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Applications of the Supra Soft sd-Closure Operator to Soft Connectedness and Compactness

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Abstract. The concepts of supra soft somewhere dense closure, also known as SS-sd-closure, are first applied in this study to the problem of connectedness in supra soft topological spaces, or SSTSs. To be more specific, we use the concepts of SS-sd-components to describe a novel method of connectedness via SS-sd-sets, which we refer to as SSL-sd-connectedness, or supra soft locally sd-connectedness. SS-sd-hyperconnectedness, a different kind of connection via SS-sd-sets in SSTSs, is shown. We investigate the master characteristics of different kinds of connectedness. We found out that, SS-sd-hyperconnected spaces are equivalent to SS-sd-connected spaces, which distinguishes our concepts from their counterparts. Furthermore, we present two new forms of compactness, called SS-sd-compactness and SS-sd-Lindelöfness, which are based on the concepts of super soft sd-sets in the context of SSTSs. We get into a lot of detail about their primary features. In particular, we demonstrate that an SS-sd-compact (Lindelöf) soft set is the soft intersection of an SS-sc-set and an SS-sd-compact (Lindelöf) soft set. Furthermore, an SS-sd-compact (Lindelöf) SSTS with the soft finite (countable) intersection property also known as SFIP (SCIP) has been demonstrated in terms of its behaviour. Finally, we compare our novel soft versions of connectedness and compactness using SS-sd-sets with earlier research and add two topological charts to Figures 1 and 2 to illustrate the main ideas of this study. Concrete examples and counterexamples have verified that the arrows in these charts are non-reversible.

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Key Words and Phrases: Supra soft sd-operators, SSL-sd-connectedness, SS-sd-hyperconnectedness, SS-almost compactness, SS-sd-compactness, SS-sd-Lindelöfness

1. Introduction

The approach of supra topological spaces [1] was defined by Mashhour et al., in 1983. Numerous applications of this study have been proposed. in [2–4]. Novel rough sets models inspired by these spaces have been explored in [5].

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A crucial alternative tool to fuzzy, crisp, and rough set theories all of which struggle with handling uncertainties the soft set theory [6]. Maji et al. [7] improved the soft set theory by defining new operations. Several concrete applications on soft sets have been introduced in rough set models [8], medical sciences [9], and decision making problems [10].

In 2011, the soft topological spaces (or STSs) [11–13] were introduced as a parameterized collections of crisp topological spaces. In the same year, the notions of soft continuity were introduced by Ahmad and Kharal [14]. After that, some classes of soft functions [15, 16] have been presented. Later, more investigations of soft continuity were disused [17, 18].

Several studies related to generalized soft open sets and generalized soft continuity, named soft b-open sets and soft b-continuous functions [19], soft semi-open sets and soft semi irresolute soft functions [20, 21], several kinds of soft continuity[22], soft sd-sets [23], and nearly soft β-open sets [24] have been presented. Al-Shami et al. [25] applied these notions in compactness and connectedness

In 2014, the soft ideal notion was introduced by Kandil et al. [26]. They used the soft local functions to define the class of soft ideal topological spaces. These notions have been generalized by using the approach of soft semi-open sets [27, 28]. Novel soft ideal rough topological spaces and application in Diabetes mellitus [29] have been explored by Abd El-latif in 2018. After that, many researchers used the soft ideal notions to generalize several weaker classes of soft open sets [30–32] and soft separation axioms [33, 34].

Since connectedness has a vital role in discriminating among different STSs, Weijian and Lin [35] presented the notions of connectedness in STSs in 2013. More investigations for this new approach have been discussed later in [36, 37]. An application of soft connectedness in decision making problems has been introduced in [38]. Kandil et al. [39] used the soft ideal notions to define a new class of soft connectedness, named soft I-connectedness. Abd El-latif [40] used the notions of soft β-open sets to present a new generalization to this notion. In 2023, Al-Ghour and Al-Saadi [41] defined a new approach of connectedness in STS, named soft weakly connected sets.

Aygünoglu and Aygün [42] in 2013, defined the notions of soft compactness. A stronger notions of soft compactness [43] were introduced, in 2014, which generalized by using using the soft ideal notion in [44]. Recently, Al-Shami et al. [45] used the notion of soft sd-sets to define six types of compactness in STS in 2021.

The SS-operators are not only important in accuracy measures [46] , but also tools to introduce many topological properties SSTSs like, generating supra soft topologies via soft set operators [47], continuity [48], connectedness [49], compactness [50],......etc. This encouraged El-Sheikh and Abd El-latif [51], in 2014, to define the notions of SSTSs as a parameterized collections of crisp supra topological spaces. They also defined several types of SS-operators. In addition, they defined different types of SS-continuity.

Later, many generalized SS-operators via SS-b-open sets [52], SS-δ_i-open sets [53, 54], SS-sw-open sets [55], and SS-sd-sets [56] have been studied. Abd El-latif and Alqahtani [57] introduced different types of SS-continuous functions inspired by SS-sd-sets and SS-sd-operators. Abd El-latif [58] presented different kinds of connectedness in SSTSs based on SS-b-open sets. He also defined different kinds of compactness [59] in SSTSs. Al-shami and El-Shafei [60] extended these ideas by using the notions of SS-pre-open set. El-Shafei and Al-Shami [61] defined the SS-sd-connectedness, as a generalization to many famous studies. Abd El-latif et al. [62] defined the notions of SS-sd-connectedness in the frame of SSTSs.

This paper is arranged as follows: In Preliminaries, we provide the approaches and terminologies which shall needed in the sequels.

In Section 3: We investigate more characterizations of the notions of SS-sd-connectedness. In special, we define the concept of SS-sd-component, and use it to define a new type of connectedness in SSTS, named SSL-sd-connectedness. Another type of connectedness in SSTSs, named SS-sd-hyperconnectedness has been defined. We show that it is equivalent to the notions of SS-sd-connectedness [62]. Moreover, the relationships among them, in addition to the relationships with previous studies, have been provided. Furthermore, we present a topological chart to declare the key concepts presented in Figure 1. Also, we provide counterexamples to elucidate that the arrows in this chart are non-reversible.

In Section 4: We introduce two different types of compactness in SSTSs, named SS-sd-compactness and SS-sd-Lindelöfness. We study several of their essential properties. Moreover, we discuss the relationships between supra soft-sd-compact (Lindelöf) SSTS and their parametric supra topological spaces, which are defined on any universal set and finite set of parameters. Finally, a topological chart to illustrate the key concepts are presented in Figure 2, and confirmed by concrete counterexamples.

2. Preliminaries

The concepts and terms that will be used in this manuscript are introduced in this section; for further information, see [11, 51, 56, 57].

Definition 1. *[6] A soft set is a pair (K, Δ); represented by K_Δ, over the initial universe U and the set of parameters Δ. It is characterized by $K_\Delta = \{K(\gamma) : \gamma \in \Delta,\ K : \Delta \to P(U)\}$. If for all $\gamma \in \Delta$, $K(\gamma) = \varphi$ (respectively, $K(\gamma) = U$), then (K, Δ) is referred to as a null (or absolute) soft set, and it is represented by $\tilde{\varphi}$ (or $\tilde{U}$, respectively). From now on, we will use $S(U)_\Delta$ to represent the class of all soft sets.*

Definition 2. *[11] If the collection $\tau \subseteq S(U)_\Delta$ comprises $\tilde{U}, \tilde{\varphi}$, and is closed under arbitrary soft union and finite soft intersection, then it is referred to as a soft topology on U. Over U, the triplet (U, τ, Δ) is referred to as an STS.*

Definition 3. *[11] Assume that $(K, \Delta) \in S(U)_\Delta$ and (U, τ, Δ) are STS. The intersection of all soft closed supersets of (K, Δ) is the soft closure of (K, Δ); it is represented by $cl(K, \Delta)$. Additionally, the union of all soft open subsets of (G, Δ) is the soft interior of (G, Δ); this is represented as $int(G, \Delta)$.*

Definition 4. *[11, 20] The soft set $(G, \Delta) \in S(U)_\Delta$ is referred to as a soft point in $\tilde{U}$; it is represented as r_γ, provided that $r \in U$ and $\gamma \in \Delta$ exist such that $G(\gamma) = \{r\}$ and $G(\gamma') = \varphi$ for every $\gamma' \in \Delta - \{\gamma\}$. Moreover, $s_\gamma \tilde{\in} (F, \Delta)$, if $G(\gamma) \subseteq F(\gamma)$ for the element $\gamma \in \Delta$.*

Theorem 1. *[14] For the soft map $\psi_{sd} : (U, \tau, \Delta) \to (V, \sigma, \Lambda)$, the following statements hold.*

(1) $\psi_{sd}^{-1}(T^{\tilde{c}}, \Lambda) = (\psi_{sd}^{-1}(T, \Lambda))^{\tilde{c}}\ \forall\ (T, \Lambda) \in S(V)_\Lambda$.

(2) $\psi_{sd}(\psi_{sd}^{-1}(T, \Lambda)) \tilde{\subseteq} (T, \Lambda)\ \forall\ (T, \Lambda) \in S(V)_\Lambda$. *In the event that ψ_{sd} is surjective, we obtain equality.*

(3) $(M, \Delta) \tilde{\subseteq} \psi_{sd}^{-1}(\psi_{sd}(M, \Delta))\ \forall\ (M, \Delta) \in S(U)_\Delta$. *In the event that ψ_{sd} is injective, we obtain equality.*

(4) $\psi_{sd}(\tilde{U}) \tilde{\subseteq} \tilde{V}$. *In the event that ψ_{sd} is surjective, we obtain equality.*

Definition 5. *[51] An SSTS on U is defined as the collection $\mu \subseteq S(U)_\Delta$ if μ contains $\tilde{U}, \tilde{\varphi}$, and is closed under an arbitrary soft union.*
Also, the SS-interior of a soft subset (T, Δ), is indicated by $int^s(T, \Delta)$, which is the soft union of all SS-open subsets of (T, Δ).
Additionally, SS-closure of (T, Δ) is represented as $cl^s(T, \Delta)$, which is the soft intersection of all supra closed soft supersets of (T, Δ).
Moreover, (T, Δ) is referred to as an SS-open set if $(T, \Delta) \in \mu$, and its soft complement $(T^{\tilde{c}}, \Delta)$ is referred to as an SS-closed set.

Definition 6. *[51] Consider the STS (U, τ, Δ) and the SSTS (U, μ, Δ). If $\tau \subset \mu$, then μ is an SSTS associated with τ.*

Definition 7. *[51] A soft map $\psi_{sd} : (U, \tau, \Delta) \to (V, \sigma, \Lambda)$ with μ an associated SSTS with τ is claimed to be SS-continuous if $\psi_{sd}^{-1}(G, \Lambda) \in \mu\ \forall\ (G, \Lambda) \in \sigma$.*

Definition 8. *[56] Given a soft set $(W, \Delta) \in S(U)_\Delta$. If $int^s(cl^s(W, \Delta))eq\tilde{\varphi}$, then (W, Δ) is called an SS-sd-set. Additionally, $(W^{\tilde{c}}, \Delta)$ is also known as SS-sc-set. Furthermore, $SD(U)_\Delta$ (or $SC(U)_\Delta$, respectively) will be used to represent the category of all SS-sd-sets (or SS-sc-sets). Moreover, (W, Δ) is referred to as SS-nowhere dense if it is not SS-sd-set. Finally, if (W, Δ) is both SS-sd-set and SS-sc-set, then it is called SS-sdc-set.*

Corollary 1. *[56] Any SS-sc-set (SS-sd-set) has a soft subset (superset) that is also SS-sc-set (SS-sd-set).*

Definition 9. *[56] The greatest SS-sd-subsets of (T, Δ) are represented by $int^s_{sd}(T, \Delta)$, which is the SS-sd-interior of a non-null soft subset (T, Δ) of an SSTS (U, μ, Δ). Additionally, $cl^s_{sd}(T, \Delta)$, the smallest SS-sc-superset of (T, Δ), represents the SS-sd-closure of (T, Δ).*

Theorem 2. *[56] Given a soft set $(T, \Delta) \in S(U)_\Delta$, we have that*

(1) $cl^s_{sd}(T^{\tilde{c}}, \Delta) = [int^s_{sd}(T, \Delta)]^{\tilde{c}}$ *and* $int^s_{sd}(T^{\tilde{c}}, \Delta) = [cl^s_{sd}(T, \Delta)]^{\tilde{c}}$.

(2) $cl^s_{sd}(T, \Delta)\tilde{\subseteq}cl^s(T, \Delta)$.

(3) $int^s(T, \Delta)\tilde{\subseteq}int^s_{sd}(T, \Delta)$.

Definition 10. *[62]*Given $(G, \Delta) \neq \tilde{\varphi}$ and $(H, \Delta) \neq \tilde{\varphi}$. If $(G, \Delta)\tilde{\cap}cl^s_{sd}(H, \Delta) = \tilde{\varphi}$ and $cl^s_{sd}(G, \Delta)\tilde{\cap}((H, \Delta)) = \tilde{\varphi}$, then (G, Δ), (H, Δ) are called SS-sd-separated.

Definition 11. *[62]*Given $(G, \Delta) \neq \tilde{\varphi}$ and $(H, \Delta) \neq \tilde{\varphi}$ are SS-sd-separated sets. If (G, Δ) and (H, Δ) cannot be expressed as a soft union of $\tilde{U}$, then the SSTS (U, μ, Δ) is considered as an SS-sd-connected. If not, (U, μ, Δ) is considered as an SS-sd-disconnected. Additionally, if $\tilde{Y}$ is SS-sd-connected subspace of $\tilde{U}$, then it is SS-sd-connected.

Definition 12. *[57] Given μ, μ^* as associated SSTSs with τ, σ, respectively, a soft map $\psi_{sd} : (U, \tau, \Delta) \to (V, \sigma, \Lambda)$ is stated to be:*

(1) *SS-sd-continuous if and only if either $\psi_{sd}^{-1}(T, \Lambda) = \tilde{\varphi}$ or $\psi_{sd}^{-1}(T, \Lambda) \in SD(U)_\Delta$, $\forall (T, \Lambda) \in \sigma$.*

(2) *SS-sd-irresolute if either $\psi_{sd}^{-1}(T, \Lambda) = \tilde{\varphi}$ or $\psi_{sd}^{-1}(T, \Lambda) \in SD(U)_\Delta$, $\forall (T, \Lambda) \in SD(V)_\Lambda$.*

Definition 13. *[62]*Given $(G, \Delta) \neq \tilde{\varphi}$ and $(H, \Delta) \neq \tilde{\varphi}$ are SS-sd-separated sets. If (G, Δ) and (H, Δ) cannot be expressed as a soft union of $\tilde{U}$, then the SSTS (U, μ, Δ) is considered as an SS-sd-connected. If not, (U, μ, Δ) is considered as an SS-sd-disconnected. Additionally, if $\tilde{Y}$ is SS-sd-connected subspace of $\tilde{U}$, then it is SS-sd-connected.

Theorem 3. *[62]* The following characteristics are equivalent for any SSTS (U, μ, Δ):

(1) $\tilde{U}$ is SS-sd-connected.

(2) The soft union of any two disjoint SS-sd-sets cannot be expressed as $\tilde{U}$.

(3) The soft union of any two disjoint SS-sc-sets cannot be expressed as $\tilde{U}$.

(4) There isn't a proper SS-sdc-subset of $\tilde{U}$.

(5) The soft union of any two non-null SS-sd-separated sets cannot be expressed as $\tilde{U}$.

3. Novel generalized connectedness types derived from the supra soft sd-closure operator

Here, we continue studying the properties of connectedness via SS-sd-sets in SSTS [62]. We define the concept of SS-sd-component. We demonstrate that an SS-sd-component's image under a bijective SS*-sd-open map is an SS-sd-component. Additionally, we prove that the category of all SS-sd-components of an SSTS (U, μ, Δ) forms a partition to it. Moreover, we apply it to define a new approach of connectedness via SS-sd-sets, named SSL-sd-connectedness. We prove that the property of SSL-sd-connectedness is hereditary w.r.t SS-sd-subspaces. Also, we define the concept of SS-sd-hyperconnectedness. We discovered that, SS-sd-hyperconnected spaces are identical to SS-sd-connected spaces, which distinguishes our notions from their counterparts. Finally, we provide a topological chart to declare the key concepts presented in this section in Figure 1. The arrows in this chart are non-reversible, as has been confirmed by concrete counterexamples.

Definition 14. *Let (U, μ, Δ) be an SSTS and $(A, \Delta)\tilde{\subseteq}\tilde{U}$ with $s_\gamma\tilde{\in}\tilde{U}$, then the SS-sd-component of (A, Δ) related to s_γ is the finer SS-sd-connected subset of (A, Δ) containing s_γ and will denoted by $\tilde{C}^s_{sd}(A_{s_\gamma}, \Delta)$.*

Example 1. *Let $U = \{x, y, z\}$, $\Delta = \{\gamma_1, \gamma_2\}$ and $\mu = \{\tilde{U}, \tilde{\varphi}, (O_i, \Delta), i = 1, 2, , ..., 5\}$ be an SSTS over N, where:*

$$O_1(\gamma_1) = U, \quad O_1(\gamma_2) = \varphi.$$

$$O_2(\gamma_1) = \varphi, \quad O_2(\gamma_2) = U.$$

$$O_3(\gamma_1) = U, \quad O_3(\gamma_2) = \{y, z\}.$$

$$O_4(\gamma_1) = \{z\}, \quad O_4(\gamma_2) = U.$$

$$O_5(\gamma_1) = \{x, y\}, \quad O_5(\gamma_2) = U.$$

Then, $\tilde{U} = (O_1, \Delta)\tilde{\cup}(O_2, \Delta)$ whereas $(O_1, \Delta), (O_2, \Delta) \in SD(U)_\Delta$ which are disjoint. It follows that, $\tilde{U}$ is an SS-sd-disconnected. Now, $\tilde{C}^s_{sd}(x_{\gamma_1}) = \tilde{C}^s_{sd}(y_{\gamma_1}) = \tilde{C}^s_{sd}(z_{\gamma_1}) = (O_1, \Delta)$ and $\tilde{C}^s_{sd}(x_{\gamma_2}) = \tilde{C}^s_{sd}(y_{\gamma_2}) = \tilde{C}^s_{sd}(z_{\gamma_2}) = (O_2, \Delta)$.

Lemma 1. *Let (U, μ, Δ) be an SSTS , then*

(1) *$\tilde{U}$ is SS-sd-connected if and only if it is SS-sd-component.*

(2) *If $\tilde{U}$ is SS-sd-connected, then $\tilde{U}$ is the only SS-sd-component of each of its soft points.*

Proof. Obvious from Definition 14.

Proposition 1. *[62] If an SSTS (U, μ, Δ) has a soft subset (G, Δ) which is SS-sd-connected, then $cl^s_{sd}(G, \Delta)$ is also.*

Lemma 2. *Let (U, μ, Δ) be an SSTS , then*

(1) *Every SS-sd-component subset of $\tilde{U}$ is the finer SS-sd-connected subset of $\tilde{U}$.*

(2) *Every SS-sd-component subset of $\tilde{U}$ is an SS-sc-set.*

Proof. Obvious from Definition 14 and Proposition 1.

Theorem 4. *For each soft point s_γ in an SSTS (U, μ, Δ), there is only one SS-sd-component subset of $\tilde{U}$ which contains s_γ.*

Proof. Consider the class $\Omega = \{(K, \Delta)\tilde{\subseteq}\tilde{U} : s_\gamma\tilde{\in}(K, \Delta),\ (K, \Delta)\ is\ an\ SS-sd-connected\}$. Since $s_\gamma\tilde{\in}(K, \Delta)$ for each $(K, \Delta) \in \Omega$, $\Omega \neq \tilde{\varphi}$. It follows that, the soft set

$$(G,\Delta)=\tilde{\bigcup}\{(K,\Delta)\tilde{\subseteq}\tilde{U}:s_\gamma\tilde{\in}(K,\Delta),\ (K,\Delta)\ is\ an\ SS-sd-connected\}$$

has a non-null soft intersection and SS-sd-connected subset of $\tilde{U}$ containing s_γ. Hence, (G,Δ) is the SS-sd-component $\tilde{C}^s_{sd}(U_{s_\gamma},\Delta)$ of $\tilde{U}$ w.r.t s_γ.

Now, if $\tilde{C}^{s*}_{sd}(U_{s_\gamma},\Delta)$ is another SS-sd-component containing s_γ, then $\tilde{C}^{s*}_{sd}(U_{s_\gamma},\Delta)$ is SS-sd-connected subset of $\tilde{U}$ containing s_γ.

However,

$$\tilde{C}^s_{sd}(U_{s_\gamma},\Delta)\text{ is an SS-sd-component, and so}$$

$$\tilde{C}^{s*}_{sd}(U_{s_\gamma},\Delta)\tilde{\subseteq}\tilde{C}^s_{sd}(U_{s_\gamma},\Delta).$$

By a similar argument we can get

$$\tilde{C}^s_{sd}(U_{s_\gamma},\Delta)\tilde{\subseteq}\tilde{C}^{s*}_{sd}(U_{s_\gamma},\Delta).$$

Thus, s_γ is contained in one SS-sd-component subset of $\tilde{U}$ only.

Proposition 2. **(1)** *Any two SS-sd-components related to two distinct soft points are either identical or disjoint.*

(2) *The class of all SS-sd-components of an SSTS (U,μ,Δ) forms a partition to $\tilde{U}$.*

Proof.

(1) Clear from Definition 14.

(2) Consider the class $\{\tilde{C}^s_{sd}(U_{s_\gamma},\Delta):s_\gamma\tilde{\in}\tilde{U}\}$ of all SS-sd-components of an SSTS (U,μ,Δ). Then, $\tilde{\bigcup}\{\tilde{C}^s_{sd}(U_{s_\gamma},\Delta):s_\gamma\tilde{\in}\tilde{U}\}=\tilde{U}$. Assume that s_γ, r_ϑ are any two distinct soft points in $\tilde{U}$. From (1), $\tilde{C}^s_{sd}(U_{s_\gamma},\Delta),\tilde{C}^s_{sd}(U_{r_\vartheta},\Delta)$ are either identical or disjoint. If $\tilde{C}^s_{sd}(U_{s_\gamma},\Delta),\tilde{C}^s_{sd}(U_{r_\vartheta},\Delta)$ are identical, then it contradicts the $\tilde{U}$ is the finer SS-sd-connected set containing s_γ, r_ϑ from Lemma 2. Hence, $\tilde{C}^s_{sd}(U_{s_\gamma},\Delta),\tilde{C}^s_{sd}(U_{r_\vartheta},\Delta)$ are disjoint, and so we get the proof.

Theorem 5. *Every non-null SS-sd-connected subset of an SSTS (U,μ,Δ) in which is SS-sdc-set is SS-sd-component.*

Proof. Let (T,Δ) is SS-sd-connected and SS-sdc-set. Assume that there is an arbitrary SS-sd-connected superset (S,Δ) of (T,Δ). Since

$$cl^s_{sd}(T,\Delta)\tilde{\cap}[(T^{\tilde{c}},\Delta)\tilde{\cap}(S,\Delta)]=\tilde{\varphi},$$

$$(T,\Delta)\tilde{\cap}cl^s_{sd}[(T^{\tilde{c}},\Delta)\tilde{\cap}(S,\Delta)]\tilde{\subseteq}(T,\Delta)\tilde{\cap}cl^s_{sd}(T^{\tilde{c}},\Delta)=\tilde{\varphi}\text{ and}$$

$$(S,\Delta)=(T,\Delta)\tilde{\cup}[(T^{\tilde{c}},\Delta)\tilde{\cap}(S,\Delta)].$$

Therefore, (T,Δ) and $[(T^{\tilde{c}},\Delta)\tilde{\cap}(S,\Delta)]$ form an SS-sd-disconnection of (S,Δ), which runs counter to our hypothesis. Thus, (T,Δ) is an SS-sd-component.

Definition 15. *Let $\psi_{sd}:(U,\tau,\Delta)\ \to\ (V,\sigma,\Lambda)$ be a soft mapping with μ,μ^* as associated SSTSs with τ,σ, respectively. Then, ψ_{sd} is called SS*-sd-open if*

$$\psi_{sd}(G,\Delta)\in SD(V)_\Lambda\ for\ each\ \tilde{\varphi}\neq(G,\Delta)\in SD(U)_\Delta.$$

Theorem 6. *Under an injective SS*-sd-open map, the pre-image of an SS-sd-connected set is SS-sd-connected.*

Proof. Let $\psi_{sd} : (U, \tau, \Delta) \rightarrow (V, \sigma, \Lambda)$ is an injective SS*-sd-open map with μ, μ^* as associated SSTSs with τ, σ, respectively, and (W, Λ) is SS-sd-connected subset of $\tilde{V}$. Assume conversely $\psi_{sd}^{-1}((W, \Lambda)$ be an SS-sd-disconnected subset of $\tilde{U}$, then there are two disjoint SS-sd-subsets (A, Δ), (B, Δ) of $\tilde{U}$ such that

$$\psi_{sd}^{-1}(W, \Lambda) = (A, \Delta)\tilde{\cup}(B, \Delta).$$

Since ψ_{sd} is SS*-sd-open, $\psi_{sd}(A, \Delta)$ and $\psi_{sd}(B, \Delta)$ are disjoint SS-sd-subsets of $\tilde{V}$. Hence,

$(W, \Lambda) = \psi_{sd}(\psi_{sd}^{-1}((W, \Lambda)) = \psi_{sd}(A, \Delta)\tilde{\cup}\psi_{sd}(B, \Delta)$, ψ_{sd} is surjective, which is a contradiction to our hypothesis.

Therefore, $\psi_{sd}^{-1}((W, \Lambda)$ is an SS-sd-connected subset of $\tilde{U}$.

Theorem 7. *Under a bijective SS*-sd-open map, the image of an SS-sd-component is SS-sd-component.*

Proof. Let $\psi_{sd} : (U, \tau, \Delta) \rightarrow (V, \sigma, \Lambda)$ is a bijective SS*-sd-open map with μ, μ^* as associated SSTSs with τ, σ, respectively, such that (G, Δ) is SS-sd-component subsets of $\tilde{U}$. Assume instead that, $\psi_{sd}(G, \Delta)$ is not SS-sd-component subsets of $\tilde{V}$. Hence, there is an SS-sd-connected subset (H, Λ) of $\tilde{V}$ such that

$$\psi_{sd}(G, \Delta)\tilde{\subseteq}(H, \Lambda) \text{ which follows}$$

$$(G, \Delta)\tilde{\subseteq}\psi_{sd}^{-1}(\psi_{sd}(G, \Delta))\tilde{\subseteq}\psi_{sd}^{-1}(H, \Lambda), \ \psi_{sd} \text{ is surjective.}$$

Since ψ_{sd} is injective SS*-sd-open, $\psi_{sd}^{-1}(H, \Lambda)$ is an SS-sd-connected subset of $\tilde{U}$, from Theorem 6, it conflicts with that (G, Δ) is an SS-sd-component. Therefore, $\psi_{sd}(G, \Delta)$ is an SS-sd-component subset of $\tilde{V}$.

Definition 16. *[58] Given the soft point s_γ in an SSTS (U, μ, Δ). Then $\tilde{U}$ is stated to be an SSL-connected at s_γ if each SS-neighbourhood (A, Δ) of s_γ there is $(G, \Delta)\tilde{\subseteq}(A, \Delta)$, which is SS-connected and containing s_γ. If $\tilde{U}$ is SSL-connected at all of its soft points, then it is stated to be an SSL-connected.*

Definition 17. *An SSTS (U, μ, Δ) is said to be SSL-sd-connected at a soft point s_γ if every SS-sd-neighbourhood (A, Δ) of s_γ there is an SS-sd-connected subset (G, Δ) of (A, Δ) containing s_γ. If $\tilde{U}$ is SSL-sd-connected at all of its soft points, then it is stated to be an SSL-sd-connected.*

Proposition 3. (1) *Every SS-sd-connected is SSL-sd-connected.*

(2) *Every SSL-sd-connected is SSL-connected.*

Proof.

(1) Let $s_\gamma\tilde{\in}\tilde{U}$ such that $\tilde{U}$ is SS-sd-connected. Consequently, $\tilde{U}$ does not have a proper SS-sdc-set. Hence, $s_\gamma\tilde{\in}\tilde{U}\tilde{\subseteq}\tilde{U}$. Therefore, $\tilde{U}$ is an SSL-sd-connected at the arbitrary soft point s_γ and hence it is an SSL-sd-connected.

(2) Follows from (1).

Remark 1. In general, the opposite of Proposition 3 is not satisfied, as demonstrated by the example that follows.

Example 2. *Let $U = \{f_1, f_2, f_3\}$, $\Delta = \{\gamma_1, \gamma_2\}$ and $\mu = \{\tilde{U}, \tilde{\varphi}, (X_i, \Delta), i = 1, 2, 3, 4\}$ be an SSTS on U, where:*

$$X_1(\gamma_1) = \{f_1\}, \quad X_1(\gamma_2) = \varphi.$$

$$X_2(\gamma_1) = \{f_1\}, \quad X_2(\gamma_2) = \{f_1\}.$$

$$X_3(\gamma_1) = \{f_1, f_2\}, \quad X_3(\gamma_2) = \{f_1, f_3\}.$$

$$X_4(\gamma_1) = \{f_2\}, \quad X_4(\gamma_2) = \{f_1, f_3\}.$$

For the soft sets $(G, \Delta), (H, \Delta)$, where:

$$G(\gamma_1) = \{f_2, f_3\}, \quad G(\gamma_2) = \{f_1, f_3\}.$$

$$H(\gamma_1) = \{f_1\}, \quad H(\gamma_2) = \{f_2\}.$$

It's simple to verify that $\tilde{U}$ is an SSL-sd-connected. However, we have $\tilde{U} = (G, \Delta)\tilde{\cup}(H, \Delta)$, whereas $(G, \Delta), (H, \Delta)$ are SS-sd-separated sets. Therefore, (U, μ, Δ) is an SS-sd-disconnected.

Theorem 8. The SS-sd-component of an SSL-sd-connected SSTS is a SS-sd-set.

Proof. Let (U, μ, Δ) be an SSL-connected SSTS and $\tilde{C}^s_{sd}(U_{s_\gamma}, \Delta)$ is SS-sd-component related to $s_\gamma \tilde{\in} \tilde{U}$. Since $\tilde{U}$ is SSL-sd-connected, every SS- sd-neighbourhood of s_γ contains an SS-sd-connected neighbourhood (G, Δ) of s_γ. However, $\tilde{C}^s_{sd}(U_{s_\gamma}, \Delta)$ is the largest SS-sd-connected neighbourhood of s_γ. Therefore, $s_\gamma \tilde{\in} (G, \Delta) \tilde{\subseteq} \tilde{C}^s_{sd}(U_{s_\gamma}, \Delta)$ which follows $\tilde{C}^s_{sd}(U_{s_\gamma}, \Delta)$ is an SS-sd-neighbourhood of s_γ. Hence, $\tilde{C}^s_{sd}(U_{s_\gamma}, \Delta)$ is an SS-sd-neighbourhood of each its soft points. Thus, $\tilde{C}^s_{sd}(U_{s_\gamma}, \Delta)$ is SS-sd-set.

Theorem 9. The property of SSL-sd-connectedness is hereditary w.r.t SS-sd-subspaces.

Proof. Assume that (U, μ_U, Δ) be an SS-sd-subspace of an SSL-sd-connected SSTS (U, μ, Δ) and $s_\gamma \tilde{\in} \tilde{U}$. Since $\tilde{U}$ is SSL-sd-connected, there is an SS-sd-connected neighbourhood (G, Δ) of s_γ such that $s_\gamma \in (G, \Delta) \tilde{\subseteq} \tilde{U}$. Since $(G, \Delta) \in SD(U)_\Delta$, $(G, \Delta) \tilde{\cap} \tilde{\tilde{U}} \in SD(U)_\Delta$ and $(G, \Delta) \cap \tilde{U}$ is SS-sd-connected subset of $\tilde{U}$. Hence, $\tilde{U}$ is an SSL-sd-connected for each $s_\gamma \tilde{\in} \tilde{U}$. Therefore, $\tilde{U}$ is an SSL-sd-connected.

Proposition 4. The SS-sd-components of every SS-sd-subspace of an SSL-sd-connected SSTS are SS-sd-set.

Proof. Direct result of Theorems 8 and 9.

Theorem 10. *[62] An SS-sd-connected set is represented by its image under an SS-sd-irresolute map.*

Theorem 11. *If $\psi_{sd} : (U, \tau, \Delta) \to (V, \sigma, \Lambda)$ is a surjective SS-sd-irresolute map with μ, μ^* as associated SSTSs with τ, σ, respectively, and $\tilde{U}$ is SSL-sd-connected, then also $\tilde{V}$.*

Proof. Let $\psi_{sd} : (U, \tau, \Delta) \to (V, \sigma, \Lambda)$ is a surjective SSL-sd-irresolute map with μ, μ^* as associated SSTSs with τ, σ, respectively, and $\tilde{U}$ is an SSL-sd-connected. Let (H, Λ) is an SS-sd-neighbourhood of $s_\lambda \tilde{\in} \tilde{V}$. It follows that,

$$\psi^{-1}_{sd}(H, \Lambda) \text{ is SS-sd-neighbourhood of } \psi^{-1}_{sd}(s_\lambda) \tilde{\in} \tilde{U}.$$

Since $\tilde{U}$ is SSL-sd-connected, there is an SS-sd-connected subset (G, Δ) of $\psi^{-1}_{sd}(H, \Lambda)$ containing $\psi^{-1}_{sd}(s_\lambda)$ which follows

$$\psi_{sd}(G, \Delta) \tilde{\subseteq} \psi_{sd}(\psi^{-1}_{sd}(H, \Lambda)) = (H, \Lambda), \ \psi_{sd} \text{ is surjective}.$$

Since ψ_{sd} is SS-sd-irresolute, $\psi_{sd}(G, \Delta)$ is SS-sd-connected neighbourhood of $\psi^{-1}_{sd}(s_\lambda)$ from Theorem 10. Therefore, $\tilde{V}$ is an SSL-sd-connected.

Definition 18. If for all $(G,\Delta),(H,\Delta)\in SD(U)_\Delta$ such that $(G,\Delta)\tilde{\cap}(H,\Delta)\neq\tilde{\varphi}$, then $\tilde{U}$ is called an SS-sd-hyperconnected.

Definition 19. *A soft subset (F,Δ) of an SSTS (U,μ,Δ) is said to be SS-sd-dense if $cl^s_{sd}(F,\Delta)=\tilde{U}$.*

Theorem 12. *An SSTS (U,μ,Δ) is SS-sd-hyperconnected if and only if every SS-sd-set (G,Δ) is SS-sd-dense.*

Proof. Necessity: Alternatively, suppose that there is an SS-sd-subset (G,Δ) of an SS-sd-hyperconnected SSTS (U,μ,Δ) such that $cl^s_{sd}(F,\Delta)\neq\tilde{U}$, then there is an SS-sc-subset (H,Δ) of $\tilde{U}$ such that $(G,\Delta)\tilde{\subseteq}(H,\Delta)$. From Corollary 1, (G,Δ) is also an SS-sc-set. Hence, (G,Δ) and $(G^{\tilde{c}},\Delta)$ are SS-sd-sets in which

$$(G,\Delta)\tilde{\cap}(G^{\tilde{c}},\Delta)=\tilde{\varphi}, \text{ which contradicts our hypothesis.}$$

Sufficient:Suppose the contrary that $\tilde{U}$ is not SS-sd-hyperconnected, then there are two SS-sd-subsets $(A,\Delta),(B,\Delta)$ of $\tilde{U}$ such that $(A,\Delta)\tilde{\cap}(B,\Delta)=\tilde{\varphi}$. Then, $(A,\Delta)\tilde{\subseteq}(B^{\tilde{c}},\Delta)$, $(B^{\tilde{c}},\Delta)$ is an SS-sc-set which follows (A,Δ) is also an SS-sc-set; from Corollary 1, and hence $cl^s_{sd}(A,\Delta)=(A,\Delta)\neq\tilde{U}$. Hence, (A,Δ) is not SS-sd-dense, which goes against what we assumed. Thus, $\tilde{U}$ is an SS-sd-hyperconnected.

Corollary 2. *An SSTS (U,μ,Δ) is SS-sd-hyperconnected if and only if every SS-open set (G,Δ) is SS-sd-dense.*

Proof. It is evident from the fact that each SS-open set is an SS-sd-set and Theorem 12.

Note 1. *The following theorem introduces a novel result that is not satisfied by its STSs and SSTSs counterparts.*

Theorem 13. An SSTS (U,μ,Δ) is SS-sd-hyperconnected if and only if it is SS-sd-connected.

Proof. Suppose the contrary that (U,μ,Δ) is SS-sd-disconnected. Given Theorem 3, there is a proper SS-sdc-subset (G,Δ) of $\tilde{U}$. Hence,

$$(G,\Delta) \text{ and } (G^{\tilde{c}},\Delta) \text{ are SS-sd-sets in which } (G,\Delta)\tilde{\cap}(G^{\tilde{c}},\Delta)=\tilde{\varphi}.$$

Thus, $\tilde{U}$ is not SS-sd-hyperconnected.
On the other hand, suppose that (U,μ,Δ) is not SS-sd-hyperconnected, then there are two disjoint SS-sd-subsets (G,Δ) and (H,Δ) of $\tilde{U}$. It follows that, $(G,\Delta)\tilde{\subseteq}(H^{\tilde{c}},\Delta)$, whereas $(H^{\tilde{c}},\Delta)$ is SS-sc-set. According to Corollary 1, (G,Δ) is SS-sd-set. This meas that, (G,Δ) is SS-sdc-set. Thus, (U,μ,Δ) is SS-sd-disconnected.

Proposition 5. *If (U,μ,Δ) is SS-sd-hyperconnected, then it is SS-hyperconnected.*

Proof. Assume the contrary that (U,μ,Δ) is not SS-hyperconnected, then there are $(G,\Delta),(H,\Delta)\in\mu$ in which $(G,\Delta)\tilde{\cap}(H,\Delta)=\tilde{\varphi}$. Hence, $(G,\Delta),(H,\Delta)\in SD(U)_\Delta$ and $(G,\Delta)\tilde{\cap}(H,\Delta)=\tilde{\varphi}$. Therefore, $\tilde{U}$ is not SS-sd-hyperconnected.

Remark 2. *The example that follows illustrates why the opposite of Theorem 5 is typically not true.*

Example 3. *Let $U=\{s_1,s_2\}$, $\Delta=\{\gamma_1,\gamma_2\}$ and consider the soft sets $(H_i,\Delta), i=1,2,....5$ over U, where:*

$$H_1(\gamma_1)=\{s_1\},\quad H_1(\gamma_2)=\{s_1\}.$$

$$H_2(\gamma_1)=U,\quad H_2(\gamma_2)=\{s_2\}.$$

$$H_3(\gamma_1)=\{s_1\},\quad H_3(\gamma_2)=U.$$

$$H_4(\gamma_1) = U, \quad H_4(\gamma_2) = \{s_1\}.$$
$$H_5(\gamma_1) = \{s_1\}, \quad H_5(\gamma_2) = \{s_2\}.$$

Consider $\mu = \{\tilde{U}, \tilde{\varphi}, (H_i, \Delta), i = 1, 2, ..., 5\}$ *is an SSTS on U, then it is easy to check that* (U, μ, Δ) *is an SS-hyperconnected. Alternatively, regarding the soft sets* $(C, \Delta), (D, \Delta)$ *where:*

$$C(\gamma_1) = \varphi, \quad C(\gamma_2) = U.$$

$$D(\gamma_1) = U, \quad D(\gamma_2) = \varphi,$$

we have $(C, \Delta), (D, \Delta) \in SD(U)_\Delta$ *in which* $(C, \Delta)\tilde{\cap}(D, \Delta) = \tilde{\varphi}$*. Therefore, U is not SS-sd-hyperconnected.*

Corollary 3. The following conclusions are valid for an SSTS (U, μ, Δ) from propositions 3, 5, and Theorem 13, which are not reversible.

$$\begin{array}{ccccc} \text{SS-sd-hyperconnected} & \Leftrightarrow \text{SS-sd-connected} & \Rightarrow & \text{SSL-sd-connected} \\ \Downarrow & \Downarrow & & \Downarrow \\ \text{SS-hyperconnected} & \Rightarrow \text{SS-connected} & \Rightarrow & \text{SSL-connected} \end{array}$$

Figure 1. The relationships between different types of connectedness via SS-sd-sets in the frame of SSTSs.

4. Compactness and Lindelöfness based on supra soft sd-sets

Herein, we define novel forms of compactness related to the notions of SS-sd-sets in the frame of SSTSs, namely SS-sd-compactness and SS-sd-Lindelöfness. We discuss their essential properties comprehensively. Specifically, we show that the soft intersection of an SS-sd-compact (Lindelöf) soft set and SS-sc-set is an SS-sd-compact (Lindelöf). Moreover, the behaviour of an SS-sd-compact (Lindelöf) SSTS with the SFIP (SCIP) has been presented. Also, we demonstrate that the image (respectively, pre-image) of each SS-sd-Lindelöf (compact) is an SS-Lindelöf (compact) under a surjective and SS-sd-continuous (respectively, an injective and SS-sd-open) map. We discuss relationships among supra soft-sd-compact (Lindelöf) SSTS and their parametric supra topological spaces, which are defined on any universal set and finite set of parameters. Furthermore, we study the relationships with previous studies and add a topological chart to illustrate the key concepts presented in this section in Figure 2. The arrows in this chart are non-reversible, as has been confirmed by concrete counterexamples.

Definition 20. *A family of soft sets* $\Psi = \{(C_\epsilon, \Delta) : \epsilon \in \varepsilon\}$ *is claimed to be a supra soft open cover, if all of the members of* Ψ *are supra open soft sets. .*

Definition 21. *A class of SS-sd-subsets* $\Psi = \{(C_\epsilon, \Delta) : \epsilon \in \varepsilon\}$ *of an SSTS* (U, μ, Δ) *is claimed to be an SS-sd-cover of soft subset* (G, Δ) *of* $\tilde{U}$*, if* $(G, \Delta)\tilde{\subseteq}\Psi$*.*

Definition 22. *A soft subset* (G, Δ) *of an SSTS* (U, μ, Δ) *is claimed to be SS-sd-compact (Lindelöf), if every SS-sd-cover* $\{(C_\epsilon, \Delta) : \epsilon \in \varepsilon\}$ *of* (G, Δ) *has a finite (countable) subclass* ε_o *of* ε *such that*

$$(G, \Delta)\tilde{\subseteq}\tilde{\bigcup}_{\epsilon\in\varepsilon_o}(C_\epsilon, \Delta).$$

The space (U, μ, Δ) *is claimed to be SS-sd-compact (Lindelöf) if* $\tilde{U}$ *is SS-sd-compact (Lindelöf) as a soft subset.*

Theorem 14. *If the components of an SS-sd-compact SSTS* (U, μ, Δ) *are SS-sd-sets, then they are finite.*

Proof. Let (U, μ, Δ) be an SS-sd-compact SSTS and assume the contrary that $\tilde{U}$ has infinite number of SS-sd-components. Then, $\tilde{C}^s_{\epsilon_{sd}}(U_{s_\gamma}, \Delta) \in SD(U)_\Delta$ for each $s_\gamma \tilde{\in} \tilde{U}$. Hence, the class

$$\{\tilde{C}^s_{\epsilon_{sd}}(U_{s_\gamma}, \Delta) : s_\gamma \tilde{\in} \tilde{U}\}$$

forms an SS-sd-cover for $\tilde{U}$. Since $\tilde{U}$ is SS-sd-compact, there is a finite subclass ε_o such that

$$\tilde{\bigcup}_{\epsilon \in \varepsilon_o} \{\tilde{C}^s_{\epsilon_{sd}}(U_{s_\gamma}, \Delta) : s_\gamma \tilde{\in} \tilde{U}\} = \tilde{U}.$$

Therefore, the cover contains some elements with non-null soft intersections. According to Proposition 2, this is contradictory. Thus, $\tilde{U}$ has a finite number of SS-sd-components.

Corollary 4. *If the components of an SS-sd-Lindelöf SSTS (U, μ, Δ) are SS-sd-sets, then they are countable.*

Proof. similarly to the method used to prove Theorem 14.

Proposition 6. *[59] Every SS-compact space is SS-Lindelöf.*

Proposition 7. (1) *Every SS-sd-compact (Lindelöf) space is SS-compact (Lindelöf).*

(2) *Every SS-sd-compact space is SS-sd-Lindelöf.*

Proof.

(1) Assume that $\{(C_\epsilon, \Delta) : \epsilon \in \varepsilon\}$ be a supra soft open cover of an SS-sd-compact space (U, μ, Δ), then Ψ is an SS-sd-cover of $\tilde{U}$. Since $\tilde{U}$ is SS-sd-compact, there is a finite subclass ε_o of ε such that

$$\tilde{U} = \tilde{\bigcup}_{\epsilon \in \varepsilon_o} (C_\epsilon, \Delta).$$

Hence, $\tilde{U}$ is an SS-compact.

(2) Follows from the fact that every finite class is countable.

Remark 3. *The examples below will confirm that the opposite of Proposition 7 is not satisfied in general.*

Examples 1. (1) *Consider any real number $e \in R$. Let $\Delta = \{\gamma_1, \gamma_2, \gamma_3, ...\}$ and $\mu = \{\tilde{U}, \tilde{\varphi}, (Y, \Delta), (Z, \Delta), (W, \Delta)\}$ be an SSTS on U, where:*

$$(Y, \Delta) = \{(\gamma_1, \{e\}), (\gamma_2, \{e\}), (\gamma_3, \{e\}), (\gamma_4, \{e\}), ...\}.$$
$$(Z, \Delta) = \{(\gamma_1, \{e\}), (\gamma_2, \varphi), (\gamma_3, \{e\}), (\gamma_4, \{e\}), ...\}.$$
$$(W, \Delta) = \{(\gamma_1, \varphi), (\gamma_2, \{e\}), (\gamma_3, \{e\}), (\gamma_4, \{e\}), ...\}.$$

We have that $\tilde{U}$ is an SS-compact (Lindelöf). On the opposite side, the class

$$\Psi = \{(H, \Delta) : H(\gamma_i) = \{e, k\},\ for\ \gamma_i \in \Delta\ and\ e, k \in R\}$$

forms an SS-sd-cover for $\tilde{U}$. However, there is no finite (countable) subclass of Ψ which cover $\tilde{R}$. Thus, $\tilde{R}$ is not SS-sd-compact (Lindelöf).

(2) *Consider any natural number $c \in N$. Let $\Delta = \{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_m, m \in N\}$ and $\mu = \{\tilde{U}, \tilde{\varphi}, (A, \Delta), (B, \Delta), (C, \Delta)\}$ be an SSTS on U, where:*

$$(A, \Delta) = \{(\gamma_1, \{c\}), (\gamma_2, \{c\}), (\gamma_3, \{c\}), (\gamma_4, \{c\}), ..., (\gamma_m, \{c\})\}.$$
$$(B, \Delta) = \{(\gamma_1, \{c\}), (\gamma_2, \varphi), (\gamma_3, \{c\}), (\gamma_4, \{c\}), ..., (\gamma_m, \{c\})\}.$$
$$(C, \Delta) = \{(\gamma_1, \varphi), (\gamma_2, \{c\}), (\gamma_3, \{c\}), (\gamma_4, \{c\}), ..., (\gamma_m, \{c\})\}.$$

Since N and Δ are countable, it is simple to confirm that $\tilde{N}$ is an SS-sd-Lindelöf space. On the opposite side,, the class

$$\Psi = \{(H, \Delta) : H(\gamma_i) = \{c, k\}, i = 1, 2, 3, ..., m \ for \ \gamma_i \in \Delta \ and \ m, c, k \in N\}$$

forms an SS-sd-cover for $\tilde{N}$. However, there is no finite subclass of Ψ which cover $\tilde{N}$. Thus, $\tilde{N}$ is not SS-sd-compact.

Proposition 8. **(1)** *Every SSTS (U, μ, Δ) defined on a finite (countable) universal set U is SS-sd-compact (Lindelöf).*

(2) *A finite (countable) soft union of SS-sd-Lindelöf (compact) subsets of an SSTS (U, μ, Δ) is SS-sd-compact (Lindelöf).*

Proof.

(1) Direct from Definition 22.

(2) Suppose that (J, Δ) and (K, Δ) are SS-sd-Lindelöf subsets of an SSTS (U, μ, Δ), and $\Psi = \{(C_\epsilon, \Delta) : \epsilon \in \varepsilon\}$ is SS-sd-cover for $(J, \Delta)\tilde{\cup}(K, \Delta)$, then Ψ is an SS-sd-cover for (J, Δ) and (K, Δ). According to the hypothesis, there are countable subclasses ε_1 and ε_2 of ε such that

$$(J, \Delta)\tilde{\subseteq}\tilde{\bigcup}_{i\in\varepsilon_1}(C_i, \Delta) \text{ and } (K, \Delta)\tilde{\subseteq}\tilde{\bigcup}_{j\in\varepsilon_2}(C_j, \Delta).$$

Hence,

$$(J, \Delta)\tilde{\cup}(K, \Delta)\tilde{\subseteq}(\tilde{\bigcup}_{i\in\varepsilon_1}(C_i, \Delta))\tilde{\cup}(\tilde{\bigcup}_{j\in\varepsilon_2}(C_j, \Delta)) \text{ which follows}$$

$$(J, \Delta)\tilde{\cup}(K, \Delta)\tilde{\subseteq}\tilde{\bigcup}_{i\in\varepsilon_1, j\in\varepsilon_2}[(C_i, \Delta)\tilde{\cup}(C_j, \Delta)] = \tilde{\bigcup}_{k\in\varepsilon_3}(C_k, \Delta), \ \varepsilon_3 \text{ is a countable subclasses of } \varepsilon.$$

Thus, $(J, \Delta)\tilde{\cup}(K, \Delta)$ is SS-sd-Lindelöf. When parenthesis are used, the case can be obtained by a similar way.

Theorem 15. *Every SS-sc-subset of an SS-sd-compact (Lindelöf) SSTS (U, μ, Δ) is SS-sd-compact (Lindelöf).*

Proof. Let $\{(C_\epsilon, \Delta) : \epsilon \in \varepsilon\}$ is SS-sd-cover for an SS-sc-subset (S, Δ) of an SS-sd-compact SSTS (U, μ, Δ), then $(S, \Delta)\tilde{\subseteq}\tilde{\bigcup}_{\epsilon\in\varepsilon}(C_\epsilon, \Delta)$ which follows

$$\tilde{U} = (S, \Delta)\tilde{\cup}(S^{\tilde{c}}, \Delta)\tilde{\subseteq}\tilde{\bigcup}_{\epsilon\in\varepsilon}(C_\epsilon, \Delta)\tilde{\cup}(S^{\tilde{c}}, \Delta).$$

Now, we have that

$$\{(C_\epsilon, \Delta) : \epsilon \in \varepsilon\}\tilde{\cup}(S^{\tilde{c}}, \Delta) \text{ is an SS-sd-cover of } \tilde{U}.$$

Since $\tilde{U}$ is SS-sd-compact, there is a finite subclass ε_o of ε such that

$$(S, \Delta)\tilde{\subseteq}\tilde{U} = \tilde{\bigcup}_{\epsilon\in\varepsilon_o}(C_\epsilon, \Delta)\tilde{\cup}(S^{\tilde{c}}, \Delta).$$

Hence,

$$(S, \Delta)\tilde{\subseteq}\tilde{\bigcup}_{\epsilon\in\varepsilon_o}(C_\epsilon, \Delta).$$

Thus, (S, Δ) is an SS-sd-compact. The case of SS-sd-Lindelöfness is similar.

Proposition 9. *The soft intersection of an SS-sd-compact (Lindelöf) soft subset (A, Δ) and SS-sc-subset (B, Δ) of an SSTS (U, μ, Δ) is SS-sd-compact (Lindelöf).*

Proof. Let (A,Δ) is SS-sd-compact, (B,Δ) is an SS-sc-set and $\{(C_\epsilon,\Delta) : \epsilon \in \varepsilon\}$ is SS-sd-cover of $(A,\Delta)\tilde{\cap}(B,\Delta)$, then

$$(A,\Delta)\tilde{\subseteq}\tilde{\bigcup}_{\epsilon\in\varepsilon}(C_\epsilon,\Delta)\tilde{\cup}(B^{\tilde{c}},\Delta).$$

Since (A,Δ) is SS-sd-compact, there is a finite subclass ε_o of ε such that

$$(A,\Delta)\tilde{\subseteq}\tilde{\bigcup}_{\epsilon\in\varepsilon_o}(C_\epsilon,\Delta)\tilde{\cup}(B^{\tilde{c}},\Delta) \text{ which follows}$$

$$(A,\Delta)\tilde{\cap}(B,\Delta)\tilde{\subseteq}\tilde{\bigcup}_{\epsilon\in\varepsilon_o}(C_\epsilon,\Delta).$$

Therefore, $(A,\Delta)\tilde{\cap}(B,\Delta)$ is an SS-sd-compact. The case of SS-sd-Lindelöfness can be achieved similarly .

Remark 4. *We will show in the following example that the contrary of Proposition 9 is not always satisfied.*

Example 4. *Let $U = \{u_1, u_2\}$, $\Delta = \{\gamma_1, \gamma_2\}$ and $(S_i,\Delta), i = 1, 2, 3, 4$ be soft sets over U, where:*
$S_1(\gamma_1) = \{u_1, u_2\}$, $S_1(\gamma_2) = \{u_2\}$,
$S_2(\gamma_1) = \varphi$, $S_2(\gamma_2) = \{u_2\}$,
$S_3(\gamma_1) = \{u_1\}$, $S_3(\gamma_2) = \{u_2\}$,
$S_4(\gamma_1) = \{u_1, u_2\}$, $S_4(\gamma_2) = \varphi$,
then $\mu = \{\tilde{U}, \tilde{\varphi}, (S_i,\Delta), i = 1, 2, 3, 4\}$ defines an SSTS on U. According to Proposition 8, it is clear that $\tilde{U}$ is an SS-sd-compact (Lindelöf). Also the soft set (W,Δ) where:
$W(\gamma_1) = \{u_1\}$, $W(\gamma_2) = \{u_2\}$ *is SS-sd-compact (Lindelöf). However, $\tilde{U}\tilde{\cap}(W,\Delta) = (W,\Delta)$ is not SS-sc-set.*

Corollary 5. *The soft difference between an SS-sd-compact (Lindelöf) soft subset (A,Δ) and SS-sd-subset (B,Δ) of an SSTS (U,μ,Δ) is SS-sd-compact (Lindelöf).*

Proof. Obvious from Theorem 9.

Definition 23. *[63] A collection Ψ of soft sets has the soft finite (countable) intersection property (briefly, SFIP (SCIP)), if the soft intersection of the finite (countable) subfamily of Ψ is non-empty.*

Theorem 16. *An SSTS (U,μ,Δ) is SS-sd-compact (Lindelöf) if and only if every family of SS-sc-subsets of $\tilde{U}$ with the SFIP (SCIP) has a non-empty intersection.*

Proof.Necessity: Assume that $\{(C_\epsilon,\Delta) : \epsilon \in \varepsilon\}$ is a class of SS-sc-sets with the SFIP (SCIP), and assume contrary that $\tilde{\bigcap}_{\epsilon\in\varepsilon}(C_\epsilon,\Delta)=\tilde{\varphi}$. It follows that,

$$\tilde{\bigcup}_{\epsilon\in\varepsilon}(C_\epsilon^{\tilde{c}},\Delta) = \tilde{U} \text{ which means}$$

the class $\{(C_\epsilon^{\tilde{c}},\Delta) : \epsilon \in \varepsilon\}$ forms an SS-sd-cover of $\tilde{U}$. Since $\tilde{U}$ is SS-sd-compact, there is a finite subclass ε_o of ε which also covers $\tilde{U}$. That's is

$$\tilde{\bigcup}_{\epsilon\in\varepsilon_o}(C_\epsilon^{\tilde{c}},\Delta)=\tilde{U}.$$

Hence,

$$\tilde{\bigcap}_{\epsilon\in\varepsilon_o}(C_\epsilon,\Delta)=\tilde{\varphi}$$

which is a contradiction with the SFIP.

Sufficient: Suppose that $\Psi = \{(C_\epsilon,\Delta) : \epsilon \in \varepsilon\}$ be an SS-sd-cover of $\tilde{U}$ and assume conversely $\tilde{U}$ is not SS-sd-compact. It follows that, for every finite subclass ε_o of ε we have

$$\tilde{\bigcup}_{\epsilon\in\varepsilon_o}(C_\epsilon,\Delta)\neq\tilde{U} \text{ and so } \tilde{\bigcap}_{\epsilon\in\varepsilon_o}(C_\epsilon^{\tilde{c}},\Delta)\neq\tilde{\varphi}.$$

Hence, $\{(C_\epsilon^{\tilde{c}},\Delta):\epsilon\in\varepsilon\}$ is a class of SS-sc-subsets of $\tilde{U}$ has the SFIP. By assumption, $\tilde{\bigcap}_{\epsilon\in\varepsilon}(C_\epsilon^{\tilde{c}},\Delta)\tilde{\neq}\ \tilde{\varphi}$ and so $\tilde{\bigcup}_{\epsilon\in\varepsilon}(C_\epsilon,\Delta)\neq\tilde{U}$, it conflicts with that Ψ is an SS-sd-cover of $\tilde{U}$. Thus, $\tilde{U}$ is an SS-sd-compact.
The case of SS-sd-Lindelöfness can be obtained by a similar way.

Theorem 17. *An SSTS (U,μ,Δ) is SS-sd-compact (Lindelöf) if and only if every class $\Psi=\{(C_\epsilon,\Delta):\epsilon\in\varepsilon\}$ of soft subsets of $\tilde{U}$ with the SFIP (SCIP) satisfies $\tilde{\bigcap}_{\epsilon\in\varepsilon}\{cl^s_{sd}(C_\epsilon,\Delta):(C_\epsilon,\Delta)\tilde{\in}\Psi\}\tilde{\neq}\ \tilde{\varphi}$.*

Proof. We prove the case of SS-sd-compactness, the case that is enclosed in parenthesis can be obtained by the same way.
Necessity: Assume that $\Psi=\{(C_\epsilon,\Delta):\epsilon\in\varepsilon\}$ is a class of soft sets with the SFIP, and suppose contrary $\tilde{\bigcap}_{\epsilon\in\varepsilon}\{cl^s_{sd}(C_\epsilon,\Delta):(C_\epsilon,\Delta)\tilde{\in}\Psi\}\tilde{=}\tilde{\varphi}$, then

$$\tilde{\bigcup}_{\epsilon\in\varepsilon}\{(cl^s_{sd}(C_\epsilon,\Delta))^{\tilde{c}}:(C_\epsilon,\Delta)\tilde{\in}\Psi\}\tilde{=}\tilde{U} \text{ which means}$$

$$\{(cl^s_{sd}(C_\epsilon,\Delta))^{\tilde{c}}:(C_\epsilon,\Delta)\tilde{\in}\Psi\}\text{is an SS-sd-cover of } \tilde{U}.$$

Since $\tilde{U}$ is SS-sd-compact, there is a finite subclass ε_o of ε such that

$$\tilde{\bigcup}_{\epsilon\in\varepsilon_o}\{(cl^s_{sd}(C_\epsilon,\Delta))^{\tilde{c}}:(C_\epsilon,\Delta)\tilde{\in}\Psi\}\tilde{=}\tilde{U}. \text{ This leads to}$$

$$\tilde{\bigcap}_{\epsilon\in\varepsilon_o}\{(C_\epsilon,\Delta):(C_\epsilon,\Delta)\in\Psi\}\tilde{\subseteq}\tilde{\bigcap}_{\epsilon\in\varepsilon_o}\{cl^s_{sd}(C_\epsilon,\Delta):(C_\epsilon,\Delta)\in\Psi\}=\tilde{\varphi},$$

which is a contradiction with the SFIP. Thus, $\tilde{\bigcup}_{\epsilon\in\varepsilon}\{cl^s_{sd}(C_\epsilon,\Delta):(C_\epsilon,\Delta)\tilde{\in}\Psi\}\tilde{\neq}\ \tilde{\varphi}$.
Sufficient: Suppose that $\Psi=\{(C_\epsilon,\Delta):\epsilon\in\varepsilon\}$ be an SS-sd-cover for $\tilde{U}$ and assume contrary that $\tilde{U}$ is not SS-sd-compact, then Ψ has not any finite subcover which cover $\tilde{U}$. Hence, for each finite subclass ε_o of ε we have

$$\tilde{\bigcup}_{\epsilon\in\varepsilon_o}(C_\epsilon,\Delta)\neq\tilde{U} \text{ which follows } \tilde{\bigcap}_{\epsilon\in\varepsilon_o}(C_\epsilon^{\tilde{c}},\Delta)\neq\tilde{\varphi}.$$

Therefore, $\{(C_\epsilon^{\tilde{c}},\Delta):(C_\epsilon,\Delta)\tilde{\in}\Psi\}$ is a class of SS-sc-sets with SFIP. From hypothesis,

$$\tilde{\bigcap}_{\epsilon\in\varepsilon}\{cl^s_{sd}(C_\epsilon^{\tilde{c}},\Delta)=(C_\epsilon^{\tilde{c}},\Delta):(C_\epsilon,\Delta)\tilde{\in}\Psi\}\tilde{\neq}\ \tilde{\varphi}.$$

Hence,

$$\tilde{\bigcup}_{\epsilon\in\varepsilon}\{(C_\epsilon,\Delta):(C_\epsilon,\Delta)\tilde{\in}\Psi\}\tilde{\neq}\ \tilde{U}, \text{ which is in opposition to that } \Psi \text{ is an SS-sd-cover for } \tilde{U}.$$

Thus, $\tilde{U}$ is an SS-sd-compact.

Theorem 18. *The image of each SS-sd-Lindelöf (compact) set is SS-Lindelöf (compact) under a surjective and SS-sd-continuous map.*

Proof. Let $\psi_{sd}:(U,\tau,\Delta)\ \rightarrow\ (V,\sigma,\Delta)$ be an SS-sd-continuous map with μ,μ^* as associated SSTSs with τ,σ, respectively, and $\{(C_\epsilon,\Delta):\epsilon\in\varepsilon\}$ is SS-cover for the image of an SS-sd-Lindelöf subset (K,Δ) of $\tilde{U}$. It follows that, $\psi^{-1}_{sd}(C_\epsilon,\Delta)\in SD(U)_\Delta$ for each $\epsilon\in\varepsilon$ with $(K,\Delta)\tilde{\subseteq}\tilde{\bigcup}_{\epsilon\in\varepsilon}[\psi^{-1}_{sd}(C_\epsilon,\Delta)]$.
Since (K,Δ) is SS-sd-Lindelöf, there is a countable subclasses ε_o of ε such that

$$(K,\Delta)\tilde{\subseteq}\tilde{\bigcup}_{\epsilon\in\varepsilon_o}[\psi^{-1}_{sd}(C_\epsilon,\Delta)] \text{ which follows}$$

$$\psi_{sd}(K,\Delta)\tilde{\subseteq}\tilde{\bigcup}_{\epsilon\in\varepsilon_o}\psi_{sd}[\psi^{-1}_{sd}(C_\epsilon,\Delta)]=\tilde{\bigcup}_{\epsilon\in\varepsilon_o}(C_\epsilon,\Delta),\ \psi_{sd} \text{ is surjective.}$$

Therefore, $\psi_{sd}(K,\Delta)$ is an SS-Lindelöf. Similarly, one can prove the case of SS-sd-compactness.

Corollary 6. *The image of each SS-sd-Lindelöf (compact) set is SS-sd-Lindelöf (compact) under a surjective and SS-sd-irresolute map.*

Proof. Immediate form Theorem 18.

Definition 24. *[57] A soft mapping $\psi_{sd} : (U, \tau, \Delta) \to (V, \sigma, \Delta)$ μ, μ^* as associated SSTSs with τ, σ, respectively, is claimed to be SS-sd-open if $\psi_{sd}(G, \Delta) \in SD(V)$ for each non-null soft open subset (G, Δ) of $\tilde{U}$.*

Theorem 19. *The pre-image of each SS-sd-Lindelöf (compact) set is SS-Lindelöf (compact) under an injective and SS-sd-open map.*

Proof. Let $\psi_{sd} : (U, \tau, \Delta) \to (V, \sigma, \Delta)$ be an SS-sd-open map with μ, μ^* as associated SSTSs with τ, σ, respectively, and (P, Δ) is the SS-sd-Lindelöf subset of $\tilde{V}$. Assume that $\{(C_\epsilon, \Delta) : \epsilon \in \varepsilon\}$ is an SS-open cover for $\psi_{sd}^{-1}(P, \Delta)$. It follows that,

$$\psi_{sd}(C_\epsilon, \Delta) \in SD(V)_\Delta \text{ for each } \epsilon \in \varepsilon \text{ with } (P, \Delta)\tilde{\subseteq}\tilde{\bigcup}_{\epsilon\in\varepsilon}[\psi_{sd}(C_\epsilon, \Delta)] \text{ given } \psi_{sd} \text{ is injective.}$$

Since (P, Δ) is SS-sd-Lindelöf, there is a countable subclasses ε_o of ε such that

$$(P, \Delta)\tilde{\subseteq}\tilde{\bigcup}_{\epsilon\in\varepsilon_o}[\psi_{sd}(C_\epsilon, \Delta)] \text{ which follows}$$

$$\psi_{sd}^{-1}(P, \Delta)\tilde{\subseteq}\tilde{\bigcup}_{\epsilon\in\varepsilon_o}\psi_{sd}^{-1}[\psi_{sd}(C_\epsilon, \Delta)] = \tilde{\bigcup}_{\epsilon\in\varepsilon_o}(C_\epsilon, \Delta).$$

Therefore, $\psi_{sd}^{-1}(P, \Delta)$ is an SS-Lindelöf. Similarly, the proof of SS-sd-compactness can obtained.

Corollary 7. *The pre-image of each SS-sd-Lindelöf (compact) set is SS-sd-Lindelöf (compact) under an injective and SS*-sd-open map.*

Proof. Immediate form Theorem 19.

Theorem 20. *Let (U, μ, Δ) is SSTS defined on any universal set U and finite set of parameters Δ. Then, (U, μ, Δ) is SS-sd-compact (Lindelöf) if every γ-parameter supra topological space is supra-sd-compact (Lindelöf), for each $\gamma \in \Delta$.*

Proof. We prove the case of supra-sd-compact, the other case is similar.
Let (U, μ, Δ) is SSTS over the set of parameter $\Delta = \{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_n, n \in N\}$ and the universal set U. Assume that (U, μ_{γ_i}) is the γ_i-parametric supra topological spaces such that

$$(U, \mu_{\gamma_i}), i = 1, 2, 3, ..., n \text{ is a supra-sd-compact, for each } \gamma \in \Delta \text{ and}$$

$$\Psi = \{(C_\epsilon, \Delta) : \epsilon \in \varepsilon\} \text{ is SS-sd-cover of } \tilde{U}.$$

Since $U = \bigcup_{\epsilon\in\varepsilon}(C_\epsilon, \Delta)(\gamma_i)$ for each $\gamma_i \in \Delta$ and (U, μ_{γ_i}) is supra-sd-compact, there is a finite subclasses ε_i of ε such that

$$U = \bigcup_{\epsilon\in\varepsilon_i}(C_\epsilon, \Delta)(\gamma_i).$$

Hence,

$$\tilde{U} = \tilde{\bigcup}_{i=1}^{n}\tilde{\bigcup}_{\epsilon\in\varepsilon_i}(C_\epsilon, \Delta).$$

That's is, $\{(C_\epsilon, \Delta) : \epsilon \in \cup_{i=1}^{n}\varepsilon_i\}$ is a finite subcover of Ψ which cover $\tilde{U}$. Thus, $\tilde{U}$ is SS-sd-compact.

Remark 5. *The following example shall show that the converse of Theorem 20 is not necessarily satisfied in general.*

Example 5. *Let the set of natural numbers N be the universal set and $\Delta = \{\gamma_1, \gamma_2\}$. Consider the classes of soft sets $\{(A_n, \Delta) : n \in N\}$ and $\{(B_n, \Delta) : n \in N\}$, where*
$(A_n, \Delta) = \{\{(\gamma_1, \{2\}), (\gamma_2, \{2\})\}, \{(\gamma_1, \{1\}), (\gamma_2, \varphi)\}, \{(\gamma_1, \{1, 2\}), (\gamma_2, \varphi)\}, \{(\gamma_1, \{1, 2, 3\}), (\gamma_2, \varphi)\}, \{(\gamma_1, \{1, 2, 3, 4\}), (\gamma_2, \varphi)\}, ..., \{(\gamma_1, \{1, 2, 3, 4, ..., n\}), (\gamma_2, \varphi)\}\}$.
$(B_n, \Delta) = \{\{(\gamma_1, \{1, 2\}), (\gamma_2, \{2\})\}, \{(\gamma_1, \{1, 2, 3\}), (\gamma_2, \{2\})\}, \{(\gamma_1, \{1, 2, 3, 4\}), (\gamma_2, \{2\})\}, ..., \{(\gamma_1, \{1, 2, 3, 4, ..., n\}), (\gamma_2, \{2\})\}\}$.
Then, $\mu = \{\tilde{N}, \tilde{\varphi}\} \cup \{(A_n, \Delta) : n \in N\} \cup \{(B_n, \Delta) : n \in N\}$ defines an SSTS on N. It is easy to check that $\tilde{N}$ is an SS-sd-compact (Lindelöf).
On the other hand, the γ_1-parametric supra topological space (U, μ_{γ_1}), where
$(N, \mu_{\gamma_1}) = \{N, \varphi, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, ..., \{1, 2, 3, 4, 5, ..., n\}, n \in N\}$ *is not supra-sd-compact (Lindelöf). Since the class*
$\Psi = \{\{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, ...\{1, 2, 3, 4, 5, 6, ..., n\}, n \in N\}$ *forms a supra-sd-cover for $\tilde{N}$. However, there is no a finite (countable) subclass of Ψ which cover $\tilde{N}$.*

Definition 25. *[64] A soft subset (G, Δ) of an SSTS (U, μ, Δ) is claimed to be SS-almost compact, if every SS-open-cover $\{(C_\epsilon, \Delta) : \epsilon \in \varepsilon\}$ for (G, Δ) has a finite subclass ε_o of ε such that*

$$(G, \Delta) \tilde{\subseteq} \tilde{\bigcup}_{\epsilon \in \varepsilon_o} cl^s(C_\epsilon, \Delta).$$

The space (U, μ, Δ) is claimed to be SS-almost compact if $\tilde{U}$ is SS-almost compact as a soft subset.

Theorem 21. Every SS-sd-hyperconnected SSTS is SS-almost compact.

Proof. Assume that $\Psi = \{(C_\epsilon, \Delta) : \epsilon \in \varepsilon\}$ be an SS-open cover for an SS-sd-hyperconnected SSTS (U, μ, Δ), then $cl^s_{sd}(C_\epsilon, \Delta) = \tilde{U}$ for each $\epsilon \in \varepsilon$, from Corollary 2. Therefore, there is a finite subclass ε_o of ε such that

$$\tilde{U} = \tilde{\bigcup}_{\epsilon \in \varepsilon_o} cl^s_{sd}(C_\epsilon, \Delta) \tilde{\subseteq} \tilde{\bigcup}_{\epsilon \in \varepsilon_o} cl^s(C_\epsilon, \Delta).$$

Thus, $\tilde{U}$ is an SS-almost compact.

Remark 6. *The following example shall confirm that the converse of Theorem 21 is not satisfied in general.*

Example 6. *Let the set of natural numbers N be the universal set and consider $n_1, n_2 \in N$ any two natural numbers. Let $\Delta = \{\gamma_1, \gamma_2\}$ and $\mu = \{\tilde{N}, \tilde{\varphi}, (X_i, \Delta), i = 1, 2, 3\}$ be an SSTS on U, where:*

$$X_1(\gamma_1) = \{n_1\}, \quad X_1(\gamma_2) = \{n_2\}.$$
$$X_2(\gamma_1) = N - \{n_1\}, \quad X_2(\gamma_2) = N - \{n_2\}.$$
$$X_3(\gamma_1) = N - \{n_1\}, \quad X_3(\gamma_2) = N.$$

It is easy to check that $\tilde{N}$ is an SS-almost compact space. On the other hand, for the soft sets (X_1, Δ) and (X_2, Δ), we have $(X_1, \Delta), (X_2, \Delta) \in SD(N)_\Delta$, whereas $(X_1, \Delta) \tilde{\cap} (X_2, \Delta) = \tilde{\varphi}$. Thus, $\tilde{N}$ is not SS-sd-hyperconnected.

Corollary 8. Let (U, μ, Δ) be an SSTS, then the following implications hold from Proposition 6, Proposition 7 and Theorem 21, which are not reversible.

$$\begin{array}{ccccc}
 & & & & \text{SS-sd-hyperconnected} \\
 & & & & \Downarrow \\
\text{SS-sd-compact} & \Rightarrow & \text{SS-compact} & \Rightarrow & \text{SS-almost compact} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{SS-sd-Lindelöf} & \Rightarrow & \text{SS-Lindelöf} & \Rightarrow & \text{SS-almost Lindelöf}
\end{array}$$

Figure 2. The relationships among different types of compactness, Lindelöfness and hyperconnectedness via SS-sd-sets in the frame of SSTSs.

5. Conclusion and future work

This manuscript is devoted to investigating more interesting properties of the notions of SS-sd-connectedness. Specifically, we use the notions of SS-sd-components to define a new type of connectedness in SSTSs, named SSL-sd-connectedness. In addition, we introduce another type, named SS-sd-hyperconnectedness. We discovered that, in contrast to their counterparts, SS-sd-hyperconnected spaces are equivalent to SS-sd-connected spaces. We discuss their basic properties in detail. Moreover, we study another topological property in an SSTS named SS-sd-Lindelöfness (compactness). The behaviour of an SS-sd-compact (Lindelöf) SSTS with the SFIP (SCIP) has been discussed. Furthermore, we show that the image (respectively, pre-image) of each SS-sd-Lindelöf (compact) is SS-Lindelöf (compact) under a surjective and SS-sd-continuous (respectively, an injective and SS-sd-open) function. Finally, we provide two topological charts in Figures 1 and 2, to illustrate the key concepts presented in this paper, and several interesting examples and counterexamples have been provided. Our upcoming work is to apply the provided notions to decision making problem, and introducing more types of compactness and separation axioms in an SSTS via SS-sd-sets. Moreover, we will apply theses notions to the fuzzy supra soft topological spaces [65].

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Conflicts of Interest

The authors declare that they have no conflict of interest regarding the publication of this paper.

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