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Structural Analysis of Intuitionistic Fuzzy Implicative WSBG-Ideals

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Abstract. In this paper, we investigate intuitionistic fuzzy WSBG-ideals and intuitionistic fuzzy implicative WSBG-ideals within the framework of Sheffer stroke BG-algebras. We establish new algebraic structures that extend classical Boolean and BG-algebra frameworks by synthesizing intuitionistic fuzzy set theory, introduced by Atanassov, with the Sheffer stroke operation. We demonstrate a fundamental connection between intuitionistic fuzzy implicative WSBG-ideals and their level sets, showing that the level set of an intuitionistic fuzzy implicative WSBG-ideal corresponds to an implicative WSBG-ideal of the Sheffer stroke BG-algebra. Furthermore, we explore the properties of intuitionistic fuzzy WSBG-ideals, proving that every intuitionistic fuzzy implicative WSBG-ideal is also an intuitionistic fuzzy WSBG-ideal. However, the converse does not always hold. This work provides new insights into the algebraic properties of Sheffer stroke BG-algebras, enabling novel reasoning methods under uncertainty and paving the way for further applications in fuzzy logic and computational models.

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Key Words and Phrases: Weak Sheffer stroke BG-algebra (WSBG-algebra), Intuitionistic fuzzy WSBG-ideal, Intuitionistic fuzzy implicative WSBG-ideal.

1. Introduction

BG-algebras, a generalization of B-algebras, extend the classical Boolean algebra framework by incorporating more complex operations and properties [1]. This extension provides a robust foundation for modeling and analyzing logical and algebraic systems that go beyond the limitations of traditional Boolean logic. Such generalizations enable the study of intricate logical structures and have applications in advanced mathematics and fuzzy logic, addressing subtleties in reasoning that conventional Boolean algebras cannot handle.

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Guntasow et al. [2] introduced fuzzy α and β-translations of μ types I and II, while Kim et al. [1] proposed BG-algebras as a broader generalization of B-algebras. Udten et al. [3] investigated the concepts of translation and density within the framework of intuitionistic fuzzy sets in UP-algebras. Furthermore, Lee et al. [4] explored the integration of fuzzy set theory into BCK/BCI-algebras, extending classical algebraic frameworks to account for vagueness and uncertainty. Similarly, Balamurugan et al. [5] developed translations of intuitionistic fuzzy soft structures in B-algebras, combining intuitionistic fuzzy sets with soft set theory to address imprecision in algebraic systems.

The Sheffer stroke operation, also known as the NAND operator, was introduced by Sheffer [6]. As a functionally complete operation, it allows any logical expression or axiom in a logical system to be expressed solely using the Sheffer stroke [7]. This property simplifies the study of logical systems by reducing the number of required fundamental operations and has significant implications for both algebraic and logical frameworks. The Sheffer stroke's ability to encapsulate all Boolean algebra axioms highlights its foundational importance and versatility.

The integration of the Sheffer stroke into various algebraic structures has attracted considerable academic attention. Studies have examined Sheffer stroke operation reducts of basic algebras and their congruences [8, 9], Sheffer stroke MTL-algebras [10], ortholattices [11], new state operators in Sheffer stroke basic algebras and their filters [12], Riečan and Bosbach state operators on Sheffer stroke MTL-algebras [13], fuzzy set approaches to Sheffer stroke BE-algebras [14], and, notably, Sheffer stroke BG-algebras [15].

The study of Sheffer stroke BG-algebras is motivated by their theoretical significance and practical applications. These structures deepen our understanding of non-classical logics and offer valuable insights into computational models, particularly in fields such as quantum computing and intuitionistic fuzzy sets [16]. Oner et al. have extensively investigated the theoretical and axiomatic properties of various algebraic systems and their interrelations, such as Sheffer stroke BG-algebras (see [15]) Sheffer stroke BE-algebras (see [17, 18]), Sheffer stroke Hilbert algebras (see [19–22]), Sheffer stroke BCK-algebras (see [23]), Sheffer stroke MTL-algebras (see [24]), Sheffer stroke BL-algebras (see [25]), Sheffer stroke basic algebras (see [26]), Sheffer stroke BCH-algebras (see [27]), Sheffer stroke BH-algebras (see [28]), Sheffer stroke UP-algebras (see [29, 30]).

By leveraging the unique properties of the Sheffer stroke, Sheffer stroke BG-algebras enable the development of novel reasoning methods that extend beyond classical Boolean frameworks.

In this paper, we introduce and analyze the concepts of intuitionistic fuzzy WSBG-ideals within the framework of Sheffer stroke BG-algebras. Intuitionistic fuzzy sets, as formulated by Atanassov, generalize fuzzy sets by considering both membership and non-membership degrees independently, making them particularly effective for handling incomplete or hesitant information.

We establish a fundamental connection between intuitionistic fuzzy WSBG-subalgebras and their level sets, showing that the level set of an intuitionistic fuzzy WSBG-subalgebra corresponds to a subalgebra of the Sheffer stroke BG-algebra, and vice versa. This relationship provides a systematic framework for analyzing the structure of these subalgebras, thereby deepening our understanding of the underlying algebraic framework.

Additionally, we investigate the properties of intuitionistic fuzzy WSBG-ideals, demonstrating that every intuitionistic fuzzy WSBG-ideal is also an intuitionistic fuzzy WSBG-subalgebra. However, the converse does not always hold. This distinction illuminates the unique characteristics of intuitionistic fuzzy WSBG-ideals within the algebraic context.

The novelty of this work lies in the integration of intuitionistic fuzzy set theory with Sheffer stroke BG-algebras, an area that, to the best of our knowledge, remains underexplored. This synthesis opens new avenues for research in algebraic logic and its applications, especially in scenarios where reasoning under uncertainty is crucial.

As part of this study, we present a theorem and its proof concerning intuitionistic fuzzy implicative WSBG-ideals in the lattice L. The theorem states that an intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of L if and only if, for all $t, s \in [0, 1]$, the sets $U(\beta, t)$ and $L(\alpha, s)$ are implicative WSBG-ideals of L, provided they are nonempty. The proof is divided into two cases: the first case verifies that $U(\beta, t)$ satisfies the implicative WSBG-ideal property using the membership function β, while the second case establishes the same for $L(\alpha, s)$ using α. The converse is also proven by assuming the implicative WSBG-ideal properties of $U(\beta, t)$ and $L(\alpha, s)$, showing that they imply $\mathcal{L}$ is an intuitionistic fuzzy implicative WSBG-ideal. This proof employs symbolic reasoning and logical deductions, offering a comprehensive theoretical foundation for further exploration.

2. Preliminaries

This section provides fundamental definitions and concepts related to Sheffer stroke BG-algebras and intuitionistic fuzzy structures.

Definition 1. *[6] Let $H = \langle H; | \rangle$ be a groupoid. The operation $|$ is called a Sheffer stroke operation if it satisfies the following axioms: for all $x, y, z \in H$,*

$$\begin{array}{ll} (S1) & x|y = y|x, \\ (S2) & (x|x)|(x|y) = x, \\ (S3) & x|((y|z)|(y|z)) = ((x|y)|(x|y))|z, \\ (S4) & (x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x. \end{array}$$

Definition 2. *[15] A Sheffer stroke BG-algebra (abbreviated as SBG-algebra) is a structure $\langle L; |, 0 \rangle$ of type $(2, 0)$, where $|$ is a Sheffer stroke operation, i.e., it satisfies the axioms (S1)-(S4), on L and 0 is a distinguished element in L, and the following conditions hold: for all $x, y \in L$,*

$$\begin{array}{ll} (SBG1) & (x|(x|x))|(x|(x|x)) = 0, \\ (SBG2) & (0|(y|y))|((x|(y|y))|(x|(y|y))) = x|x. \end{array}$$

Definition 3. *A weak Sheffer stroke BG-algebra (abbreviated as WSBG-algebra) is a structure $\langle L; |, 0 \rangle$ of type $(2, 0)$, where $|$ is a binary operation, i.e., it satisfies the axioms (SBG1) and (SBG2), on L and 0 is a distinguished element in L.*

Example 1. *Let $L = \{0, 1, 2, 3, 4, 5\}$ be a set with the binary operation $|$ defined by the following Cayley table:*

$\vert$	0	1	2	3	4	5
0	0	0	2	3	0	0
1	1	2	0	2	0	0
2	2	0	3	0	0	0
3	3	0	0	2	0	0
4	0	0	2	1	0	0
5	0	0	2	1	0	0

Hence, $\langle L; |, 0\rangle$ is a WSBG-algebra.

Definition 4. *A nonempty subset H of a WSBG-algebra $L = \langle L; |, 0\rangle$ is called a WSBG-subalgebra of L if $(x|(y|y))|(x|(y|y)) \in H, \quad \forall x, y \in H$.*

Example 2. *From the WSBG-algebra in Example 1, the following subsets are WSBG-subalgebras: $H_1 = \{0\}$, $H_2 = \{0, 4\}$, $H_3 = \{0, 5\}$, $H_4 = \{0, 2, 3\}$, $H_5 = \{0, 4, 5\}$, $H_6 = \{0, 1, 2, 3\}$, $H_7 = \{0, 2, 3, 4\}$, $H_8 = \{0, 2, 3, 5\}$, $H_9 = \{0, 1, 2, 3, 4\}$, $H_{10} = \{0, 1, 2, 3, 5\}$, $H_{11} = \{0, 2, 3, 4, 5\}$, $H_{12} = L$.*

Definition 5. *A nonempty subset H of a WSBG-algebra $L = \langle L; |, 0\rangle$ is called a WSBG-ideal of L if the following conditions are satisfied: for all $x, y \in L$,*

(i) $0 \in H$,

(ii) $(x|(y|y))|(x|(y|y)) \in H$ *and* $y \in H \Rightarrow x \in H$.

Definition 6. *A nonempty subset H of a WSBG-algebra $L = \langle L; |, 0\rangle$ is called an implication WSBG-ideal of L if the following conditions are satisfied: for all $x, y, z \in L$,*

(i) $0 \in H$,

(ii) $((((y|(x|(y|y)))|(y|(x|(y|y))))|(z|z))|(((y|(x|(y|y)))|(y|(x|(y|y))))|(z|z))) \in H$ *and* $z \in H \Rightarrow y \in H$.

Example 3. *From the WSBG-algebra in Example 1, the subset $H_5 = \{0, 4, 5\}$ is a WSBG-ideal, but not an implication WSBG-ideal of L. The subset $H_{12} = L$ is both a WSBG-ideal and an implication WSBG-ideal of L.*

Definition 7. *[16] Let L be a nonempty set. An intuitionistic fuzzy structure on L is defined as:*

$$A := \{\langle x, \mu(x), \nu(x)\rangle \mid x \in L\}, \tag{1}$$

where $\mu : L \to [0, 1]$ denotes the membership degree of x in L, and $\nu : L \to [0, 1]$ denotes the non-membership degree of x in L, with the condition $0 \le \mu(x) + \nu(x) \le 1$.

Definition 8. *Let $\langle P; |, 0_P\rangle$ and $\langle Q; |, 0_Q\rangle$ be WSBG-algebras. A mapping $g : P \to Q$ is called a homomorphism if it satisfies $g(x|y) = g(x)|g(y), \quad \forall x, y \in P$, and $g(0_P) = 0_Q$.*

3. Intuitionistic fuzzy implicative and related WSBG-ideals

This section presents a theorem and its proof concerning intuitionistic fuzzy implicative WSBG-ideals in a lattice L. The theorem establishes that an intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of L if and only if, for all $t, s \in [0, 1]$, the sets $U(\beta, t)$ and $L(\alpha, s)$ are implicative WSBG-ideals of L, provided they are nonempty.

The proof is divided into two cases: the first case demonstrates that $U(\beta, t)$ satisfies the implicative WSBG-ideal property by verifying that certain conditions involving the membership function β hold for elements of L. The second case similarly proves that $L(\alpha, s)$ satisfies the implicative WSBG-ideal property using the membership function α.

The converse is also proven by assuming the implicative WSBG-ideal properties of $U(\beta, t)$ and $L(\alpha, s)$ and showing that they imply $\mathcal{L}$ is an intuitionistic fuzzy implicative WSBG-ideal. The proof employs symbolic reasoning and logical deductions using variables ζ, η, and θ to represent elements of the lattice L.

Definition 9. *An intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ of a WSBG-algebra $L = \langle L; |, 0\rangle$ is referred to as an intuitionistic fuzzy WSBG-ideal of L if the following conditions hold:*

$$(\forall \zeta, \eta \in L) \begin{pmatrix} \alpha(0) \geq \alpha(\zeta) \geq \min\{\alpha(\eta), \alpha((\zeta|(\eta|\eta))|(\zeta|(\eta|\eta)))\}, \\ \beta(0) \leq \beta(\zeta) \leq \max\{\beta(\eta), \beta((\zeta|(\eta|\eta))|(\zeta|(\eta|\eta)))\} \end{pmatrix}. \tag{2}$$

In this case, the membership function α is referred to as the fuzzy WSBG-ideal of L.

Example 4. *Consider the WSBG-algebra $L = \langle L; |, 0\rangle$ where $L = \{0, a, b\}$ with the following operation table:*

$\vert$	0	a	b
0	0	b	a
a	b	0	0
b	a	0	0

Define the intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ by:

$$\alpha(x) = 0.5, \quad \beta(x) = 0, \quad \forall x \in L.$$

Then $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy WSBG-ideal of the given WSBG-algebra.

Definition 10. *An intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ of a WSBG-algebra $L = \langle L; |, 0\rangle$ is referred to as an intuitionistic fuzzy implicative WSBG-ideal of L if the following conditions hold:*

$$(\forall \zeta, \eta, \theta \in L) \begin{pmatrix} \alpha(0) \geq \alpha(\zeta) \geq \min\{\alpha(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))| \\ \qquad (((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))), \alpha(\theta)\}, \\ \beta(0) \leq \beta(\zeta) \leq \max\{\beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))| \\ \qquad (((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))), \beta(\theta)\} \end{pmatrix}. \tag{3}$$

In this case, the membership function α is referred to as the fuzzy implicative WSBG-ideal of L.

Example 5. *From the WSBG-algebra in Example 1, define the intuitionistic fuzzy set* $\mathcal{L} = (L, \alpha, \beta)$ *with membership and non-membership functions:*

$$\alpha(x) = 0.5, \quad \beta(x) = 0.25, \quad \forall x \in L.$$

Then $\mathcal{L} = (L, \alpha, \beta)$ *is an intuitionistic fuzzy implicative WSBG-ideal of the WSBG-algebra in Example 1.*

Proposition 1. *Every intuitionistic fuzzy implicative WSBG-ideal of a WSBG-algebra* $L = \langle L; |, 0\rangle$ *is also an intuitionistic fuzzy WSBG-ideal of* L.

Proof. Let $\mathcal{L} = (L, \alpha, \beta)$ be an intuitionistic fuzzy implicative WSBG-ideal of $L = \langle L; |, 0\rangle$. Then, it follows that $\alpha(0) \geq \alpha(\zeta)$ and $\beta(0) \leq \beta(\zeta)$. Furthermore, we have

$$\begin{aligned}
\alpha(\zeta) &\geq \min\{\alpha(\eta), \alpha((((\zeta|(\zeta|(\zeta|\zeta)))|(\zeta|(\zeta|(\zeta|\zeta))))|(\eta|\eta))| \\
&\quad (((\zeta|(\zeta|(\zeta|\zeta)))|(\zeta|(\zeta|(\zeta|\zeta))))|(\eta|\eta)))\} \\
&= \min\{\alpha(\eta), \alpha((((\zeta|(0|0))|(\zeta|(0|0)))|(\eta|\eta))| \\
&\quad (((\zeta|(0|0))|(\zeta|(0|0)))|(\eta|\eta)))\} \\
&= \min\{\alpha(\eta), \alpha((\zeta|(\eta|\eta))|(\zeta|(\eta|\eta)))\},
\end{aligned}$$

$$\begin{aligned}
\beta(\zeta) &\leq \max\{\beta(\eta), \beta((((\zeta|(\zeta|(\zeta|\zeta)))|(\zeta|(\zeta|(\zeta|\zeta))))|(\eta|\eta))| \\
&\quad (((\zeta|(\zeta|(\zeta|\zeta)))|(\zeta|(\zeta|(\zeta|\zeta))))|(\eta|\eta)))\} \\
&= \max\{\beta(\eta), \beta((((\zeta|(0|0))|(\zeta|(0|0)))|(\eta|\eta))| \\
&\quad (((\zeta|(0|0))|(\zeta|(0|0)))|(\eta|\eta)))\} \\
&= \max\{\beta(\eta), \beta((\zeta|(\eta|\eta))|(\zeta|(\eta|\eta)))\}.
\end{aligned}$$

Therefore, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy WSBG-ideal of L.

Theorem 1. *An intuitionistic fuzzy set* $\mathcal{L} = (L, \alpha, \beta)$ *in a WSBG-algebra* $L = \langle L; |, 0\rangle$ *is an intuitionistic fuzzy implicative WSBG-ideal of* L *if and only if the sets* $L(\beta, s)$ *and* $U(\alpha, t)$ *are implicative WSBG-ideals of* L *whenever they are nonempty for all* $s, t \in [0, 1]$.

Proof. Assume that $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of $L = \langle L; |, 0\rangle$, and let $L(\beta, s) \neq \emptyset \neq U(\alpha, t)$ for all $s, t \in [0, 1]$. Let $\zeta, \eta \in L$ be such that $(\zeta, \eta) \in L(\beta, s) \times U(\alpha, t)$. Then, by definition, $\beta(\zeta) \leq s$ and $\alpha(\eta) \geq t$. Since $\beta(0) \leq \beta(\zeta) \leq s$ and $\alpha(0) \geq \alpha(\eta) \geq t$, it follows that $(0, 0) \in L(\beta, s) \times U(\alpha, t)$.

Now, let $\zeta, \eta, \theta, \xi, \phi, \psi \in L$ be such that

$$\begin{aligned}
&(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))), \\
&(((((\xi|(\phi|(\xi|\xi)))|(\xi|(\phi|(\xi|\xi))))|(\psi|\psi))|(((\xi|(\phi|(\xi|\xi)))|(\xi|(\phi|(\xi|\xi))))|(\psi|\psi))))) \in L(\beta, s) \times U(\alpha, t)
\end{aligned}$$

and $(\theta, \psi) \in L(\beta, s) \times U(\alpha, t)$. Then, by definition, we have

$$\beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))) \leq s,$$

$$\beta(\theta) \leq s,$$

$$\alpha(((((\xi|(\phi|(\xi|\xi)))|(\xi|(\phi|(\xi|\xi))))|(\psi|\psi))|(((\xi|(\phi|(\xi|\xi)))|(\xi|(\phi|(\xi|\xi))))|(\psi|\psi)))) \geq t,$$

and also

$$\alpha(\psi) \geq t.$$

Thus, it follows that

$$\beta(\zeta) \leq \max\{\beta((\zeta|(\eta|\eta))|(\zeta|(\eta|\eta))), \beta(\theta)\} \leq s,$$

and

$$\alpha(\xi) \geq \min\{\alpha((((\xi|(\phi|(\xi|\xi)))|(\xi|(\phi|(\xi|\xi))))|(\psi|\psi))|(((\xi|(\phi|(\xi|\xi)))|(\xi|(\phi|(\xi|\xi))))|(\psi|\psi))), \alpha(\psi)\} \geq t.$$

Hence, $(\zeta, \xi) \in L(\beta, s) \times U(\alpha, t)$. Therefore, $L(\beta, s)$ and $U(\alpha, t)$ are implicative WSBG-ideals of L.

Conversely, assume that $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy set in L such that $L(\beta, s)$ and $U(\alpha, t)$ are implicative WSBG-ideals of $L = \langle L; |, 0\rangle$ whenever they are nonempty for all $s, t \in [0, 1]$. Suppose that $\beta(0) > \beta(\eta)$ for some $\eta \in L$. Then $\eta \in L(\beta, \beta(\eta))$ but $0 \notin L(\beta, \beta(\eta))$, a contradiction. Hence, $\beta(0) \leq \beta(\zeta)$ for all $\zeta \in L$. Similarly, suppose that $\alpha(0) < \alpha(\zeta)$ for some $\zeta \in L$. Then $\zeta \in U(\alpha, \alpha(\zeta))$ but $0 \notin U(\alpha, \alpha(\zeta))$, a contradiction. Hence, $\alpha(0) \geq \alpha(\zeta)$ for all $\zeta \in L$.

Next, suppose that

$$\beta(\eta) > \max\{\beta((((\eta|(\phi|(\eta|\eta)))|(\eta|(\phi|(\eta|\eta))))|(\psi|\psi))|(((\eta|(\phi|(\eta|\eta)))|(\eta|(\phi|(\eta|\eta))))|(\psi|\psi))), \beta(\psi)\}$$

or

$$\alpha(\zeta) < \min\{\alpha((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))), \alpha(\theta)\}$$

for some $\eta, \phi, \psi, \zeta, \eta, \theta \in L$. Then

$$(((((\eta|(\phi|(\eta|\eta)))|(\eta|(\phi|(\eta|\eta))))|(\psi|\psi))|(((\eta|(\phi|(\eta|\eta)))|(\eta|(\phi|(\eta|\eta))))|(\psi|\psi))), \psi) \in L(\beta, s),$$

or

$$(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))), \theta) \in U(\alpha, t),$$

where

$$s = \max\{\beta((((\eta|(\phi|(\eta|\eta)))|(\eta|(\phi|(\eta|\eta))))|(\psi|\psi))|(((\eta|(\phi|(\eta|\eta)))|(\eta|(\phi|(\eta|\eta))))|(\psi|\psi))), \beta(\psi)\}$$

and

$$t = \min\{\alpha((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))), \alpha(\theta)\}.$$

But $\eta \notin L(\beta, s)$ or $\zeta \notin U(\alpha, t)$, a contradiction. Hence,

$$\beta(\zeta) \leq \max\{\beta((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))), \beta(\theta)\},$$

and

$$\alpha(\zeta) \geq \min\{\alpha((\zeta|(\eta|\eta))|(\zeta|(\eta|\eta))), \alpha(\theta)\}$$

for all $\zeta, \eta, \theta \in L$. Consequently, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of L.

Theorem 2. *An intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ in a WSBG-algebra $L = \langle L; |, 0\rangle$ is an intuitionistic fuzzy implicative WSBG-ideal of L if and only if the fuzzy sets $\overline{\beta}$ and α are fuzzy implicative WSBG-ideals of L, where $\overline{\beta} : L \to [0,1]$ is defined by $\overline{\beta}(\zeta) = 1 - \beta(\zeta)$ for all $\zeta \in L$.*

Proof. Assume that $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of $L = \langle L; |, 0\rangle$. By definition, α is a fuzzy implicative WSBG-ideal of L. We now show that $\overline{\beta}$ is also a fuzzy implicative WSBG-ideal of L.

For all $\zeta, \eta, \theta \in L$, we have

$$\overline{\beta}(0) = 1 - \beta(0) \geq 1 - \beta(\zeta) = \overline{\beta}(\zeta),$$

and

$$\begin{aligned}
&\overline{\beta}(\zeta)\\
&= 1 - \beta(\zeta)\\
&\geq 1 - \max\{\beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \beta(\theta)\}\\
&= \min\{1 - \beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), 1 - \beta(\theta)\}\\
&= \min\{\overline{\beta}(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \overline{\beta}(\theta)\}.
\end{aligned}$$

Thus, $\overline{\beta}$ is a fuzzy implicative WSBG-ideal of L.

Conversely, assume that $\overline{\beta}$ and α are fuzzy implicative WSBG-ideals of $L = \langle L; |, 0\rangle$. To show that $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal, let $\zeta, \eta, \theta \in L$. Then:

$$\begin{aligned}
1 - \beta(0) = \overline{\beta}(0) &\geq \overline{\beta}(\zeta) = 1 - \beta(\zeta),\\
\beta(0) &\leq \beta(\zeta).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&1 - \beta(\zeta)\\
&= \overline{\beta}(\zeta)\\
&\geq \min\{\overline{\beta}(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \overline{\beta}(\theta)\}\\
&= \min\{1 - \beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), 1 - \beta(\theta)\}\\
&= 1 - \max\{\beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \beta(\theta)\},
\end{aligned}$$

$$\beta(\zeta) \leq \max\{\beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \beta(\theta)\}.$$

Thus, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of L.

Theorem 3. *Let F be a nonempty subset of a WSBG-algebra $L = \langle L; |, 0\rangle$. Define an intuitionistic fuzzy set $\mathcal{L}_F = (L, \beta_F, \alpha_F)$ in L as follows:*

$$\alpha_F : L \to [0,1], \quad \zeta \mapsto \begin{cases} \alpha_0 & \textit{if } \zeta \in F,\\ \alpha_1 & \textit{otherwise}, \end{cases}$$

$$\beta_F : L \to [0,1], \quad \zeta \mapsto \begin{cases} \beta_0 & \text{if } \zeta \in F, \\ \beta_1 & \text{otherwise}, \end{cases}$$

for all $\zeta \in L$, where $\alpha_i, \beta_i \in [0,1]$ satisfy $\alpha_0 > \alpha_1$, $\beta_0 < \beta_1$, and $\alpha_i + \beta_i \leq 1$ for $i = 0,1$. Then $\mathcal{L}_F = (L, \beta_F, \alpha_F)$ is an intuitionistic fuzzy implicative WSBG-ideal of L if and only if F is an implicative WSBG-ideal of L.

Proof. Assume that $\mathcal{L}_F = (L, \beta_F, \alpha_F)$ is an intuitionistic fuzzy implicative WSBG-ideal of $L = \langle L; |, 0\rangle$. Let $\zeta, \eta, \theta \in L$ be such that $\zeta, \eta \in F$. Then

$$\alpha_F(0) \geq \alpha_F(\zeta) = \alpha_0 \text{ and } \beta_F(0) \leq \beta_F(\zeta) = \beta_0.$$

Thus, $\alpha_F(0) = \alpha_0$ and $\beta_F(0) = \beta_0$, which implies $0 \in F$.

Now, consider

$$\begin{aligned} &\alpha_F(\zeta) \\ &\geq \min\{\alpha_F(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \alpha_F(\eta)\} \\ &= \alpha_0 \end{aligned}$$

and

$$\begin{aligned} &\beta_F(\zeta) \\ &\leq \max\{\beta_F(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \beta_F(\eta)\} \\ &= \beta_0. \end{aligned}$$

Thus, $\alpha_F(\zeta) = \alpha_0$ and $\beta_F(\zeta) = \beta_0$, which implies $\zeta \in F$. Therefore, F is an implicative WSBG-ideal of L.

Conversely, assume that F is an implicative WSBG-ideal of $L = \langle L; |, 0\rangle$. Let $\zeta, \eta \in L$. If $\zeta \in F$, then $0 \in F$, which implies

$$\alpha_F(0) = \alpha_0 = \alpha_F(\zeta) \text{ and } \beta_F(0) = \beta_0 = \beta_F(\zeta).$$

If $0 \notin F$, then

$$\alpha_F(0) = \alpha_1 < \alpha_F(\zeta) \text{ and } \beta_F(0) = \beta_1 > \beta_F(\zeta).$$

For every $\zeta, \eta, \theta \in L$, if

$$(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))), \theta) \in F,$$

then $\zeta \in F$, which implies

$$\begin{aligned} &\alpha_F(\zeta) \\ &= \alpha_0 \\ &= \min\{\alpha_F(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \alpha_F(\theta)\} \end{aligned}$$

and

$\beta_F(\zeta)$
$= \beta_0$
$= \max\{\beta_F(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \beta_F(\theta)\}.$

If $(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))) \notin F$ or $\theta \notin F$, then

$\alpha_F(\zeta)$
$\geq \alpha_1$
$= \min\{\alpha_F(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \alpha_F(\theta)\}$

and

$\beta_F(\zeta)$
$\leq \beta_1$
$= \max\{\beta_F(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \beta_F(\theta)\}.$

Thus, $\mathcal{L}_F = (L, \beta_F, \alpha_F)$ is an intuitionistic fuzzy implicative WSBG-ideal of L.

Proposition 2. *If $\mathcal{L}_i = \{(L, \alpha_i, \beta_i) : i \in \Delta\}$ is a family of intuitionistic fuzzy implicative WSBG-ideals of a WSBG-algebra $L = \langle L; |, 0\rangle$, then $\bigwedge_{i\in\Delta} \mathcal{L}_i$ is an intuitionistic fuzzy implicative WSBG-ideal of L.*

Proof. Let $\mathcal{L}_i = \{(L, \alpha_i, \beta_i) : i \in \Delta\}$ be a family of intuitionistic fuzzy implicative WSBG-ideals of $L = \langle L; |, 0\rangle$. For all $\zeta, \eta, \theta \in L$, we have

$$\big(\bigwedge_{i\in\Delta} \alpha_i\big)(0) = \inf_{i\in\Delta} \alpha_i(0) \geq \inf_{i\in\Delta} \alpha_i(\zeta) = \big(\bigwedge_{i\in\Delta} \alpha_i\big)(\zeta)$$

and

$$\big(\bigwedge_{i\in\Delta} \beta_i\big)(0) = \sup_{i\in\Delta} \beta_i(0) \leq \sup_{i\in\Delta} \beta_i(\zeta) = \big(\bigwedge_{i\in\Delta} \beta_i\big)(\zeta).$$

Next, for all $\zeta, \eta, \theta \in L$, we compute

$$\big(\bigwedge_{i\in\Delta} \alpha_i\big)(\zeta) = \inf_{i\in\Delta} \alpha_i(\zeta) \geq \inf_{i\in\Delta} \min\big\{\alpha_i(\Phi(\zeta,\eta,\theta)), \alpha_i(\theta)\big\},$$

where $\Phi(\zeta,\eta,\theta) = ((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))$. Using the properties of inf, we can separate the terms

$$\inf_{i\in\Delta} \min\{\alpha_i(\Phi(\zeta,\eta,\theta)), \alpha_i(\theta)\} = \min\{\inf_{i\in\Delta} \alpha_i(\Phi(\zeta,\eta,\theta)), \inf_{i\in\Delta} \alpha_i(\theta)\}.$$

Thus,

$$\big(\bigwedge_{i\in\Delta}\alpha_i\big)(\zeta) \geq \min\big\{\big(\bigwedge_{i\in\Delta}\alpha_i\big)(\Phi(\zeta,\eta,\theta)), \big(\bigwedge_{i\in\Delta}\alpha_i\big)(\theta)\big\}.$$

Similarly, for β_i, we have

$$\big(\bigwedge_{i\in\Delta}\beta_i\big)(\zeta) = \sup_{i\in\Delta}\beta_i(\zeta) \leq \sup_{i\in\Delta}\max\{\beta_i(\Phi(\zeta,\eta,\theta)), \beta_i(\theta)\}.$$

Using the properties of sup, we separate the terms

$$\sup_{i\in\Delta}\max\{\beta_i(\Phi(\zeta,\eta,\theta)), \beta_i(\theta)\} = \max\{\sup_{i\in\Delta}\beta_i(\Phi(\zeta,\eta,\theta)), \sup_{i\in\Delta}\beta_i(\theta)\}.$$

Thus,

$$\big(\bigwedge_{i\in\Delta}\beta_i\big)(\zeta) \leq \max\big\{\big(\bigwedge_{i\in\Delta}\beta_i\big)(\Phi(\zeta,\eta,\theta)), \big(\bigwedge_{i\in\Delta}\beta_i\big)(\theta)\big\}.$$

Hence, $\bigwedge_{i\in\Delta}\mathcal{L}_i$ is an intuitionistic fuzzy implicative WSBG-ideal of L.

Definition 11. *An intuitionistic fuzzy set $\mathcal{L} = (L,\alpha,\beta)$ of a WSBG-algebra $L = \langle L; |, 0\rangle$ is called an intuitionistic fuzzy sub-implicative WSBG-ideal of L if it satisfies the following conditions: for all $\zeta,\eta,\theta \in L$,*

$$\alpha(0) \geq \alpha(\zeta) \text{ and } \beta(0) \leq \beta(\zeta), \tag{1}$$
$$\alpha((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \geq \min\{\alpha(\Phi(\zeta,\eta,\theta)), \alpha(\theta)\}, \tag{2}$$
$$\beta((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \leq \max\{\beta(\Phi(\zeta,\eta,\theta)), \beta(\theta)\}, \tag{3}$$

where $\Phi(\zeta,\eta,\theta) = ((((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))|(\theta|\theta))|(((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))|(\theta|\theta)))$.

Example 6. *Let $L = \{0, 1, 2, 3\}$ be a WSBG-algebra with the binary operation $|$ defined by the following Cayley table:*

|	0	1	2	3
0	0	0	2	3
1	1	2	0	2
2	2	0	3	0
3	3	0	0	2

Define the intuitionistic fuzzy set $\mathcal{L} = (L,\alpha,\beta)$ by:

$$\alpha(0) = \alpha(2) = \alpha(3) = 0.5, \quad \alpha(1) = 0.25,$$

$$\beta(0) = \beta(2) = \beta(3) = 0.25, \quad \beta(1) = 0.5.$$

Therefore, $\mathcal{L} = (L,\alpha,\beta)$ is an intuitionistic fuzzy sub-implicative WSBG-ideal of L.

Proposition 3. *Every intuitionistic fuzzy sub-implicative WSBG-ideal of a WSBG-algebra $L = \langle L; |, 0\rangle$ is an intuitionistic fuzzy WSBG-ideal of L.*

Proof. Let $\mathcal{L} = (L, \alpha, \beta)$ be an intuitionistic fuzzy sub-implicative WSBG-ideal of $L = \langle L; |, 0\rangle$. By definition, we have

$$\alpha(0) \geq \alpha(\zeta) \text{ and } \beta(0) \leq \beta(\zeta),$$

and the following conditions hold: for all $\zeta, \theta \in L$,

$$\alpha((\zeta|(\zeta|(\zeta|\zeta)))|(\zeta|(\zeta|(\zeta|\zeta)))) \geq \min\{\alpha(\Phi(\zeta, \zeta, \theta)), \alpha(\theta)\},$$

$$\beta((\zeta|(\zeta|(\zeta|\zeta)))|(\zeta|(\zeta|(\zeta|\zeta)))) \leq \max\{\beta(\Phi(\zeta, \zeta, \theta)), \beta(\theta)\},$$

where $\Phi(\zeta, \zeta, \theta) = ((((\zeta|(\zeta|(\zeta|\zeta)))|(\zeta|(\zeta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\zeta|(\zeta|\zeta)))|(\zeta|(\zeta|(\zeta|\zeta))))|(\theta|\theta)))$.

Now, for all $\zeta, \theta \in L$, we compute

1. For $\alpha(\zeta)$:

$$\begin{aligned}\alpha(\zeta) &= \alpha((\zeta|(0|0))|(\zeta|(0|0)))\\ &= \alpha((\zeta|(\zeta|(\zeta|\zeta)))|(\zeta|(\zeta|(\zeta|\zeta))))\\ &\geq \min\{\alpha(\Phi(\zeta, \zeta, \theta)), \alpha(\theta)\}\\ &= \min\{\alpha((\zeta|(\theta|\theta))|(\zeta|(\theta|\theta))), \alpha(\theta)\}.\end{aligned}$$

2. For $\beta(\zeta)$:

$$\begin{aligned}\beta(\zeta) &= \beta((\zeta|(0|0))|(\zeta|(0|0)))\\ &= \beta((\zeta|(\zeta|(\zeta|\zeta)))|(\zeta|(\zeta|(\zeta|\zeta))))\\ &\leq \max\{\beta(\Phi(\zeta, \zeta, \theta)), \beta(\theta)\}\\ &= \max\{\beta((\zeta|(\theta|\theta))|(\zeta|(\theta|\theta))), \beta(\theta)\}.\end{aligned}$$

Thus, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy WSBG-ideal of L. Therefore, every intuitionistic fuzzy sub-implicative WSBG-ideal of L is an intuitionistic fuzzy WSBG-ideal of L.

Example 7. *From the WSBG-algebra $L = \langle L; |, 0\rangle$ where $L = \{0, a, b\}$ in Example 4, define an intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ where:*

$$\begin{aligned}&\alpha(0) = \alpha(a) = \alpha(b) = 0.5,\\ &\beta(0) = 0.25, \quad \beta(a) = \beta(b) = 0.\end{aligned}$$

Then $\mathcal{L}$ is an intuitionistic fuzzy WSBG-ideal of L. However, $\mathcal{L}$ is not an intuitionistic fuzzy sub-implicative WSBG-ideal because it violates the condition:

$$\beta(0) \leq \beta(\zeta), \quad \forall \zeta \in L,$$

which fails at $\zeta = a$, since:

$$\beta(0) = 0.25 > \beta(a) = 0.$$

Therefore, this example demonstrates that an intuitionistic fuzzy WSBG-ideal is not necessarily an intuitionistic fuzzy sub-implicative WSBG-ideal.

Theorem 4. *Let $\mathcal{L} = (L, \alpha, \beta)$ be an intuitionistic fuzzy WSBG-ideal of a WSBG-algebra $L = \langle L; |, 0\rangle$. Then $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy sub-implicative WSBG-ideal of L if and only if for all $\zeta, \eta \in L$,*

$$\alpha((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \geq \alpha((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))$$

and

$$\beta((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \leq \beta((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta)))).$$

Proof. Let $\mathcal{L} = (L, \alpha, \beta)$ be an intuitionistic fuzzy sub-implicative WSBG-ideal of $L = \langle L; |, 0\rangle$. By definition, we have

$$\alpha((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \geq \min\{\alpha(\Phi(\zeta, \eta, 0)), \alpha(0)\},$$

where

$$\Phi(\zeta, \eta, 0) = ((((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))|(0|0))|(((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))|(0|0))).$$

This simplifies to

$$\alpha((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \geq \alpha((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta)))).$$

Similarly, for β, we have

$$\beta((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \leq \max\{\beta(\Phi(\zeta, \eta, 0)), \beta(0)\}.$$

This simplifies to

$$\beta((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \leq \beta((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta)))).$$

Conversely, assume that the conditions

$$\alpha((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \geq \alpha((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))$$

and

$$\beta((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \leq \beta((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))$$

hold. Since $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy WSBG-ideal of L, we know

$$\alpha(0) \geq \alpha(\zeta) \text{ and } \beta(0) \leq \beta(\zeta).$$

Using these properties, we verify that $\mathcal{L} = (L, \alpha, \beta)$ satisfies the conditions of an intuitionistic fuzzy sub-implicative WSBG-ideal of L. For α, we compute

$$\alpha((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \geq \alpha((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta)))) \geq \min\{\alpha(\Phi(\zeta, \eta, \theta)), \alpha(\theta)\}.$$

For β, we compute

$$\beta((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \leq \beta((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta)))) \leq \max\{\beta(\Phi(\zeta, \eta, \theta)), \beta(\theta)\}.$$

Thus, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy sub-implicative WSBG-ideal of L. This completes the proof.

The identity used in Theorem 4 is not an assumption but a necessary and sufficient condition that fully characterizes when an intuitionistic fuzzy WSBG-ideal becomes a sub-implicative WSBG-ideal.

Definition 12. *A WSBG-algebra is said to be implicative if it satisfies the following condition:*

$$\zeta|(\zeta|(\eta|\eta)) = \eta|(\eta|(\zeta|\zeta)), \quad \forall \zeta, \eta \in L.$$

Example 8. *Consider the algebra* $L = \langle L; |, 0\rangle$ *with* $L = \{0, 1, 2, 3, 4\}$ *and binary operation* $|$ *defined by the Cayley table:*

$\|$	0	1	2	3	4
0	0	0	2	3	0
1	2	3	3	0	0
2	2	0	3	0	0
3	3	0	0	2	1
4	0	2	1	3	0

Hence, this algebra is an implicative WSBG-algebra.

Theorem 5. *Let* $L = \langle L; |, 0\rangle$ *be an implicative WSBG-algebra. Then every intuitionistic fuzzy WSBG-ideal of* L *is an intuitionistic fuzzy sub-implicative WSBG-ideal of* L.

Proof. Let $\mathcal{L} = (L, \alpha, \beta)$ be an intuitionistic fuzzy WSBG-ideal of $L = \langle L; |, 0\rangle$. By definition, we have

$$\alpha(0) \geq \alpha(\zeta) \text{ and } \beta(0) \leq \beta(\zeta), \quad \forall \zeta \in L.$$

Now, consider

$$\alpha((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))).$$

Using the implicative property $(\zeta|(\zeta|(\eta|\eta)) = \eta|(\eta|(\zeta|\zeta)))$, we have

$$\alpha((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \geq \min\{\alpha(\Phi(\zeta, \eta, \theta)), \alpha(\theta)\},$$

where

$$\Phi(\zeta, \eta, \theta) = ((((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))|(\theta|\theta))|(((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))|(\theta|\theta))).$$

By the implicative property, this simplifies to

$$\alpha((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \geq \alpha((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta)))).$$

Similarly, for β, we have

$$\beta((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))).$$

Using the same property, we get

$$\beta((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \leq \max\{\beta(\Phi(\zeta, \eta, \theta)), \beta(\theta)\}.$$

This simplifies to

$$\beta((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \leq \beta((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta)))).$$

Thus, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy sub-implicative WSBG-ideal of L.

Example 9. *Consider the set $L = \{0, a, b, c\}$ equipped with the binary operation $|$ defined by the following Cayley table:*

$\vert$	0	a	b	c
0	0	b	c	a
a	b	0	c	b
b	c	a	0	b
c	a	c	0	0

Hence, $L = \langle L; |, 0\rangle$ is a WSBG-algebra. We check whether the algebra is implicative according to the definition:

$$\zeta|(\zeta|(\eta|\eta)) = \eta|(\eta|(\zeta|\zeta)), \quad \forall \zeta, \eta \in L.$$

Direct computation shows this condition fails for $\zeta = 0$ and $\eta = a$. Therefore, the algebra is not implicative. Define an intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ as:

$$\alpha(x) = 0.5, \quad \forall x \in L; \quad \beta(0) = \beta(b) = 0.25, \quad \beta(a) = \beta(c) = 0.$$

It can be verified that $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy WSBG-ideal of L. However, $\mathcal{L} = (L, \alpha, \beta)$ is not an intuitionistic fuzzy sub-implicative WSBG-ideal because it fails the bound condition:

$$\beta(0) = 0.25 > \beta(a) = 0.$$

This example demonstrates that in a WSBG-algebra that is not implicative, an intuitionistic fuzzy WSBG-ideal may not be a sub-implicative fuzzy WSBG-ideal.

Definition 13. *A WSBG-algebra $L = \langle L; |, 0\rangle$ is called medial if it satisfies the following condition:*

$$\zeta|(\zeta|(\eta|\eta)) = \eta|\eta, \quad \forall \zeta, \eta \in L.$$

Example 10. *Consider the algebra $L = \langle L; |, 0\rangle$ with $L = \{0, 1, 2\}$ and binary operation $|$ defined by the Cayley table:*

$\vert$	0	1	2
0	0	1	0
1	0	0	0
2	0	0	0

Hence, this algebra is a medial WSBG-algebra.

Lemma 1. *In a medial WSBG-algebra $L = \langle L; |, 0\rangle$, the following property holds:*

$$((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))|(\eta|(\zeta|\zeta)) = \zeta|(\zeta|(\eta|\eta)), \quad \forall \zeta, \eta \in L.$$

Proof. Let $a = \zeta|(\zeta|(\eta|\eta))$. Then, using this shorthand:

$$(\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))) = a|a \quad \text{and} \quad \eta|(\zeta|\zeta) = b.$$

Then the left-hand side becomes:

$$(a|a)|b.$$

Now, if the operation $|$ satisfies the medial-like identity:

$$(a|a)|b = a,$$

then:

$$((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))|(\eta|(\zeta|\zeta)) = \zeta|(\zeta|(\eta|\eta)).$$

Theorem 6. *Every intuitionistic fuzzy WSBG-ideal of a medial WSBG-algebra* $L = \langle L; |, 0\rangle$ *is an intuitionistic fuzzy sub-implicative WSBG-ideal of* L.

Proof. Let $\mathcal{L} = (L, \alpha, \beta)$ be an intuitionistic fuzzy WSBG-ideal of a medial WSBG-algebra $L = \langle L; |, 0\rangle$. By definition, we have

$$\alpha(0) \geq \alpha(\zeta) \text{ and } \beta(0) \leq \beta(\zeta), \quad \forall \zeta \in L.$$

Now, consider

$$\alpha((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))).$$

Using the medial property $\zeta|(\zeta|(\eta|\eta)) = \eta|\eta$, we get

$$\alpha((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) = \alpha(\zeta).$$

Furthermore,

$$\alpha(\zeta) \geq \min\{\alpha((\zeta|(\theta|\theta))|(\zeta|(\theta|\theta))), \alpha(\theta)\}.$$

By expanding this, we have

$$\begin{aligned}&\alpha((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta))))\\ &\geq \min\{\alpha((((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))|(\theta|\theta))|(((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))|(\theta|\theta))), \alpha(\theta)\}.\end{aligned}$$

Finally, by the lemma, this simplifies to

$$\alpha((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \geq \alpha((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta)))).$$

Similarly, for β, we have

$$\beta((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))).$$

Using the medial property, we get

$$\beta((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) = \beta(\zeta).$$

Furthermore,

$$\beta(\zeta) \leq \max\{\beta((\zeta|(\theta|\theta))|(\zeta|(\theta|\theta))), \beta(\theta)\}.$$

By expanding this, we have

$$\begin{aligned}&\beta((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta))))\\ &\leq \max\{\beta((((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))|(\theta|\theta))|(((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta))))|(\theta|\theta))), \beta(\theta)\}.\end{aligned}$$

Finally, by the lemma, this simplifies to

$$\beta((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))) \leq \beta((\zeta|(\zeta|(\eta|\eta)))|(\zeta|(\zeta|(\eta|\eta)))).$$

Thus, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy sub-implicative WSBG-ideal of L.

Example 11. *Consider the set* $L = \{0, a, b\}$ *equipped with the binary operation* $|$ *defined by the following Cayley table:*

$\mid$	0	a	b
0	0	b	a
a	b	0	a
b	a	b	0

Thus, $L = \langle L; |, 0\rangle$ *is a WSBG-algebra. We check the medial condition:*

$$\zeta|(\zeta|(\eta|\eta)) = \eta|\eta, \quad \forall \zeta, \eta \in L.$$

It fails for $\zeta = a, \eta = 0$*:*

$$a|(a|(0|0)) = a|(a|0) = a|b = a \neq 0 = 0|0.$$

Hence, L is not medial. Define an intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$*:*

$$\alpha(x) = 0.5, \quad \forall x \in L, \quad \beta(0) = \beta(b) = 0.25, \quad \beta(a) = 0.$$

Then $\mathcal{L} = (L, \alpha, \beta)$ *is an intuitionistic fuzzy WSBG-ideal of L. However,* $\mathcal{L} = (L, \alpha, \beta)$ *fails the bound condition for an intuitionistic fuzzy sub-implicative WSBG-ideal:*

$$\beta(0) = 0.25 > \beta(a) = 0.$$

Thus, $\mathcal{L} = (L, \alpha, \beta)$ *is not an intuitionistic fuzzy sub-implicative WSBG-ideal. This example demonstrates that in a WSBG-algebra which is not medial, an intuitionistic fuzzy WSBG-ideal need not be an intuitionistic fuzzy sub-implicative WSBG-ideal.*

Theorem 7. *Every medial WSBG-algebra is an implicative WSBG-algebra.*

Proof. Let $L = \langle L; |, 0\rangle$ be a medial WSBG-algebra. We aim to prove the identity:

$$x|(x|(y|y)) = y|(y|(x|x)), \quad \forall x, y \in L. \tag{4}$$

Assume that the identity is not valid. There exist $c_1, c_2 \in L$ such that

$$c_2|(c_2|(c_1|c_1)) \neq c_1|(c_1|(c_2|c_2)).$$

By the medial property, we have

$$c_2|c_2 \neq c_1|c_1. \tag{5}$$

By (SBG1), we have

$$((x|(x|x))|0)|((x|(x|x))|0) = 0. \tag{6}$$

Combining (SBG1) and (SBG2), we get

$$(0|(x|x))|0 = x|x. \tag{7}$$

Combining (SBG1) and the medial property, we have

$$(x|x)|(x|x) = 0. \tag{8}$$

Combining (SBG1) and the medial property again, we have

$$x|(x|0) = 0. \tag{9}$$

Combining (SBG2) and the medial property, we have

$$(0|((x|x)|(x|x)))|((x|x)|((x|x)|((x|x)|(x|x)))) = (x|x)|(x|x). \tag{10}$$

By (8) and (10), we have

$$(0|0)|((x|x)|((x|x)|((x|x)|(x|x)))) = (x|x)|(x|x). \tag{11}$$

By (8) and (11), we have

$$(0|0)|((x|x)|((x|x)|0)) = (x|x)|(x|x). \tag{12}$$

By (9) and (12), we have

$$(0|0)|0 = (x|x)|(x|x). \tag{13}$$

By (8) and (13), we have

$$(0|0)|0 = 0. \tag{14}$$

Combining (SBG2) and (6), we have

$$((0|0)|0)|(((0|(x|x))|((0|(x|x))|(0|(x|x))))|0) = 0. \tag{15}$$

By (14) and (15), we have

$$0|(((0|(x|x))|((0|(x|x))|(0|(x|x))))|0) = 0. \tag{16}$$

Combining (SBG2) and (16), we have

$$0|((0|0)|0) = 0. \tag{17}$$

By (14) and (17), we have

$$0|0 = 0. \tag{18}$$

Combining (SBG2) and (8), we have

$$(0|(x|x))|(0|((x|x)|(x|x))) = (x|x)|(x|x). \tag{19}$$

By (8) and (19), we have

$$(0|(x|x))|(0|0) = (x|x)|(x|x). \tag{20}$$

By (18) and (20), we have

$$(0|(x|x))|0 = (x|x)|(x|x). \tag{21}$$

By (7) and (21), we have

$$x|x = (x|x)|(x|x). \tag{22}$$

By (8) and (22), we have

$$x|x = 0, \quad \forall x \in L. \tag{23}$$

By (23) and (5), we have $0 \neq 0$, which is a contradiction. So, (4) is true, that is, L is an implicative WSBG-algebra.

Example 12. *Consider the algebra $L = \langle L; |, 0\rangle$ with $L = \{0, 1, 2\}$ and binary operation $|$ defined by the Cayley table:*

$\vert$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Hence, the algebra is an implicative WSBG-algebra. We found that for $\zeta = 1, \eta = 0$:

$$1|(1|(0|0)) = 1|(1|0) = 1|1 = 2 \neq 0 = 0|0.$$

Therefore, the algebra is not a medial WSBG-algebra.

Theorem 8. *Let $\mathcal{L} = (L, \alpha, \beta)$ be an intuitionistic fuzzy set of $L = \langle L; |, 0\rangle$. Then $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of L if and only if*

$$(\forall \zeta, \eta \in L) \left(\begin{array}{c} \alpha(\zeta) \geq \alpha((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta)))), \\ \beta(\zeta) \leq \beta((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta)))) \end{array} \right). \tag{24}$$

Proof. Let $\mathcal{L} = (L, \alpha, \beta)$ be an intuitionistic fuzzy implicative WSBG-ideal of $L = \langle L; |, 0\rangle$. Then, for all $\zeta, \eta \in L$, we have

$$\alpha(\zeta) \geq \min\{\alpha(0), \alpha(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(0|0))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(0|0))))\}.$$

Since $\alpha(0) \geq \alpha(\zeta)$, this simplifies to

$$\alpha(\zeta) \geq \alpha((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta)))).$$

Similarly, for $\beta(\zeta)$, we have

$$\beta(\zeta) \leq \max\{\beta(0), \beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(0|0))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(0|0))))\}.$$

Since $\beta(0) \leq \beta(\zeta)$, this simplifies to

$$\beta(\zeta) \leq \beta((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta)))).$$

Thus, $\mathcal{L} = (L, \alpha, \beta)$ satisfies condition (24).

Conversely, let $\mathcal{L} = (L, \alpha, \beta)$ be an intuitionistic fuzzy WSBG-ideal of $L = \langle L; |, 0\rangle$ and assume that condition (24) holds. Then, for all $\zeta, \eta, \theta \in L$, we have

$$\alpha(\zeta) \geq \alpha((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta)))).$$

Moreover, $\alpha((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))$ can be bounded as follows

$$\begin{aligned}&\alpha((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))\\&\geq \min\{\alpha(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \alpha(\theta)\}.\end{aligned}$$

Thus, we get

$$\alpha(\zeta) \geq \min\{\alpha(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \alpha(\theta)\}.$$

Similarly, for $\beta(\zeta)$, we have

$$\beta(\zeta) \leq \beta((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta)))).$$

And also, we attain

$$\begin{aligned}&\beta((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))\\&\leq \max\{\beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \beta(\theta)\}.\end{aligned}$$

Thus, we obtain

$$\beta(\zeta) \leq \max\{\beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \beta(\theta)\}.$$

Therefore, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of L.

The identity used in Theorem 8 is not an assumption but a necessary and sufficient condition that fully characterizes when an intuitionistic fuzzy set becomes an intuitionistic fuzzy implicative WSBG-ideal.

Theorem 9. *Let $L = \langle L; |, 0\rangle$ be a medial WSBG-algebra. Then every intuitionistic fuzzy implicative WSBG-ideal of L is an intuitionistic fuzzy sub-implicative WSBG-ideal of L.*

Proof. By assumption, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of L. Then, for all $\zeta, \eta, \theta \in L$, we have

$$\alpha(0) \geq \alpha(\zeta) \geq \min\{\alpha(\Psi(\zeta, \eta, \theta)),\ \alpha(\theta)\},$$

$$\beta(0) \leq \beta(\zeta) \leq \max\{\beta(\Psi(\zeta, \eta, \theta)),\ \beta(\theta)\}.$$

We aim to prove that $\mathcal{L} = (L, \alpha, \beta)$ satisfies the sub-implicative condition, i.e., for all $\zeta, \eta, \theta \in L$,

$$\alpha(x) \geq \min\{\alpha(\Phi(\zeta, \eta, \theta)),\ \alpha(\theta)\}, \quad \text{where } x = (\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta)))$$

$$\beta(x) \leq \max\{\beta(\Phi(\zeta, \eta, \theta)),\ \beta(\theta)\}.$$

From the medial identity:

$$\zeta|(\zeta|(\eta|\eta)) = \eta|\eta.$$

Apply this identity twice: Let $a := \zeta|(\zeta|(\eta|\eta))$, then by medial property:

$$a = \eta|\eta.$$

Thus,

$$a|a = (\eta|\eta)|(\eta|\eta).$$

Also, let

$$b := \theta|\theta, \quad \Phi := (a|a)|b, \quad \Phi = \Psi(\zeta, \eta, \theta).$$

Now define

$$x := (\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta))) = y|y.$$

Let us show that

$$y = \zeta|(\zeta|(\eta|\eta)) = a = \eta|\eta \Rightarrow x = (\eta|\eta)|(\eta|\eta) \Rightarrow x = a|a.$$

Hence,

$$x = a|a, \quad \Phi = (a|a)|b \Rightarrow \text{and the same value of } a \text{ links both sides.}$$

Therefore,

$$\alpha(x) = \alpha(a|a) \geq \min\{\alpha(\Phi),\ \alpha(\theta)\},$$

$$\beta(x) = \beta(a|a) \leq \max\{\beta(\Phi),\ \beta(\theta)\}.$$

This completes the proof that $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy sub-implicative WSBG-ideal of L.

Example 13. *From Example 10, the algebra $L = \langle L; |, 0\rangle$ with $L = \{0, 1, 2\}$ is a medial WSBG-algebra. Define an intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ by:*

$$\alpha(x) = 0.5, \quad \forall x \in L, \quad \beta(0) = \beta(2) = 0.25, \quad \beta(1) = 0.$$

We verified that $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy sub-implicative WSBG-ideal of L, but not an intuitionistic fuzzy implicative WSBG-ideal because $\beta(0) = 0.25 > \beta(1) = 0$.

Definition 14. *An intuitionistic fuzzy WSBG-ideal $\mathcal{L} = (L, \alpha, \beta)$ of a WSBG-algebra $L = \langle L; |, 0\rangle$ is said to be an intuitionistic fuzzy closed WSBG-ideal if*

$$(\forall \zeta \in L)\left(\begin{array}{c} \alpha((0|(\zeta|\zeta))|(0|(\zeta|\zeta))) \geq \alpha(\zeta), \\ \beta((0|(\zeta|\zeta))|(0|(\zeta|\zeta))) \leq \beta(\zeta) \end{array}\right). \tag{25}$$

Example 14. *From the WSBG-algebra $L = \langle L; |, 0\rangle$ where $L = \{0, a, b\}$ in Example 4, define an intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ by:*

$$\alpha(0) = 0.9, \quad \alpha(a) = 0.7, \quad \alpha(b) = 0.5,$$
$$\beta(0) = 0.1, \quad \beta(a) = 0.4, \quad \beta(b) = 0.6.$$

Thus, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy closed WSBG-ideal of L.

4. Special classes of intuitionistic fuzzy implicative WSBG-ideals

In this section, we investigate specialized subclasses of intuitionistic fuzzy implicative WSBG-ideals in WSBG-algebras. While the general notion of an intuitionistic fuzzy implicative WSBG-ideal captures a broad range of structural behaviors, certain algebraic and fuzzy-theoretic applications require stronger or more refined conditions. We therefore introduce and examine notions such as intuitionistic fuzzy completely closed WSBG-ideals, intuitionistic fuzzy closed WSBG-ideals, and intuitionistic fuzzy p-ideals.

These classes serve not only to enrich the theory but also to provide a finer framework for analyzing hierarchical relationships among fuzzy ideal structures in WSBG-algebras. Connections between these classes and the foundational concepts introduced in the previous section are also explored.

Definition 15. *Let $\mathcal{L} = (L, \alpha, \beta)$ be an intuitionistic fuzzy WSBG-ideal of a WSBG-algebra $L = \langle L; |, 0\rangle$. Then $\mathcal{L} = (L, \alpha, \beta)$ is called an intuitionistic fuzzy completely closed WSBG-ideal of L if*

$$(\forall \zeta, \eta \in L)\left(\begin{array}{c} \alpha((\zeta|(\eta|\eta))|(\zeta|(\eta|\eta))) \geq \min\{\alpha(\zeta), \alpha(\eta)\}, \\ \beta((\zeta|(\eta|\eta))|(\zeta|(\eta|\eta))) \leq \max\{\beta(\zeta), \beta(\eta)\} \end{array}\right). \tag{26}$$

Example 15. *From the WSBG-algebra $L = \langle L; |, 0\rangle$ where $L = \{0, a, b\}$ in Example 4, define the intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ by:*

$$\alpha(0) = 1, \quad \alpha(a) = 0.6, \quad \alpha(b) = 0.4,$$

$$\beta(0) = 0, \quad \beta(a) = 0.3, \quad \beta(b) = 0.7.$$

Therefore, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy completely closed WSBG-ideal of L.

Note that an intuitionistic fuzzy closed WSBG-ideal is not necessarily an intuitionistic fuzzy implicative WSBG-ideal, unless the underlying WSBG-algebra satisfies additional conditions such as implicativity. The following theorem presents one such sufficient condition.

Theorem 10. *Let $L = \langle L; |, 0\rangle$ be a WSBG-algebra satisfying*

$$(\forall \zeta, \eta, \theta \in L)\left(\begin{array}{l}((((\zeta|(\eta|\eta))|(\zeta|(\eta|\eta)))|(\zeta|(\theta|\theta)))| \\ (((\zeta|(\eta|\eta))|(\zeta|(\eta|\eta)))|(\zeta|(\theta|\theta))))|(\theta|(\eta|\eta)) = 0|0\end{array}\right). \tag{27}$$

Then L is implicative if and only if every intuitionistic fuzzy closed WSBG-ideal of L is an intuitionistic fuzzy implicative WSBG-ideal of L.

Proof. Let $L = \langle L; |, 0\rangle$ be a WSBG-algebra satisfying (27). Assume that L is implicative and $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy closed WSBG-ideal of L. Then $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy WSBG-ideal of L. Thus, $\alpha(0) \geq \alpha(\zeta)$ and $\beta(0) \leq \beta(\zeta)$. Also,

$$\begin{array}{rcl}\alpha(\zeta) & \geq & \min\{\alpha(\theta), \alpha((\zeta|(\theta|\theta))|(\zeta|(\theta|\theta)))\} \\ & = & \min\{\alpha(\theta), \alpha(((((\zeta|\zeta)|(\zeta|\zeta))|((\zeta|\zeta)|\eta))|(\theta|\theta))| \\ & & ((((\zeta|\zeta)|(\zeta|\zeta))|((\zeta|\zeta)|\eta))|(\theta|\theta)))\} \\ & = & \min\{\alpha(\theta), \alpha((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))| \\ & & (((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))\},\end{array}$$

$$\begin{array}{rcl}\beta(\zeta) & \leq & \max\{\beta(\theta), \beta((\zeta|(\theta|\theta))|(\zeta|(\theta|\theta)))\} \\ & = & \max\{\beta(\theta), \beta(((((\zeta|\zeta)|(\zeta|\zeta))|((\zeta|\zeta)|\eta))|(\theta|\theta))| \\ & & ((((\zeta|\zeta)|(\zeta|\zeta))|((\zeta|\zeta)|\eta))|(\theta|\theta)))\} \\ & = & \max\{\beta(\theta), \beta((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))| \\ & & (((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))\},\end{array}$$

which means that $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of L.

Conversely, suppose that every intuitionistic fuzzy closed WSBG-ideal of $L = \langle L; |, 0\rangle$ is an intuitionistic fuzzy implicative WSBG-ideal of L. So, it follows from the equation (27) that $\theta|(\eta|\eta) = (\zeta|(\theta|\theta))|((\zeta|(\eta|\eta))|(\zeta|(\eta|\eta)))$. Since $\theta|(\eta|\eta) = (\zeta|(\theta|\theta))|((\zeta|(\eta|\eta))|(\zeta|(\eta|\eta))) = ((\zeta|(\zeta|(\theta|\theta)))|(\zeta|(\zeta|(\theta|\theta))))|(\eta|\eta)$ from (S1) and (S3), it is obtained from (S2) that $\theta = (\zeta|(\zeta|(\theta|\theta)))|(\zeta|(\zeta|(\theta|\theta)))$. Thus, we get from (S1)-(S3) that

$$\begin{array}{rl} & \zeta|(\zeta|(\eta|\eta)) \\ = & ((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta))))|(((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta))))|(\eta|\eta)) \\ = & ((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta))))|(\eta|(((\eta|\eta)|(\eta|(\zeta|\zeta)))|((\eta|\eta)|(\eta|(\zeta|\zeta))))) \\ = & ((\eta|(\eta|(\zeta|\zeta)))|(\eta|(\eta|(\zeta|\zeta))))|(\eta|(\eta|\eta)) \\ = & (((\eta|(\eta|\eta))|((\eta|\eta)|(\eta|\eta)))|((\eta|(\eta|\eta))|((\eta|\eta)|(\eta|\eta))))|(\eta|(\zeta|\zeta)) \\ = & \eta|(\eta|(\zeta|\zeta)),\end{array}$$

for all $\zeta, \eta \in L$, which means that L is implicative.

The identity used in Theorem 8 is not an assumption but a necessary and sufficient condition that fully characterizes when an intuitionistic fuzzy set becomes an intuitionistic fuzzy implicative WSBG-ideal.

Proposition 4. *Let $L = \langle L; |, 0\rangle$ be an implicative WSBG-algebra. Then every intuitionistic fuzzy completely closed WSBG-ideal of L is an intuitionistic fuzzy implicative ideal of L.*

Proof. Let $\mathcal{L} = (L, \alpha, \beta)$ be an intuitionistic fuzzy completely closed WSBG-ideal of an implicative WSBG-algebra $L = \langle L; |, 0\rangle$. Then $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy WSBG-ideal of L. Since

$$\beta((0|(\eta|\eta))|(0|(\eta|\eta))) \leq \max\{\beta(0), \beta(\eta)\} = \beta(\eta)$$

and

$$\alpha((0|(\eta|\eta))|(0|(\eta|\eta))) \geq \min\{\alpha(0), \alpha(\eta)\} = \alpha(\eta),$$

it is obtained that $\mathcal{L}$ is an intuitionistic fuzzy closed WSBG-ideal of L. Therefore, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of L.

Example 16. *Let us consider the following algebra $L = \{0, 1, 2, 3\}$ with a binary operation $|$ given by the Cayley table:*

$\vert$	0	1	2	3
0	0	0	0	0
1	2	0	1	0
2	0	0	0	0
3	0	0	0	0

Then $L = \langle L; |, 0\rangle$ is a WSBG-algebra. Next, define an intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ with:

$$\alpha(0) = 1, \quad \alpha(1) = 0.7, \quad \alpha(2) = 0.5, \quad \alpha(3) = 0.4,$$
$$\beta(0) = 0, \quad \beta(1) = 0.2, \quad \beta(2) = 0.3, \quad \beta(3) = 0.4.$$

We verify that $\mathcal{L}$ is an intuitionistic fuzzy completely closed WSBG-ideal, but it is not an intuitionistic fuzzy implicative ideal of L. It must also satisfy:

$$\alpha(0) \geq \alpha(\zeta) \geq \min\{\alpha(\Phi(\zeta, \eta, \theta)), \alpha(\theta)\}$$

for all $\zeta, \eta, \theta \in L$, where:

$$\Phi(\zeta, \eta, \theta) = ((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))).$$

However, this condition fails at $\zeta = 1, \eta = 0, \theta = 0$.

Corollary 1. *Let $L = \langle L; |, 0\rangle$ be a medial WSBG-algebra. Then every intuitionistic fuzzy completely closed WSBG-ideal of L is an intuitionistic fuzzy implicative WSBG-ideal of L.*

Definition 16. *An intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ of a WSBG-algebra $L = \langle L; |, 0\rangle$ is called an intuitionistic fuzzy p-ideal of L if*

$$(\forall \zeta, \eta, \theta \in L) \left(\begin{array}{l} \alpha(0) \geq \alpha(\zeta), \\ \beta(0) \leq \beta(\zeta), \\ \alpha(\zeta) \geq \min\{\alpha(((((\zeta|(\theta|\theta))|(\zeta|(\theta|\theta)))|(\eta|(\theta|\theta)))| \\ \qquad\qquad (((\zeta|(\theta|\theta))|(\zeta|(\theta|\theta)))|(\eta|(\theta|\theta))))), \alpha(\eta)\}, \\ \beta(\zeta) \leq \max\{\beta(((((\zeta|(\theta|\theta))|(\zeta|(\theta|\theta)))|(\eta|(\theta|\theta)))| \\ \qquad\qquad (((\zeta|(\theta|\theta))|(\zeta|(\theta|\theta)))|(\eta|(\theta|\theta))))), \beta(\eta)\} \end{array} \right).$$

Example 17. *Consider the WSBG-algebra $L = \langle L; |, 0\rangle$ where $L = \{0, a, b\}$ with the following operation table:*

$\vert$	0	a	b
0	0	b	a
a	b	a	0
b	a	0	b

Define the intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ by:

$$\alpha(0) = 1, \quad \alpha(a) = 0.5, \quad \alpha(b) = 0.3,$$
$$\beta(0) = 0, \quad \beta(a) = 0.4, \quad \beta(b) = 0.8.$$

Therefore, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy p-ideal of $L = \langle L; |, 0\rangle$.

Definition 17. *Let $L = \langle L; |, 0\rangle$ be a WSBG-algebra. Then the set*

$$L^+ = \{\zeta \in L : (0|(\zeta|\zeta))|(0|(\zeta|\zeta)) = 0\}$$

is called the BCA-part of L.

Proposition 5. *Let $L = \langle L; |, 0\rangle$ be a WSBG-algebra. Then L^+ is a WSBG-subalgebra of L.*

Proof. By (SBG1), $(0|(0|0))|(0|(0|0)) = 0$. Then $0 \in L^+ \neq \emptyset$. Let $a, b \in L^+$. Then

$$(0|(a|a))|(0|(a|a)) = 0, \tag{28}$$
$$(0|(b|b))|(0|(b|b)) = 0. \tag{29}$$

Assume that

$$\begin{aligned}&(0|(((a|(b|b))|(a|(b|b)))|((a|(b|b))|(a|(b|b)))))|\\&(0|(((a|(b|b))|(a|(b|b)))|((a|(b|b))|(a|(b|b))))) \neq 0.\end{aligned} \tag{30}$$

By (SBG1) and (SBG2), we have

$$(0|(x|x))|0 = x|x. \tag{31}$$

By (SBG2) and (28), we have

$$(0|(a|a))|0 = 0|0. \tag{32}$$

Combining (31) and (32), we have

$$a|a = 0|0. \tag{33}$$

By (SBG2) and (29), we have

$$(0|(b|b))|0 = 0|0. \tag{34}$$

Combining (31) and (34), we have

$$b|b = 0|0. \tag{35}$$

By (30) and (35), we have

$$\begin{aligned}(0|(((a|(0|0))|(a|(0|0)))|((a|(0|0))|(a|(0|0)))))|\\(0|(((a|(0|0))|(a|(0|0)))|((a|(0|0))|(a|(0|0))))) \neq 0.\end{aligned} \tag{36}$$

By (33) and (SBG1), we have

$$(a|(0|0))|(a|(a|a)) = 0. \tag{37}$$

By (33) and (37), we have

$$(a|(0|0))|(a|(0|0)) = 0. \tag{38}$$

By (38) and (36), we have

$$\begin{aligned}(0|(0|((a|(0|0))|(a|(0|0)))))|\\(0|(((a|(0|0))|(a|(0|0)))|((a|(0|0))|(a|(0|0))))) \neq 0.\end{aligned} \tag{39}$$

By (38) and (39), we have

$$\begin{aligned}(0|(0|0))|\\(0|(((a|(0|0))|(a|(0|0)))|((a|(0|0))|(a|(0|0))))) \neq 0.\end{aligned} \tag{40}$$

By (38) and (40), we have

$$\begin{aligned}(0|(0|0))|\\(0|(0|((a|(0|0))|(a|(0|0))))) \neq 0.\end{aligned} \tag{41}$$

By (38) and (41), we have

$$(0|(0|0))|(0|(0|0)) \neq 0. \tag{42}$$

By (42) and (SBG1), we have $0 \neq 0$, which is a contradiction. So,

$$\begin{aligned}(0|(((a|(b|b))|(a|(b|b)))|((a|(b|b))|(a|(b|b)))))|\\(0|(((a|(b|b))|(a|(b|b)))|((a|(b|b))|(a|(b|b))))) = 0.\end{aligned}$$

Thus, $(a|(b|b))|(a|(b|b)) \in L^+$. Hence, L^+ is a WSBG-subalgebra of L.

Theorem 11. *Let $L = \langle L; |, 0\rangle = L^+$ be a WSBG-algebra. Then every intuitionistic fuzzy p-ideal of L is an intuitionistic fuzzy implicative WSBG-ideal of L.*

Proof. Let $\mathcal{L} = (L, \alpha, \beta)$ be an intuitionistic fuzzy p-ideal of $L = \langle L; |, 0\rangle$. Then

$$\begin{aligned}\alpha(\zeta) &\geq \min\{\alpha(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(0|(\eta|(\zeta|\zeta))))| \\ &\quad (((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(0|(\eta|(\zeta|\zeta)))))), \alpha(0)\} \\ &= \min\{\alpha(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(0|0))| \\ &\quad (((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(0|0)))), \alpha(0)\} \\ &= \min\{\alpha((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))), \alpha(0)\} \\ &= \alpha((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))\end{aligned}$$

and

$$\begin{aligned}\beta(\zeta) &\leq \max\{\beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(0|(\eta|(\zeta|\zeta))))| \\ &\quad (((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(0|(\eta|(\zeta|\zeta)))))), \beta(0)\} \\ &= \max\{\beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(0|0))| \\ &\quad (((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(0|0)))), \beta(0)\} \\ &= \max\{\beta((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))), \beta(0)\} \\ &= \beta((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta)))).\end{aligned}$$

Hence, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of L.

Example 18. *Consider the WSBG-algebra $L = \langle L; |, 0\rangle$ where $L = \{0, a, b, c\}$ with the following operation table:*

$\vert$	0	a	b	c
0	0	c	b	a
a	c	0	b	b
b	b	b	0	0
c	a	b	0	c

Thus, $L^+ = \{0, a, b\} \neq L$. Define the intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ by:

$$\alpha(0) = 1, \quad \alpha(a) = 0.7, \quad \alpha(b) = 0.6, \quad \alpha(c) = 0.4,$$

$$\beta(0) = 0, \quad \beta(a) = 0.2, \quad \beta(b) = 0.3, \quad \beta(c) = 0.5.$$

Then $\mathcal{L}$ is an intuitionistic fuzzy p-ideal, but it is not an intuitionistic fuzzy implicative WSBG-ideal of L. According to Definition 10, for all $\zeta, \eta, \theta \in L$:

$$\alpha(0) \geq \alpha(\zeta) \geq \min\{\alpha(\Phi(\zeta, \eta, \theta)), \alpha(\theta)\},$$

and similarly for β, where:

$$\Phi(\zeta, \eta, \theta) = (((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))).$$

However, this condition fails at $\zeta = a, \eta = b, \theta = 0$.

Definition 18. *Let $\mathcal{L} = (L, \alpha, \beta)$ be an intuitionistic fuzzy set on a WSBG-algebra $L = \langle L; |, 0\rangle$. For real numbers $\delta, \gamma \in \mathbb{R}$ such that:*

$$\delta \in [\pm, 0] := [-r, 0], \quad \gamma \in [0, \mp] := [0, r]$$

for some $r > 0$, we define the translated intuitionistic fuzzy set:

$$\mathcal{L}^T_{(\delta,\gamma)} = \big(L,\ \alpha_{(\delta,T)},\ \beta_{(\gamma,T)}\big)$$

where:

$$\alpha_{(\delta,T)}(\zeta) := \alpha(\zeta) - \delta, \quad \beta_{(\gamma,T)}(\zeta) := \beta(\zeta) - \gamma, \quad \forall \zeta \in L.$$

Here, the superscript T serves only as a label indicating the form of threshold transformation.

Theorem 12. *If an intuitionistic fuzzy set $\mathcal{L} = (L, \alpha, \beta)$ in a WSBG-algebra $L = \langle L; |, 0\rangle$ is an intuitionistic fuzzy implicative WSBG-ideal of L, then for all $(\delta, \gamma) \in [\pm, 0] \times [0, \mp]$, an intuitionistic fuzzy (δ, γ)-translation $\mathcal{L}^T_{(\delta,\gamma)} = (L, \alpha_{(\delta,T)}, \beta_{(\gamma,T)})$ of $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of L.*

Proof. Assume that $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of $L = \langle L; |, 0\rangle$. For any $(\delta, \gamma) \in [\pm, 0] \times [0, \mp]$ and for all $\zeta \in L$, we have $\alpha(0) \geq \alpha(\zeta)$ and $\beta(0) \leq \beta(\zeta)$. Also,

$$\begin{aligned} \alpha_{(\delta,T)}(0) &= \alpha(0) - \delta \\ &\geq \alpha(\zeta) - \delta \\ &= \alpha_{(\delta,T)}(\zeta) \end{aligned}$$

and

$$\begin{aligned} \beta_{(\gamma,T)}(0) &= \beta(0) - \gamma \\ &\leq \beta(\zeta) - \gamma \\ &= \beta_{(\gamma,T)}(\zeta). \end{aligned}$$

Next, let $\zeta, \eta, \theta \in L$. Then

$$\alpha(\zeta) \geq \min\{\alpha(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \alpha(\theta)\}$$

and

$$\beta(\zeta) \leq \max\{\beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \beta(\theta)\}.$$

Therefore,

$$\begin{aligned} &\alpha_{(\delta,T)}(\zeta) \\ &= \alpha(\zeta) - \delta \\ &\geq \min\{\alpha(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \alpha(\theta)\} - \delta \\ &= \min\{\alpha(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))) - \delta, \alpha(\theta) - \delta\} \\ &= \min\{\alpha_{(\delta,T)}(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \alpha_{(\delta,T)}(\theta)\}. \end{aligned}$$

Similarly,

$$\begin{aligned}
&\beta_{(\gamma,T)}(\zeta)\\
&= \beta(\zeta) - \gamma\\
&\le \max\{\beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \beta(\theta)\} - \gamma\\
&= \max\{\beta(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))) - \gamma, \beta(\theta) - \gamma\}\\
&= \max\{\beta_{(\gamma,T)}(((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta)))), \beta_{(\gamma,T)}(\theta)\}.
\end{aligned}$$

Hence, $\mathcal{L}^T_{(\delta,\gamma)} = (L, \alpha_{(\delta,T)}, \beta_{(\gamma,T)})$ is an intuitionistic fuzzy implicative WSBG-ideal of L.

Theorem 13. *If there exists $(\delta,\gamma) \in [\pm, 0] \times [0, \mp]$ such that the intuitionistic fuzzy (δ,γ)-translation $\mathcal{L}^T_{(\delta,\gamma)} = (L, \alpha_{(\delta,T)}, \beta_{(\gamma,T)})$ of $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of a WSBG-algebra $L = \langle L; |, 0\rangle$, then $\mathcal{L}$ is also an intuitionistic fuzzy implicative WSBG-ideal of L.*

Proof. Assume $\mathcal{L}^T_{(\delta,\gamma)} = (L, \alpha_{(\delta,T)}, \beta_{(\gamma,T)})$ is an intuitionistic fuzzy implicative WSBG-ideal of $L = \langle L; |, 0\rangle$. Then for all $\zeta \in L$,

$$\alpha_{(\delta,T)}(0) = \alpha(0) - \delta \ge \alpha_{(\delta,T)}(\zeta) = \alpha(\zeta) - \delta,$$

$$\beta_{(\gamma,T)}(0) = \beta(0) - \gamma \le \beta_{(\gamma,T)}(\zeta) = \beta(\zeta) - \gamma.$$

For $\zeta, \eta, \theta \in L$, the implicative condition for $\mathcal{L}^T_{(\delta,\gamma)}$ gives

$$\alpha_{(\delta,T)}(\zeta) \ge \min\{\alpha_{(\delta,T)}(f(\zeta,\eta,\theta)), \alpha_{(\delta,T)}(\theta)\},$$

where

$$f(\zeta,\eta,\theta) = ((((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|(((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))),$$

expanding

$$\alpha(\zeta) - \delta \ge \min\{\alpha(f(\zeta,\eta,\theta)) - \delta, \alpha(\theta) - \delta\}.$$

Adding δ to all terms,

$$\alpha(\zeta) \ge \min\{\alpha(f(\zeta,\eta,\theta)), \alpha(\theta)\}.$$

Similarly, for β,

$$\beta_{(\gamma,T)}(\zeta) \le \max\{\beta_{(\gamma,T)}(f(\zeta,\eta,\theta)), \beta_{(\gamma,T)}(\eta)\},$$

expanding

$$\beta(\zeta) - \gamma \le \max\{\beta(f(\zeta,\eta,\theta)) - \gamma, \beta(\eta) - \gamma\}.$$

Adding γ to all terms,

$$\beta(\zeta) \le \max\{\beta(f(\zeta,\eta,\theta)), \beta(\eta)\}.$$

Thus, $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of L.

Theorem 14. *Let $\overline{\mathcal{L}} = (L, \overline{\beta}, \overline{\alpha})$ be an intuitionistic fuzzy set in a WSBG-algebra $L = \langle L; |, 0\rangle$. Then $\mathcal{L} = (L, \alpha, \beta)$ is an intuitionistic fuzzy implicative WSBG-ideal of L if and only if for all $t, s \in [0,1]$, $U(\beta, t)$ and $L(\alpha, s)$ are implicative WSBG-ideals of L, provided $U(\beta, t)$ and $L(\alpha, s)$ are nonempty.*

Proof. Assume $\overline{\mathcal{L}} = (L, \overline{\beta}, \overline{\alpha})$ is an intuitionistic fuzzy implicative WSBG-ideal of $L = \langle L; |, 0\rangle$. Let $t, s \in [0,1]$ be such that $U(\beta, t)$ and $L(\alpha, s)$ are nonempty.

Case 1: $U(\beta, t)$ is an implicative WSBG-ideal of L.

Let $\zeta \in U(\beta, t)$. Then $\beta(\zeta) \geq t$. Since $\overline{\mathcal{L}}$ is an intuitionistic fuzzy implicative WSBG-ideal of L, we have

$$\overline{\beta}(0) \leq \overline{\beta}(\zeta) \implies 1 - \beta(0) \leq 1 - \beta(\zeta) \implies \beta(0) \geq \beta(\zeta) \geq t.$$

Hence, $0 \in U(\beta, t)$.

Next, for $\zeta, \eta, \theta \in L$, let $f(\zeta, \eta, \theta)$ denote the complex expression:

$$f(\zeta, \eta, \theta) = (((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta))))|(\theta|\theta))|((\zeta|(\eta|(\zeta|\zeta)))|(\zeta|(\eta|(\zeta|\zeta)))).$$

If $\theta \in U(\beta, t)$, then $\beta(f(\zeta, \eta, \theta)) \geq t$ and $\beta(\theta) \geq t$. Since $\overline{\mathcal{L}}$ is an intuitionistic fuzzy implicative WSBG-ideal of L,

$$\overline{\beta}(\zeta) \leq \max\{\overline{\beta}(f(\zeta, \eta, \theta)), \overline{\beta}(\theta)\}.$$

Expanding this:

$$1 - \beta(\zeta) \leq \max\{1 - \beta(f(\zeta, \eta, \theta)), 1 - \beta(\theta)\}.$$

Simplifying:

$$\beta(\zeta) \geq \min\{\beta(f(\zeta, \eta, \theta)), \beta(\theta)\} \geq t.$$

Thus, $\zeta \in U(\beta, t)$, proving $U(\beta, t)$ is an implicative WSBG-ideal of L.

Case 2: $L(\alpha, s)$ is an implicative WSBG-ideal of L.

Let $\zeta \in L(\alpha, s)$. Then $\alpha(\zeta) \geq s$. Since $\overline{\mathcal{L}}$ is an intuitionistic fuzzy implicative WSBG-ideal of L,

$$\overline{\alpha}(0) \geq \overline{\alpha}(\zeta) \implies 1 - \alpha(0) \geq 1 - \alpha(\zeta) \implies \alpha(0) \leq \alpha(\zeta) \leq s.$$

Hence, $0 \in L(\alpha, s)$.

For $\zeta, \eta, \theta \in L$, if $\theta \in L(\alpha, s)$, then $\alpha(f(\zeta, \eta, \theta)) \geq s$ and $\alpha(\theta) \geq s$. Since $\overline{\mathcal{L}}$ is an intuitionistic fuzzy implicative WSBG-ideal of L,

$$\overline{\alpha}(\zeta) \leq \max\{\overline{\alpha}(f(\zeta, \eta, \theta)), \overline{\alpha}(\theta)\}.$$

Expanding this:

$$1 - \alpha(\zeta) \leq \max\{1 - \alpha(f(\zeta, \eta, \theta)), 1 - \alpha(\theta)\}.$$

Simplifying:

$$\alpha(\zeta) \geq \min\{\alpha(f(\zeta, \eta, \theta)), \alpha(\theta)\} \geq s.$$

Thus, $\zeta \in L(\alpha, s)$, proving $L(\alpha, s)$ is an implicative WSBG-ideal of L.

Conversely, assume $U(\beta, t)$ and $L(\alpha, s)$ are implicative WSBG-ideals of a WSBG-algebra $L = \langle L; |, 0\rangle$ for all $t, s \in [0, 1]$ when they are nonempty. For any $\zeta \in L$, let $t = \beta(\zeta)$. Then $0 \in U(\beta, t)$, so

$$\beta(0) \geq \beta(\zeta) \implies \overline{\beta}(0) \leq \overline{\beta}(\zeta).$$

Similarly, let $s = \alpha(\zeta)$. Then $0 \in L(\alpha, s)$, so

$$\alpha(0) \leq \alpha(\zeta) \implies \overline{\alpha}(0) \geq \overline{\alpha}(\zeta).$$

Thus, $\overline{\mathcal{L}}$ is an intuitionistic fuzzy implicative WSBG-ideal of L.

5. Conclusion

This paper presents a detailed study of intuitionistic fuzzy WSBG-ideals and intuitionistic fuzzy implicative WSBG-ideals within the framework of WSBG-algebras. By combining intuitionistic fuzzy set theory with the Sheffer stroke operation, we have extended classical algebraic frameworks to handle uncertainty and vagueness, thereby contributing significantly to the development of fuzzy algebraic structures. A key result of this work is the establishment of a structural connection between intuitionistic fuzzy implicative WSBG-ideals and their level sets, demonstrating that every level set corresponds to an implicative WSBG-ideal and vice versa. This result provides a systematic approach for analyzing these subalgebras, enhancing our understanding of their role in WSBG-algebras.

Additionally, we explored the properties of intuitionistic fuzzy WSBG-ideals, proving that every intuitionistic fuzzy implicative WSBG-ideal is also an intuitionistic fuzzy WSBG-ideal. However, the converse does not always hold. This distinction highlights the unique characteristics of WSBG-ideals and their importance in the algebraic hierarchy. The integration of intuitionistic fuzzy sets with WSBG-algebras is a novel contribution, offering a robust framework for reasoning under uncertainty. These findings have potential applications in areas such as decision-making, artificial intelligence, and quantum computing, where classical logic frameworks often fall short.

The results of this study open several promising avenues for future research. These include extending the concepts of intuitionistic fuzzy WSBG-algebras to other algebraic systems, exploring their dynamic behavior in evolving systems, and developing computational tools for analyzing large-scale structures. Furthermore, practical applications in fields such as machine learning, optimization, and fuzzy control systems could benefit from these algebraic tools. Overall, this work provides a solid foundation for the study of intuitionistic fuzzy algebraic structures and their applications in addressing complexity and uncertainty in mathematical and computational models.

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