minimum domination numbers, and strong and effective degrees in Signed-FGs contributes to the advancement of fuzzy graph theory by strengthening its mathematical foundations. This theoretical framework is essential for developing algorithms that optimize network performance.

1.2. Organization of Content

The first section gives a detailed introduction to graph theory, establishing a solid foundation for the discussions that follow. The second section introduces the fundamentals of fuzzy graphs and signed fuzzy graphs, emphasizing key concepts such as minimum degree, maximum degree, effective degree, and standard results relevant to these structures. The third section, titled "Theoretical Framework," outlines the properties of signed fuzzy graphs and explores the interrelationships between strong arcs, effective degrees, and membership values, thereby providing a rigorous mathematical foundation for analyzing these graphs. In addition to these foundational concepts, essential terminology is introduced, including domination, domination numbers, and the strong domination number, to prepare the reader for the analysis presented in the fourth section. This section focuses on results related to domination in signed fuzzy graphs. The fifth section examines the application of domination in signed fuzzy graphs, illustrated with a numerical example that highlights its relevance to wireless sensor networks (WSNs) for environmental monitoring systems. This discussion demonstrates how domination sets can enhance communication within a WSN consisting of multiple sensors. Finally, the document concludes by summarizing the key findings and proposing directions for future research.

1.3. Preliminary Concepts and Results

Let $\Im$ be a nonempty set of nodes and $\mathfrak{E} \subseteq \Im \times \Im$ a set of edges. A crisp graph is $\Sigma^* = (\Im, \mathfrak{E})$. Following Rosenfeld, a fuzzy graph on $\Im$ is a pair of functions $\Sigma = (\sigma, \mu)$, $\quad \sigma : \Im \to [0,1], \quad \mu : \Im \times \Im \to [0,1]$, such that, for all $\beth_1, \beth_2 \in \Im$, $\mu(\beth_1, \beth_2) \leq \sigma(\beth_1) \wedge \sigma(\beth_2)$. (When the graph is undirected, assume $\mu(\beth_1, \beth_2) = \mu(\beth_2, \beth_1)$ and $\mu(\beth, \beth) = 0$.) A fuzzy subgraph $\Gamma = (\tau, \xi)$ of $\Sigma = (\sigma, \mu)$ satisfies $\tau(\beth) \leq \sigma(\beth)$ for all $\beth \in \Im$, $\quad \xi(\beth_1, \beth_2) \leq \mu(\beth_1, \beth_2)$ for all $(\beth_1, \beth_2) \in \Im \times \Im$, together with $\xi(\beth_1, \beth_2) \leq \tau(\beth_1) \wedge \tau(\beth_2)$. It is spanning if $\tau = \sigma$ (i.e., both fuzzy graphs have the same vertex memberships). A simple path of length n is a sequence of distinct nodes $\beth_0, \beth_1, \ldots, \beth_n$ with $\mu(\beth_i, \beth_{i+1}) > 0$ for all i. The strength of a path is $\min_{0 \leq i < n} \mu(\beth_i, \beth_{i+1})$.

The connectedness between $\beth_1, \beth_2 \in \Im$ is $\mathrm{CONN}_\Sigma(\beth_1, \beth_2) = \max_{P: \beth_1 \rightsquigarrow \beth_2} \min_{e \in P} \mu(e)$, the max–min strength over all paths P from $\beth_1$ to $\beth_2$. Σ is connected if every pair of nodes has $\mathrm{CONN}_\Sigma(\beth_1, \beth_2) > 0$. Σ is strong if, for every edge (i.e., for every pair with positive membership), $\mu(\beth_1, \beth_2) = \sigma(\beth_1) \wedge \sigma(\beth_2)$. It is complete if the above holds for all unordered pairs $\{\beth_1, \beth_2\}$ with $\beth_1 \neq \beth_2$.

The order and size of Σ are $|\Sigma|_V = \sum_{\beth \in \Im} \sigma(\beth), |\Sigma|_E = \sum_{\{\beth_1, \beth_2\} \subseteq \Im} \mu(\beth_1, \beth_2)$. The complement $\overline{\Sigma} = (\overline{\sigma}, \overline{\mu})$ is given by $\overline{\sigma} = \sigma, \overline{\mu}(\beth_1, \beth_2) = \sigma(\beth_1) \wedge \sigma(\beth_2) - \mu(\beth_1, \beth_2)$ for all distinct $\beth_1, \beth_2 \in \Im$.

A Signed-FG is a fuzzy graph whose memberships can be positive or negative. Formally, let

$$\Sigma_{\pm} = (\Im, \sigma, \mu), \qquad \sigma : \Im \to [-1,1], \quad \mu : \Im \times \Im \to [-1,1],$$

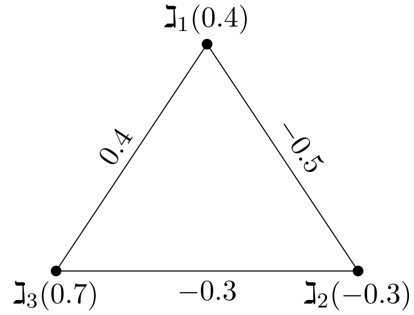

Figure 1: A Signed-FG example.

with symmetry and zero diagonals as appropriate, and satisfying the constraint

$$\mu(\gimel_1, \gimel_2) \;\le\; \sigma(\gimel_1) \wedge \sigma(\gimel_2), \quad \text{for all } (\gimel_1, \gimel_2) \in \Im \times \Im.$$

Here the sign encodes the type of relation (positive or negative), while the magnitude encodes its strength.

Definition 1. *An arc* $(\gimel_1, \gimel_2)$ *in a Signed-FG* $\sum_{\pm} = (\Im, \sigma, \mu)$ *is*

- α*-strong if* $\mu(\gimel_1, \gimel_2) > CONN_{\sum_{\pm} - (\gimel_1, \gimel_2)}(\gimel_1, \gimel_2)$.
- β*-strong if* $\mu(\gimel_1, \gimel_2) = CONN_{\sum_{\pm} - (\gimel_1, \gimel_2)}(\gimel_1, \gimel_2)$.
- δ*-arc if* $\mu(\gimel_1, \gimel_2) < CONN_{\sum_{\pm} - (\gimel_1, \gimel_2)}(\gimel_1, \gimel_2)$.

Definition 2. *An arc in Signed-FG is a strong arc if it is either an* $\alpha-$*strong or a* $\beta-$*strong arc.*

Definition 3. *In a Signed-FG, if* $\mu(\gimel_1, \gimel_2) = \sigma(\gimel_1) \wedge \sigma(\gimel_2)$ *for* $(\gimel_1, \gimel_2) \in \xi$*, then it is a effective arc.*

Definition 4. *This vertex* $\gimel_1$ *has degree* $\sum\limits_{\gimel_1 \neq \gimel_2} \mu(\gimel_1, \gimel_2)$*. The sum* $\sum_{\pm}$ *has minimal degree* $\delta_{\pm}(\gimel) = \wedge\{d_{\pm}(\gimel)/\gimel \in \Im\}$ *and maximum degree* $\Delta_{\pm}(\gimel) = \vee\{d_{\pm}(\gimel)/\gimel \in \Im\}$.

Example 1. *Consider the Signed-FG* $\Sigma_{\pm} = (\Im, \sigma, \mu)$*, where the vertex set* $\Im = \{\gimel_1, \gimel_2, \gimel_3\}$ *together with the edge set* $\Im \times \Im = \{(\gimel_1, \gimel_2), (\gimel_2, \gimel_3), (\gimel_1, \gimel_3)\}$ *whose vertex values are as follows:* $\sigma(\gimel_1) = 0.4, \sigma(\gimel_2) = -0.3$*, and* $\sigma(\gimel_3) = 0.7$ *each. Figure 1 illustrates the edge values, which are* $\mu(\gimel_1, \gimel_2) = -0.5, \mu(\gimel_2, \gimel_3) = -0.3$*, and* $\mu(\gimel_1, \gimel_3) = 0.4$*. These degrees are* $d_{\pm}(\gimel_1) = -0.1, d_{\pm}(\gimel_2) = -0.8$*, and* $d_{\pm}(\gimel_3) = 0.1$ *for* $\sum_{\pm}$*. The degree ranges from* $\delta_{\pm}(\sum_{\pm}) = -0.8$ *at the minimum to* $\Delta_{\pm}(\sum_{\pm}) = 0.1$ *at the maximum.*

Definition 5. $d_{\pm\xi}(\gimel)$ *represents the effective degree of a vertex* $\gimel$*, which is the sum of the weights of the effective arcs incident to* $\gimel$*.* $\delta_{\pm\xi}(\sum_{\pm}) = \wedge\{d_{\pm\xi}(\gimel)/\gimel \in \Im\}$ *gives the minimum sum of the degree* $\sum_{\pm}$*, and* $\Delta_{\pm\xi}(\sum_{\pm}) = \vee\{d_{\pm\xi}(\gimel)/\gimel \in \Im\}$ *gives the maximum degree.*

Example 2. *The effective arcs in Figure 1 are* $(\daleth_1, \daleth_3)$ *and* $(\daleth_2, \daleth_3)$. $d_{\pm\xi}(\daleth_1) = 0.4, d_{\pm\xi}(\daleth_2) = -0.3$, *and* $d_{\pm\xi}(\daleth_3) = 0.1$ *are the effective degrees of* $\sum_{\pm}$. $\delta_{\pm\xi}(\sum_{\pm}) = -0.3$ *is the minimum effective degree, while* $\Delta_{\pm\xi}(\sum_{\pm}) = 0.4$ *is the maximum effective degree.*

Definition 6. *The cumulative membership values of all strong arcs occurring to a node* $\daleth$ *determine its strong degree.* $d_{\pm S}(\daleth) = \sum_{\daleth \in N_s(\daleth)} \mu(\daleth, \daleth_i)$ *is the representation for it.* $\delta_{\pm S}(\sum_{\pm}) = \wedge \{d_{\pm S}(\daleth)/\daleth \in \Im\}$ *is the minimum strong degree of the sum* $\sum_{\pm}$, *and* $\Delta_{\pm S}(\sum_{\pm}) = \vee \{d_{\pm S}(\daleth)/\daleth \in \Im\}$ *is the maximum strong degree.*

Example 3. *The strong arcs in Figure 1 are* $(\daleth_1, \daleth_3)$ *and* $(\daleth_2, \daleth_3)$. $\sum_{\pm}$ *has the following strong degrees:* $d_{\pm S}(\daleth_1) = 0.4$, $d_{\pm S}(\daleth_2) = -0.3$, *and* $d_{\pm S}(\daleth_3) = 0.1$. *Strong degrees range from* $\delta_{\pm S}(\sum_{\pm}) = -0.3$ *at the minimum to* $\Delta_{\pm S}(\sum_{\pm}) = 0.4$ *at the maximum.*

Definition 7. *The definition of a vertex* $\daleth$*'s neighborhood degree is* $d_{\pm N}(\daleth) = \sum_{\daleth \in N(\daleth)} \sigma(\daleth)$. *The sum* $\sum_{\pm}$ *has a minimum neighborhood degree:* $\delta_{\pm N}(\sum_{\pm}) = \wedge \{d_{\pm N}(\daleth)/\daleth \in \Im\}$, *and a maximum neighborhood degree:* $\Delta_{\pm N}(\sum_{\pm}) = \vee \{d_{\pm N}(\daleth)/\daleth \in \Im\}$.

Example 4. *The neighborhood degrees of* $\sum_{\pm}$ *in Figure 1 are* $d_{\pm N}(\daleth_1) = 0.4, d_{\pm N}(\daleth_2) = 1.1$, *and* $d_{\pm N}(\daleth_3) = 0.1$. *Neighborhood degree is minimum* $\delta_{\pm N}(\sum_{\pm}) = 0.1$, *and effective degree is maximum* $\Delta_{\pm N}(\sum_{\pm}) = 1.1$.

Definition 8. *The formula for a vertex* $\daleth$*'s effective neighborhood degree is* $d_{\pm EN}(\daleth) = \sum_{\daleth \in EN(\daleth)} \sigma(\daleth)$. *The sum* $\sum_{\pm}$ *has a minimum effective neighborhood degree of* $\delta_{\pm EN}(\sum_{\pm}) = \wedge \{d_{\pm EN}(\daleth)/\daleth \in \Im\}$, *and a maximum effective neighborhood degree of* $\Delta_{\pm EN}(\sum_{\pm}) = \vee \{d_{\pm EN}(\daleth)/\daleth \in \Im\}$.

Example 5. *The effective arcs in Figure 1 are* $(\daleth_1, \daleth_3)$ *and* $(\daleth_2, \daleth_3)$. $d_{\pm EN}(\daleth_1) = 0.7$, $d_{\pm EN}(\daleth_2) = 0.7$, *and* $d_{\pm EN}(\daleth_3) = 0.1$ *are the effective neighborhood degrees of* $\sum_{\pm}$. $\delta_{\pm EN}(\sum_{\pm}) = 0.1$ *is the smallest effective neighborhood degree, while* $\Delta_{\pm EN}(\sum_{\pm}) = 0.7$ *is the maximum.*

Definition 9. *The definition of a vertex* $\daleth$*'s strong neighborhood degree is* $d_{\pm SN}(\daleth) = \sum_{\daleth \in N_S(\daleth)} \sigma(\daleth)$. *The sum* $\sum_{\pm}$ *has a minimum strong neighborhood degree of* $\delta_{\pm SN}(\sum_{\pm}) = \wedge \{d_{\pm SN}(\daleth)/\daleth \in \Im\}$, *and a maximum strong neighborhood degree of* $\Delta_{\pm SN}(\sum_{\pm}) = \vee \{d_{\pm SN}(\daleth)/\daleth \in \Im\}$.

Example 6. $(\daleth_1, \daleth_3)$ *and* $(\daleth_2, \daleth_3)$ *are the strong arcs in Figure 1.* $d_{\pm SN}(\daleth_1) = 0.7, d_{\pm SN}(\daleth_2) = 0.7$, *and* $d_{\pm SN}(\daleth_3) = 0.1$ *are the strong neighborhood degrees of* $\sum_{\pm}$. $\delta_{\pm SN}(\sum_{\pm}) = 0.1$ *is the smallest strong neighborhood degree, while* $\Delta_{\pm SN}(\sum_{\pm}) = 0.7$ *is the maximum.*

Theorem 1. *In a Signed-FG* $\sum_{\pm} = (\Im, \sigma, \mu)$,

(i) $d_{\pm E}(\daleth) \leq d_{\pm S}(\daleth) \leq d_{\pm}(\daleth), \forall \daleth \in \Im$.

(ii) $d_{\pm EN}(\gimel) \leq d_{\pm SN}(\gimel) \leq d_{\pm N}(\gimel)$, $\forall\ \gimel \in \Im$.

Proof. The total of the edge membership values of the effective arcs incident to $\gimel$ and the total of the strong edge membership values of the strong arcs incident to $\gimel$ makes up the effective degree of vertex $\gimel$. $d_{\pm E}(\gimel) \leq d_{\pm S}(\gimel)$ indicates that strong arcs do not necessarily have to be effective arcs, but effective arcs can be strong arcs. In a similar vein, not every arc that intersects a vertex must be a strong one. Because of this, $d_{\pm E}(\gimel) \leq d_{\pm S}(\gimel) \leq d_{\pm}(\gimel)$.The same is true for all $\gimel \in \Im$: $d_{\pm EN}(\gimel) \leq d_{\pm SN}(\gimel) \leq d_{\pm N}(\gimel)$.

Theorem 2. *In a Signed-FG* $\sum_{\pm} = (\Im, \sigma, \mu)$,

(i) $d_{\pm E}(\gimel) \leq d_{\pm EN}(\gimel)$, $\forall\ \gimel \in \Im$.

(ii) $d_{\pm}(\gimel) \leq d_{\pm N}(\gimel)$, $\forall\ \gimel \in \Im$.

(iii) $d_{\pm S}(\gimel) \leq d_{\pm SN}(\gimel)$, $\forall\ \gimel \in \Im$.

Proof. Effective arcs are defined as the total membership values of the effective arcs incident to $\gimel$, while the vertex membership values of the end vertices of these effective edges constitute the effective neighborhood degree of the vertex $\gimel$. It follows that $d_{\pm E}(\gimel) \leq d_{\pm EN}(\gimel)$ for all $\gimel \in \Im$, since the edge membership value is less than or equal to the membership values of its end vertices. Similarly, for every $\gimel \in \Im$, we have $d_{\pm}(\gimel) \leq d_{\pm N}(\gimel)$ and $d_{\pm S}(\gimel) \leq d_{\pm SN}(\gimel)$. Hence, the stated results hold.

Theorem 3. *In a Signed-FG* $\sum_{\pm} = (\Im, \sigma, \mu)$*, the total degrees of all nodes are equal to twice the total membership values of all arcs in* $\sum_{\pm}$.

Proof. Let $\Sigma_{\pm} = (\Im, \sigma, \mu)$ be a signed fuzzy graph, where $\Im = \{\gimel_1, \gimel_2, \ldots, \gimel_n\}$, σ is a signed fuzzy subset of $\Im$, and μ is a signed fuzzy subset of $\Im \times \Im$.

By Definition 4, the degree of a vertex $\gimel \in \Im$ is $d(\gimel) = \sum_{\substack{\gimel_i \in \Im \\ \gimel_i \neq \gimel}} \mu(\gimel, \gimel_i)$. Hence, $\sum_{\gimel \in \Im} d(\gimel) = \sum_{i=1}^{n} d(\gimel_i)$. Now, expanding each degree: $d(\gimel_1) = \sum_{j=1}^{n} \mu(\gimel_1, \gimel_j)$, $\quad d(\gimel_2) = \sum_{j=1}^{n} \mu(\gimel_2, \gimel_j)$, $\ldots$, $\quad d(\gimel_n) = \sum_{j=1}^{n} \mu(\gimel_n, \gimel_j)$. Thus, $\sum_{i=1}^{n} d(\gimel_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} \mu(\gimel_i, \gimel_j)$.

Since every arc $\gimel_i\gimel_j$ contributes to the degree of both $\gimel_i$ and $\gimel_j$, each membership value $\mu(\gimel_i, \gimel_j)$ is counted twice in the above sum. Therefore, $\sum_{\gimel \in \Im} d(\gimel) \ = \ 2 \sum_{\substack{\gimel_i, \gimel_j \in \Im \\ i<j}} \mu(\gimel_i, \gimel_j)$.

That is, the total degree of all vertices is equal to twice the total membership value of all arcs in $\Sigma_{\pm}$.

Theorem 4. *A Signed-FG* $\sum_{\pm}$*. The sum of the strong degrees of all vertex is twice the total membership values of all strong edges,* $\sum_{\pm} = (\Im, \sigma, \mu)$.

Proof. We give a formal, stepwise argument. Let $\Im = \{\gimel_1, \ldots, \gimel_n\}$. For ordered indices write $\varphi_{ij} = \mu(\gimel_i, \gimel_j)$. Denote by $\mathcal{S}$ the set of unordered strong edges: $\mathcal{S} = \{\{\gimel_i, \gimel_j\} : 1 \leq i < j \leq n,\ \varphi_{ij} \neq 0\}$. For each vertex $\gimel_i$ the strong degree is $d_S(\gimel_i) = \sum_{j=1, j \neq i}^{n} \varphi_{ij}$, where the sum ranges only over j such that $\{\gimel_i, \gimel_j\} \in \mathcal{S}$ (and $\varphi_{ii} = 0$ by assumption).

Summing over all vertices yields $\sum_{i=1}^{n} d_S(\beth_i) = \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \varphi_{ij} = \sum_{\substack{1\leq i,j\leq n\\ i\neq j}} \varphi_{ij}$.

By symmetry $\varphi_{ij} = \varphi_{ji}$ for every $i \neq j$. Hence each unordered strong edge $\{\beth_i, \beth_j\} \in \mathcal{S}$ contributes the two equal terms φ_{ij} and φ_{ji} to the double sum.

Therefore $\sum_{\substack{1\leq i,j\leq n\\ i\neq j}} \varphi_{ij} = 2 \sum_{\substack{1\leq i<j\leq n\\ \{\beth_i,\beth_j\}\in\mathcal{S}}} \varphi_{ij} = 2 \sum_{1\leq i<j\leq n} \varphi_{ij}$, the latter equality since $\varphi_{ij} = 0$ when $\{\beth_i, \beth_j\} \notin \mathcal{S}$.

Combining the above gives $\sum_{i=1}^{n} d_S(\beth_i) = 2 \sum_{1\leq i<j\leq n} \varphi_{ij}$, which is the claimed identity.

Theorem 5. *In a signed-FG $\Sigma_\pm = (\Im, \sigma, \mu)$, the sum of the effective degrees of all nodes is equal to twice the total membership value of all effective arcs.*
That is, $\sum_{\beth\in\Im} d_\pm(\beth) = 2 \sum_{\{\beth_1,\beth_2\}\subseteq\Im} \mu(\beth_1, \beth_2)$, *where the effective degree of a vertex $\beth \in \Im$ is defined as* $d_\pm(\beth) = \sum_{\substack{\beth_i\in\Im\\ \beth_i\neq\beth}} \mu(\beth, \beth_i)$.

2. Domination of Signed Fuzzy Graphs

In this section, we delve into the theoretical foundations of signed fuzzy graphs (Signed-FGs), emphasizing key concepts such as domination sets and the properties that govern their behavior. Critical terminology relevant to understanding the dynamics of interactions within Signed-FGs, including the strong domination number and effective degrees of nodes, is clearly defined. By presenting key theorems and assertions, we establish a structured framework for studying domination in these fuzzy graphs, paving the way for investigating their applications in complex network systems.

Definition 10. *Let $\Sigma_\pm = (\Im, \sigma, \mu)$ be a signed-FG, and let $\beth_1, \beth_2 \in \Im$. We say that $\beth_1$ dominates $\beth_2$ in $\Sigma_\pm$ if the ordered pair $(\beth_1, \beth_2)$ is a strong arc of $\Sigma_\pm$.*

A subset $\mathfrak{D} \subseteq \Im$ is a domination set of $\Sigma_\pm$ if $\forall\, \beth_2 \in \Im - \mathfrak{D}, \exists\, \beth_1 \in \mathfrak{D}$, such that $(\beth_1, \beth_2)$ is a strong arc of $\Sigma_\pm$.

Definition 11. *Let $\Sigma_\pm = (\Im, \sigma, \mu)$ be a signed-FG. A domination set $\mathfrak{D} \subseteq \Im$ is called a* minimum domination set *if no proper subset of $\mathfrak{D}$ is also a domination set of $\Sigma_\pm$.*

The domination number *of $\Sigma_\pm$, denoted $\gamma(\Sigma_\pm)$, is defined as* $\gamma(\Sigma_\pm) = \min\{\, |\mathfrak{D}| : \mathfrak{D}$ *is a domination set of* $\Sigma_\pm \,\}$.

Any domination set $\mathfrak{D}$ of size $|\mathfrak{D}| = \gamma(\Sigma_\pm)$ is called a minimum domination set.

Example 7. *Consider the signed-FG $\Sigma_\pm = (\Im, \sigma, \mu)$ with vertex set $\Im = \{\beth_1, \beth_2, \beth_3, \beth_4, \beth_5\}$ and edge set $E = \{(\beth_1, \beth_2), (\beth_2, \beth_3), (\beth_2, \beth_4), (\beth_4, \beth_5), (\beth_1, \beth_5), (\beth_2, \beth_5)\}$. The vertex membership values are $\sigma(\beth_1) = -0.3, \sigma(\beth_2) = 0.4, \sigma(\beth_3) = -0.3, \sigma(\beth_4) = 0.5, \sigma(\beth_5) = 0.2$. The* *edge membership values are* $\mu(\gimel_1, \gimel_2) = -0.4, \mu(\gimel_1, \gimel_5) = -0.3, \mu(\gimel_2, \gimel_4) = 0.4, \mu(\gimel_2, \gimel_3) = -0.3, \mu(\gimel_2, \gimel_5) = 0.2, \mu(\gimel_4, \gimel_5) = 0.1$. *The strong arcs in* $\Sigma_\pm$ *are* $(\gimel_1, \gimel_5), (\gimel_2, \gimel_5), (\gimel_2, \gimel_4), (\gimel_2, \gimel_3)$. *Possible dominating sets include* $\{\gimel_1, \gimel_2\}, \{\gimel_2, \gimel_5\}, \{\gimel_1, \gimel_3, \gimel_4\}, \{\gimel_3, \gimel_4, \gimel_5\}$. *Among these, the smallest domination set is* $\{\gimel_1, \gimel_3, \gimel_4\}$. *Therefore, the domination number of* $\Sigma_\pm$ *is* $\gamma(\Sigma_\pm) = \min\{|\mathfrak{D}| : \mathfrak{D}$ *is a domination set of* $\Sigma_\pm\} = 3$.

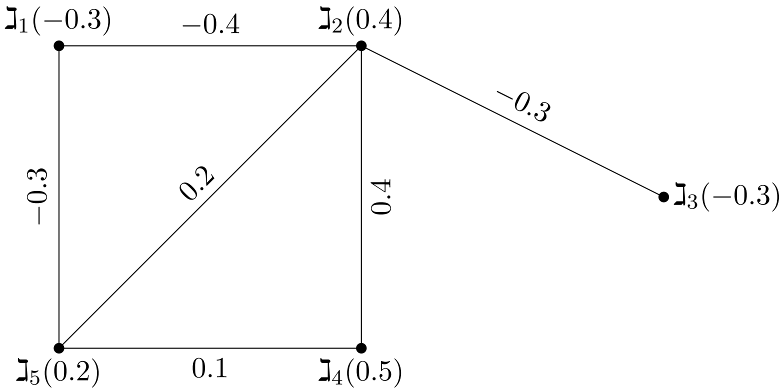

Figure 2: Domination in a Signed-FG example.

Definition 12. *Consider the Signed-FG* $\sum_\pm$*, which has no isolated vertices. If all of the vertices in* $\Im$ *are dominated by a single vertex in* $\mathfrak{D}$*, then subset* $\mathfrak{D}$ *of* $\Im$ *is referred to as a total domination set.*

A total domination set's minimal fuzzy cardinality is γ_t*, which is the total domination number of* $\sum_\pm$.

Theorem 6. *Let* $\Sigma_\pm = (\Im, \sigma, \mu)$ *be a non-trivial signed-FG, and let* $\mathfrak{D}$ *be a dominating set of* $\Sigma_\pm$*. Then* $\Im - \mathfrak{D}$ *is also a dominating set.*

Proof. Let $F_\pm$ be a spanning tree of $\Sigma_\pm$, and let $\gimel$ be any vertex of $F_\pm$. We can partition the vertex set $\Im$ into two disjoint subsets according to the parity of the distance from $\gimel$ in $F_\pm$: vertices at even distance from $\gimel$ form the set $\mathfrak{D}$, vertices at odd distance from $\gimel$ form the set $\Im - \mathfrak{D}$.

By the properties of a tree, every vertex in $\mathfrak{D}$ is adjacent to at least one vertex in $\Im - \mathfrak{D}$, and vice versa. Hence, each of $\mathfrak{D}$ and $\Im - \mathfrak{D}$ dominates the graph. Therefore, both $\mathfrak{D}$ and its complement are dominating sets in $\Sigma_\pm$.

Theorem 7. *If* $\mathfrak{D}$ *is a minimal dominating set and* $\Sigma_\pm = (\Im, \sigma, \mu)$ *is a Signed-FG without isolated nodes, then* $\Im - \mathfrak{D}$ *is also a dominating set.*

Proof. Let $\mathfrak{D}$ be a minimal dominating set of $\Sigma_\pm$. Suppose, for the sake of contradiction, that $\Im - \mathfrak{D}$ is not a dominating set. Then there exists a vertex $\gimel \in \mathfrak{D}$ which is not adjacent to any vertex in $\Im - \mathfrak{D}$. This means that $\gimel$ is adjacent only to vertices in $\mathfrak{D}$.

Since $\Sigma_{\pm}$ has no isolated vertices, $\beth$ must be a strong neighbor of at least one vertex in $\mathfrak{D}-\{\beth\}$. Therefore, $\mathfrak{D}-\{\beth\}$ would still be a dominating set, contradicting the minimality of $\mathfrak{D}$.

Hence, every vertex in $\mathfrak{D}$ has at least one strong neighbor in $\Im-\mathfrak{D}$, implying that $\Im-\mathfrak{D}$ is indeed a dominating set.

Theorem 8. *If $\Sigma_{\pm}=(\Im,\sigma,\mu)$ is a complete Signed-FG, then $\gamma(\sum_{\pm})=min\{\sigma(\beth)\}, \forall \beth \in \Im$.*

Proof. All of the arcs in $\sum_{\pm}$ are strong, and since $\sum_{\pm}$ is a complete Signed-FG, every node in $\sum_{\pm}$ is incident to every other node. Every $\beth \in \Im$ has a domination set $\mathfrak{D}=\{\beth\}$. This means that $\gamma(\sum_{\pm})=min\{\sigma(\beth)\}, \forall \beth \in \Im$.

Theorem 9. *Let $\Sigma_{\pm}=(\Im,\sigma,\mu)$ be a nontrivial Signed-FG of order $\beth$. $\gamma(\sum_{\pm})$ is $\beth$ if all nodes are isolated.*

Proof. If each node is isolated, it is clear that $\gamma(\sum_{\pm})=\beth$. However, to show that each node is isolated, suppose that $\gamma(\sum_{\pm})=\beth$. If at all feasible, let $\beth_1$ be a non-isolated node of $\sum_{\pm}$. The arc $(\beth_1,\beth_2)$ is then strong due to a node $\beth$. Therefore, there are a maximum of $(n-1)$ nodes for each dominant set of $\sum_{\pm}$, where n is the number of nodes in $\sum_{\pm}$. Consequently, $\gamma(\sum_{\pm})<\beth$ is a contradiction. Thus, each node is segregated.

Theorem 10. *If $\sum_{\pm}$ is a complete bipartite Signed-FG, then*
$\gamma(K_{\sigma_{\pm 1},\sigma_{\pm 2}})=min\left\{|\Im_1|,|\Im_2|,\min_{\beth\in\Im_1}\sigma(\beth)+\min_{\beth\in\Im_2}\sigma(\beth)\right\}$.

Proof. They are all strong arcs in $K_{\sigma_{\pm 1},\sigma_{\pm 2}}$. All nodes in $\Im_1$ are also connected to all nodes in $\Im_2$. Therefore, the domination set in $K_{\sigma_{\pm 1},\sigma_{\pm 2}}$ are $\Im_1$ and $\Im_2$, and any set that has two nodes, one in $\Im_1$ and the other in $\Im_2$, do so. Therefore,
$\gamma(K_{\sigma_{\pm 1},\sigma_{\pm 2}})=min\left\{|\Im_1|,|\Im_2|,\min_{\beth\in\Im_1}\sigma(\beth)+\min_{\beth\in\Im_2}\sigma(\beth)\right\}$.

Theorem 11. *For any nontrivial Signed-FG $\sum_{\pm}$, $\gamma(\sum_{\pm})+\overline{\gamma(\sum_{\pm})}<2\beth$.*

Proof. A domination set in $\sum_{\pm}$ has a minimum scalar cardinality of γ, therefore $\gamma\leq\beth$. Likewise, the smallest scalar cardinality for $\overline{\sum_{\pm}}$ is $\overline{\gamma}\leq\beth$. $\gamma+\overline{\gamma}\leq\beth+\beth=2\beth$, as a result. Assume that $\gamma+\overline{\gamma}\neq 2\beth$ in order to demonstrate that $\gamma+\overline{\gamma}=2\beth$. Since maximal scalar cardinality equals $\beth$, if $\gamma<\beth$, then $\overline{\gamma}>\beth$, preventing the possibility of reaching $\gamma+\overline{\gamma}=2\beth$. Consequently, both $\gamma=\beth$ and $\overline{\gamma}=\beth$. It is claimed that each node is isolated if $\gamma=\beth$. By this definition, $\overline{\sum_{\pm}}$ is a complete Signed-FG. This leads to the failure of $\overline{\gamma}=\min_{\beth_1\in V}\sigma(\beth_1)<\beth$. Likewise, a contradiction exists: $\gamma=\min_{\beth_1\in V}\sigma(\beth_1)<\beth$, as $\overline{\gamma}=\beth$ suggests. Therefore, our claim is accurate. Thus, $\gamma+\overline{\gamma}<2\beth$ for every nontrivial Signed-FG $\sum_{\pm}$.

Theorem 12. *If $\sum_{\pm}$ is any Signed-FG without isolated nodes, then $\gamma \leq \frac{\daleth}{2}$.*

Proof. In $\sum_{\pm}$, $\mathfrak{D}$ would be the smallest dominant set. Consequently, Theorem 7 states that $V - \mathfrak{D}$ is a dominant set of $\sum_{\pm}$ itself. It follows that $\gamma \leq |\mathfrak{D}|$ and $\gamma \leq |\Im - \mathfrak{D}|$. Therefore, $\gamma \leq \frac{\daleth}{2}$ results from $2\gamma \leq |\mathfrak{D}| + |\Im - \mathfrak{D}| = \daleth$.

Corollary 1. *Assume that $\sum_{\pm}$ is a Signed-FG that contains neither $\sum_{\pm}$ nor $\overline{\sum_{\pm}}$ isolated nodes. Thus, $\gamma + \overline{\gamma} \leq \daleth$. If and only if $\gamma = \overline{\gamma} = \frac{\daleth}{2}$, then additional equality is preserved.*

Proof. With no isolated nodes, $\sum_{\pm}$ is a Signed-FG. Theorem 12 thus states that $\gamma \leq \frac{\daleth}{2}$ and $\overline{\gamma} \leq \frac{\daleth}{2}$. This means that $\gamma + \overline{\gamma} \leq \frac{\daleth}{2} + \frac{\daleth}{2} = \daleth.\gamma + \overline{\gamma} = \daleth$ if $\gamma = \overline{\gamma} = \frac{\daleth}{2}$. If $\gamma + \overline{\gamma} = \daleth$, on the other hand. $\gamma \leq \frac{\daleth}{2}$ and $\overline{\gamma} \leq \frac{\daleth}{2}$ according to Theorem 12. $\gamma + \overline{\gamma} < \daleth$, which is contrary to our premise, if $\gamma \leq \frac{\daleth}{2}$ or $\overline{\gamma} \leq \frac{\daleth}{2}$. $\gamma = \overline{\gamma} = \frac{\daleth}{2}$, as a result.

Theorem 13. *Let $\Sigma_{\pm} = (\Im, \sigma, \mu)$ be a Signed-FG of order $\wp$. Then $\min_{\daleth \in \Im} \sigma(\daleth) \leq \gamma \leq \wp - \Delta_{SN}(\sum_{\pm})$.*

Proof. Suppose that the initial component $\min_{\daleth \in \Im} \sigma(\daleth) \leq \gamma$. If $\min_{\daleth \in \Im} \sigma(\daleth)$ is true, then if $\daleth$ are any vertices in $\sum_{\pm}$, then $d_{SN}(\daleth) = \Delta_{SN}(\sum_{\pm})$. $\Im - N_s(\daleth)$ is a domination set of $\sum_{\pm}$, in this case. It follows that $\gamma(\sum_{\pm}) \leq \wp - \Delta_{SN}(\sum_{\pm})$.

Remark 1. *If $\Sigma_{\pm} = (\mu, \sigma, \mu)$ is a complete Signed-FG of order $\wp$. Then $min(\sigma(\daleth)) = \gamma = \wp - \Delta_{SN}(\sum_{\pm})$.*

Remark 2. *Clearly $\Delta_S(\sum_{\pm}) \leq \Delta_{SN}(\sum_{\pm})$.*

3. Domination in Signed Fuzzy Trees

The notion of domination in signed fuzzy trees, a particular class of signed fuzzy graphs distinguished by their hierarchical structure and lack of cycles, is explored in this section. The discussion is on the distinctive characteristics of signed fuzzy trees, such as their dominating sets, signed fuzzy cut nodes, and signed fuzzy end nodes, building on the fundamental ideas of signed fuzzy graphs and dominance. To demonstrate how dominance ensures connectedness and effective representation of these trees, theorems and proofs are provided. Analyzing the interactions between signed fuzzy trees and their uses in structured network environments is made easier by these revelations.

Definition 13. *If removing $(\daleth_1, \daleth_2)$ reduces the strength between some node pairs, then an arc $(\daleth_1, \daleth_2)$ is a signed fuzzy bridge of $\Sigma_{\pm} = (\Im, \sigma, \mu)$.*

Definition 14. *In $\Sigma_{\pm} = (\Im, \sigma, \mu)$, a node is a signed fuzzy cutnode if its removal reduces the strength between some pair of nodes.*

Definition 15. *A connected Signed-FG $\Sigma_{\pm} = (\Im, \sigma, \mu)$ has a signed fuzzy spanning subgraph $F_{\pm} = (\zeta, \sigma, \mu)$, such that $\mu(\daleth_1, \daleth_2) < CONN(\daleth_1, \daleth_2)$ for all edges $(\daleth_1, \daleth_2)$ not included in $F_{\pm} = (\zeta, \sigma, \mu)$. In this case, $\Sigma_{\pm} = (\Im, \sigma, \mu)$ is classified as a signed fuzzy tree.*

Theorem 14. *$\Sigma_{\pm} = (\Im, \sigma, \mu)$ involves a signed fuzzy bridge connecting each node in a domination set of a non isolated signed fuzzy tree $\sum_{\pm}$.*

Proof. It is assumed that $\gimel_1$ is an element of $\mathfrak{D}$ and that $\sum_{\pm}$ is a dominanting set. due to the fact that $(\gimel_1, \gimel_2)$ is a strong arc since $\gimel_2 \in \Im - \mathfrak{D}$. arc,$(\gimel_1, \gimel_2)$, denotes a unique MST $F_{\pm}$ of $\sum_{\pm}$. The signed fuzzy bridge $(\gimel_1, \gimel_2)$ is therefore an example. Because $\gimel_1$ is arbitrary, every node in the domination set of $\sum_{\pm}$ has this attribute.

Theorem 15. *In a non isolated signed fuzzy tree, $F_{\pm} = (\Im, \sigma, \mu)$, the set of all signed fuzzy cut nodes is a dominant set of $\Sigma_{\pm} = (\Im, \sigma, \mu)$.*

Proof. The set of all the signed fuzzy cut nodes of $\sum_{\pm}$ is represented by $\mathfrak{D}$. The set of all $\sum_{\pm}$ signed fuzzy end nodes is then $\Im - \mathfrak{D}$. Every $\gimel \in \Im - \mathfrak{D}$ has a strong neighbor $h \in D$. Each $\gimel \in \Im - \mathfrak{D}$ is therefore dominated by a node in $\mathfrak{D}$. $\mathfrak{D}$ is therefore a dominant set of $\sum_{\pm}$.

Theorem 16. *A nontrivial signed fuzzy tree, ignoring K_2, is $\Sigma_{\pm} = (\Im, \sigma, \mu)$.There must be at least one fuzzy end node as a strong neighbor for every fuzzy cut node for the set of all fuzzy end nodes to be a dominant set.*

Proof. Assume for the moment that every signed fuzzy cut node in $\sum_{\pm}$ has at least one signed fuzzy end node as a strong neighbor. Let $\mathfrak{D}$ represent the set of all signed fuzzy end nodes in $\sum_{\pm}$. After that, $\Im - \mathfrak{D}$ represents the set of signed fuzzy cut nodes of $\sum_{\pm}$. It is assumed that for each $\gimel_1 \in \Im - \mathfrak{D}$, there is a $\gimel_2 \in \mathfrak{D}$ such that (β_1, β_2) is a strong arc. Consequently, by definition of a domination set, $\mathfrak{D}$ is a domination set of $\sum_{\pm}$.

Conversely, let's say that the set $\mathfrak{D}$, which includes all fuzzy end nodes, is a dominant set of $\sum_{\pm}$. Consider a signed fuzzy cut node of $\sum_{\pm}$, $\gimel_1$. $\gimel_1 \in \Im - \mathfrak{D}$, therefore. Since $\mathfrak{D}$ is a domination set of $\sum_{\pm}$, there exists $\gimel_1 \in \mathfrak{D}$ such that $(\gimel_1, \gimel_2)$ is a strong arc. Consequently, $\gimel_2$ is a strong neighbor of $\gimel_1$. Furthermore, since $\gimel_2 \in \mathfrak{D}$, $\gimel_2$ is a signed fuzzy end node of $\sum_{\pm}$. Thus, the signed fuzzy end node of $\gimel_1$ has a strong neighbor in $\gimel_2$. Since $\gimel_1$ is random, every signed fuzzy cut node in $\sum_{\pm}$ has at least one signed fuzzy end node that is a strong neighbor.

Theorem 17. *Let $\Sigma_{\pm} = (\Im, \sigma, \mu)$ be a signed fuzzy tree that is not trivial. An $\sum_{\pm}$ fuzzy bridge is then connected to each node in a domination set.*

Proof. Let $\gimel_1 \in \mathfrak{D}$ and $\mathfrak{D}$ be a dominant set of $\sum_{\pm}$. There is $\gimel_2 \in \Im - \mathfrak{D}$ such that $(\gimel_1, \gimel_2)$ is a strong arc since $\mathfrak{D}$ is a domination set. This means that in the special minimum spanning tree $F_{\pm}$ of $\sum_{\pm}$, $(\gimel_1, \gimel_2)$ is an arc. The signed fuzzy bridge of $\sum_{\pm}$ is thus $(\gimel_1, \gimel_2)$. As $\gimel_1$ was selected at random, this is true for all nodes in the domination set of $\sum_{\pm}$.

4. Domination in Signed Fuzzy Graphs Applied to Wireless Sensor Networks for Environmental Monitoring Systems

Signed-FGs are incredibly helpful tools for using dominance sets in wireless sensor networks, especially in real-time applications where the network must be dependable, adaptive, and efficient. An outline of a real-world use of this combination is provided below:

When people are unable to attend events that need the movement and transmission of data, networks of sensors are dispersed throughout different regions. Real-time communication, fault tolerance, power efficiency, and data quality are issues for WSNs. In a WSN, the domination set, which is a subset of sensor nodes, ensures that each non-set node is connected to at least one set node. As global communication hubs, domination set nodes use less energy than other nodes by accepting data from other nodes and sending it to the main station.

Fuzzy and signed graphs with degrees of membership and edge signs combined. Therefore, complicated node interactions, including varying degrees of trust, depending on one another for vital information about the task being done, or conversing amongst themselves, can be simulated using these graphs.

For example, to track temperature, air quality, and humidity, a WSN should be placed in a far-off forest. This network must be adaptable enough to guarantee dependable data transfer and strong performance in a variety of environmental circumstances. Sensor nodes are dispersed throughout the forest. A fuzzy membership value that reflects the nodes' initial positions, energy levels, and trustworthiness is used to create a domination set. These nodes are known as dominant nodes because they have the ability to aggregate data by collecting it from neighboring sensors and then transmitting it to a base station or central station. The network's lifespan is extended by lowering the energy usage of non-dominant nodes.

A Signed-FG uses edges to show the connections between its nodes. Positive edges indicate relationships that are going well, whereas negative edges indicate potential issues like interference or mistrust. The strength or dependability of the associations is demonstrated by the membership degree, which is impacted by variables such as energy levels and signal intensity. The fuzzy values on the edges can be quickly changed by changes in the surroundings. Lower fuzzy membership in connections can result from an external interference source that lowers a node's energy level or adds a negative sign to some edges in some of these interactions. Accordingly, only the most dependable and low-energy nodes will be included in the dominance set for any modifications to the Signed-FG.

The Signed-FG has been utilized to ensure network integrity by reducing the time required to identify a replacement node in the event that one of the sensor nodes in the dominance set fails or becomes unreliable. Depending on how dependable their connections with other devices are, nodes can alter their communication patterns in the Signed-FG. This is relevant when specific sensors are vulnerable to hacking or when unstable data transfer results from environmental factors.

Because dominant sets minimize the number of active nodes, they conserve energy in long-distance communication. To provide real-time energy management, nodes with limited energy reserves can gradually reduce their involvement in the fuzzy aspect domination set. In order to achieve a balanced network energy consumption and a longer lifespan for the WSN, the domination set is constantly adjusted in reaction to real-time data. Since the Signed-FG is dynamic, data is sent through the most reliable and accurate pathways, enhancing data quality. In order to respond to real-time environmental changes, the network can adjust sample rates by focusing resources where they are most needed, such as increasing sampling during a suspected fire outbreak.

5. Numerical Illustration of Domination in Signed Fuzzy Graphs

Wireless Sensor Networks are widely used in environmental monitoring systems to collect data from a range of sensors positioned around an area. In a Signed-FG model of such a network, sensors are the vertices, and communication links are the edges. Signs indicate the quality of the connection (negative for poor quality, and positive for high quality), while membership values indicate the reliability of the connection.

Consider a system that is placed in a forest to monitor the temperature, humidity, and air quality. In the WSN, there are five sensors: $\gimel_1, \gimel_2, \gimel_3, \gimel_4$, and $\gimel_5$. The identification of a dominance set of sensors is necessary to ensure that each sensor in the network is either a member of the domination set or has a robust, reliable link to a sensor in the domination set.

Let $\Sigma_{\pm} = (\Im, \sigma, \mu)$ be a Signed-FG, where the vertex set $\Im = \{\gimel_1, \gimel_2, \gimel_3, \gimel_4, \gimel_5\}$ and edge set $\Im \times \Im = \{(\gimel_1, \gimel_2), (\gimel_1, \gimel_3), (\gimel_1, \gimel_4), (\gimel_1, \gimel_4), (\gimel_2, \gimel_3), (\gimel_2, \gimel_4), (\gimel_2, \gimel_5), (\gimel_3, \gimel_4), (\gimel_3, \gimel_5), (\gimel_4, \gimel_5)\}$ with the following vertex values $\sigma(\gimel_1) = 0.8, \sigma(\gimel_2) = 0.7, \sigma(\gimel_3) = 0.6, \sigma(\gimel_4) = 0.5$ and $\sigma(\gimel_5) = 0.7$. The edge values are $\mu(\gimel_1, \gimel_2) = 0.5, \mu(\gimel_1, \gimel_3) = -0.3, \mu(\gimel_1, \gimel_4) = 0.5, , \mu(\gimel_1, \gimel_5) = 0.2, \mu(\gimel_2, \gimel_3) = 0.4, \mu(\gimel_2, \gimel_4) = -0.5, \mu(\gimel_2, \gimel_5) = 0.5, \mu(\gimel_3, \gimel_4) = -0.3, \mu(\gimel_3, \gimel_5) = 0.6$ and $\mu(\gimel_4, \gimel_5) = -0.4$ as shown in Figure 3. The strong arcs are $(\gimel_1, \gimel_3), (\gimel_1, \gimel_4), (\gimel_2, \gimel_5)$ and $(\gimel_3, \gimel_5)$. The possible domination sets are $\{\gimel_1, \gimel_3\}$, $\{\gimel_1, \gimel_5\}$, $\{\gimel_1, \gimel_5, \gimel_2\}$ and $\{\gimel_3, \gimel_2, \gimel_4\}$. The minimum domination set is $\{\gimel_2, \gimel_4\}$ and the domination number is $\gamma(\sum_{\pm}) = 1.2$.

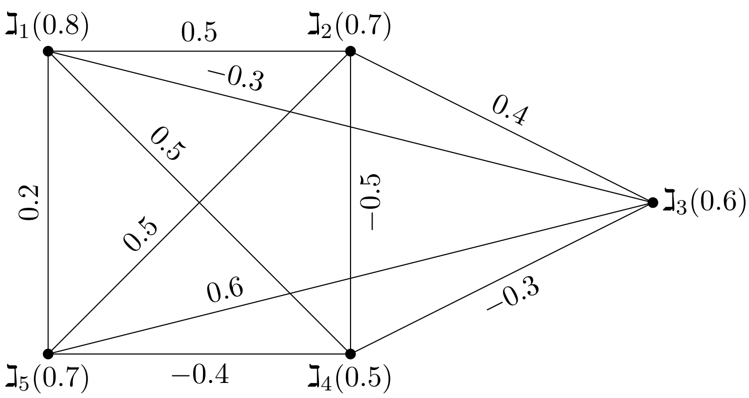

Figure 3: Domination in a Signed-FG in WSN is demonstrated.

For this environmental monitoring system, a dominant collection of sensors that can cover the entire network is essential to guarantee that data from every place is sent efficiently, even in the case that certain links are defective or of low quality.Whether directly or through reliable linkages, the objective is to determine the bare minimum of sensors needed to monitor the entire network. Start by figuring out whether one sensor is enough to cover the entire network. One sensor is not enough since no single sensor has links to all other sensors with high membership levels. Then examine a pair of sensors; the potential dominant sets are $\{\gimel_1, \gimel_3\}$, $\{\gimel_1, \gimel_5\}$, $\{\gimel_1, \gimel_5, \gimel_2\}$, and $\{\gimel_3, \gimel_2, \gimel_4\}$.

The minimum domination set is $\{\gimel_2, \gimel_4\}$ and the domination number is $\gamma(\sum_{\pm}) = 1.2$.

In this scenario, maintaining the functionality of sensors $\gimel_2$ and $\gimel_4$ ensures that the entire network is covered, even if certain connections are not ideal (such as the negative connections). Since environmental factors may have an impact on sensor dependability and connection strength, this is particularly useful for environmental monitoring. The majority of the sensors are kept in operation to enable efficient monitoring and data collection throughout the entire region.

To ensure full network coverage, the dominance number helps determine the minimum number of sensors needed in an environmental monitoring system. Keeping the sensors in the domination set, which in this case comprises of $\gimel_2$ and $\gimel_4$, allows for whole-region monitoring and data collection even when certain links are weak or unreliable.

6. Conclusion

Bounds for the concept of domination in signed-FGs are examined across different classes of such graphs, and the notion is introduced with illustrative examples. In particular, bounds for the domination number of signed-FGs, complete signed-FGs, and complete bipartite signed-FGs are established. The strong domination number is further defined in terms of strong arc membership values. In addition, the relationship between the domination number of a signed FG and that of its complement is investigated.

The study also identifies the classes of signed-FGs that admit strong domination and employs strong arcs to explore domination in greater depth. The investigation is extended to strong domination in signed fuzzy trees, where an upper bound for the strong domination number is determined. It is shown that every node belonging to a strong dominating set in a signed fuzzy tree is either a signed fuzzy cut node or a signed fuzzy end node. Moreover, each such node has a signed fuzzy bridge incident with it. We also establish the unique properties of a strong dominating set and its counterpart in a signed fuzzy graph. With promising implications for broader applications in intelligent systems, these findings demonstrate the potential of signed fuzzy graph–based approaches in the design of reliable and energy-efficient wireless sensor networks (WSNs).

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