EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 5899
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On $g\mu$-Paracompact Sets

Heyam H. Al-Jarrah[1], Amani Rawshdeh[2,*], Khalid Y. Al-Zoubi[1], Shefa A. Bani Melhem[1]

[1] *Department of Mathematics, Faculty of Science, Yarmouk University, Irbid, Jordan*
[2] *Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Alsalt, Jordan*

Abstract. In this work, we use the notion of the $g\mu$-paracompact space [6] to introduce two types of $g\mu$-paracompact sets called α-$g\mu$-paracompact and β-$g\mu$-paracompact. We show that every α-$g\mu$-paracompact set is β-$g\mu$-paracompact and if a generalized topological space (S, μ) is $g\mu$-paracompact, then every μ-closed subset in (S, μ) is α-$g\mu$-paracompact while every μg-closed subset in (S, μ) is β-$g\mu$-paracompact. Finally, we introduce the notion of *co*-α-$g\mu$-paracompact set as an application of α-$g\mu$-paracompact and study some of its features.

2020 Mathematics Subject Classifications: 54A05, 54C08, 54D10.

Key Words and Phrases: generalized topological space, $g\mu$-paracompact, α-$g\mu$-paracompact, β-$g\mu$-paracompact.

1. Introduction

In 1944, Dieudonné [7] introduced a broader class of compact spaces, namely paracompact spaces. Sorgenfrey [15] and Stone [16] investigated the behavior of paracompact spaces within the product space. Michail [10] defined paracompactness in the sense of regular topological spaces and demonstrated how metrizability implies paracompactness. Therefore, paracompactness is one of the most essential concepts and possibly the most successful generalization of compactness, which is introduced not only in general topology but also in other structures, such as generalized topological spaces (see [6, 9]).

The study of generalized topological spaces (briefly, GTS) was first initiated by Császár [4]. The pair (S, μ) is called GTS if $\mu \subseteq P(S)$ with $\phi \in \mu$ and μ is closed under the arbitrary union where $P(S)$ denotes the power set of S. A GTS in turn motivated other researchers to generalize the topological concepts including covering properties of generalized topology. For instance, in [6] the authors defined μ-paracompact spaces and $g\mu$-paracompact which are a generalization of paracompactness in GTS, where a GTS (S, μ)

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.5899

Email addresses: heyam@yu.edu.jo (H. H. Al-Jarrah), amanirawshdeh@bau.edu.jo (A. Rawshdeh), khalidz@yu.edu.jo (K. Y. Al-Zoubi), shefa.bm@yu.edu.jo (Sh. A. Bani Melhem)

is called μ-paracompact (resp. $g\mu$-paracompact) if every (S,μ)-cover of S has a $\mu-LF_{(S,\mu)}$ (resp. $g\mu-LF_{(S,\mu)}$) (S,μ)-refinement. In [11], Qahis and Noiri investigated the concept of $\mu-$paracompact spaces with respect to hereditary class $\mathcal{H}$, which is a generalization for a μ-paracompact space.

In classical topology, studying the covering definitions for sets after studying the concept of space is an area that has found interest among some authors, such as: based on the definition of I-Lindelöf space [1] the author presented the definition of I-Lindelöf sets [2]. In addition, the authors used the concept of paracompactness to study and introduce different notions of paracompact sets such as α-paracompact and β-paracompact [see [3]]. Therefore, in this work, we employ the definition of $g\mu$-paracompact spaces that are defined in [6] to introduce the notions of α-$g\mu$-paracompact and β-$g\mu$-paracompact subsets and provide some illustrative examples to elucidate the relationship between them and demonstrate the results that were achieved.

For a GTS (S,μ) the elements of μ are called μ-open sets and the collection of all μ-open sets containing $s \in S$ will be denoted by $\mu(s)$. The complement of a μ-open set is called a $\mu-$closed set, and the intersection of all μ-closed sets containing E will be denoted by $c_\mu(E)$. A subset E of GTS (S,μ) is called a generalized closed set [13], denoted by μg-closed set, if $c_\mu(E) \subseteq G$ whenever $E \subseteq G$ and G is μ-open. A GTS (S,μ) is called μ-T_2-space [14] if for each $s, e \in S$ with $s \neq e$, there are $G \in \mu(s)$ and $H \in \mu(e)$ with $G \cap H = \phi$. For $E \subseteq S$, the subspace of (S,μ) in E is denoted by (E,μ_E).

2. Preliminaries

In this section, we recall the main concepts and properties which will be needed in this work.

Definition 1. *[6] Let (S,μ) be a GTS. Then:*

(i) $\mu^*(s) = \{\cap_{i=1}^{n} G_i : G_i \in \mu(s),\ \forall i = 1, ..., n \in \mathbb{N}\}$ *for each* $s \in S$.

(ii) $\gamma_\mu(E) = \{s \in S : H \cap E \neq \phi$ *for all* $H \in \mu^*(s)\}$ *for each* $E \subseteq S$.

In [6], the authors show that the operator $\gamma_\mu(S)$ created a topology on S defined by $\mu^* = \{E \subseteq S : \gamma_\mu(S - E) = S - E\}$ that is finer than μ, and if μ is a topology on S then $\mu = \mu^*$. The elements of μ^* are called μ^*-open sets and their complements are called μ^*-closed sets. For each $E \subseteq S$ the subspace of (S,μ^*) on E is denoted by (E,μ^*_E).

Definition 2. *[6] A GTS (S,μ) is called γ_μ-regular if for each $s \in S$ and $G \in \mu(s)$, there is $H \in \mu(s)$ with $\gamma_\mu(H) \subseteq G$.*

Definition 3. *[6] Let (S,μ) be a GTS. Then a collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ is called:*

(i) μ-locally finite in (S,μ) (resp. (E,μ_E)), denoted by $\mu-LF_{(S,\mu)}$ (resp. $\mu-LF_{(E,\mu_E)}$), if for each $s \in S$ (resp. $s \in E$) there is $H \in \mu(s)$ (resp. $H \in \mu_E(s)$) with the set $\{\eta : H \cap G_\eta \neq \phi\}$ is finite.

(ii) $g\mu$*-locally finite in* (S,μ) *(resp.* (E,μ_E)*), denoted by* $g\mu{-}LF_{(S,\mu)}$ *(resp.* $g\mu{-}LF_{(E,\mu_E)}$*), if for each* $s \in S$ *(resp.* $s \in E$*) there is* $H \in \mu^*(s)$ *(resp.* $H \in \mu_E^*(s)$*) with the set* $\{\eta : H \cap G_\eta \neq \phi\}$ *is finite.*

It follows from the above definition that each $\mu{-}LF_{(S,\mu)}$ is $g\mu{-}LF_{(S,\mu)}$ but the converse need not be true in general (see Example 2.2 of [6]).

Theorem 1. *[6] If* $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ *is a* $g\mu{-}LF_{(S,\mu)}$*. Then:*

(i) $\gamma_\mu(\mathcal{G}) = \{\gamma_\mu(G_\alpha) : \alpha \in \Delta\}$ *is* $g\mu{-}LF_{(S,\mu)}$*.*

(ii) $\mathcal{G}$ *is* γ_μ*-closure preserving, i.e.* $\gamma_\mu(\cup_{\alpha\in\Delta} G_\alpha) = \cup_{\alpha\in\Delta}\gamma_\mu(G_\alpha)$*.*

Definition 4. *[11] Let* (S,μ) *be a* GTS *and* $E \subseteq Z \subseteq S$*. Then:*

(i) A collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ *is cover of* E *if* $E \subseteq \cup_{\alpha\in\Delta} G_\alpha$ *and if* $G_\alpha \in \mu$ *(resp.,* $G_\alpha \in \mu_Z$*) for each* $\alpha \in \Delta$*, then* $\mathcal{G}$ *is called* (S,μ)*-cover (resp.,* (Z,μ_Z)*-cover) of* E*.*

(ii) If $\mathcal{G}$ *and* $\mathcal{H}$ *are covers of* E*, then* $\mathcal{H}$ *is called a refinement of* $\mathcal{G}$ *if there is* $G \in \mathcal{G}$ *with* $H \subseteq G$ *for each* $H \in \mathcal{H}$*. Moreover, if* $\mathcal{H}$ *is a refinement of* $\mathcal{G}$ *with* $H \in \mu$ *(resp.,* $H \in \mu_Z$*), then* $\mathcal{H}$ *is called* (S,μ)*-refinement (resp.,* (Z,μ_Z)*-refinement) of* $\mathcal{G}$*.*

(iii) A subset E *is called* μ*-paracompact relative to* S *(or* μ*-paracompact subset) if each* (S,μ)*-cover of* E *has* $\mu{-}LF_{(S,\mu)}$ (S,μ)*-refinement and* E *is called* μ_E*-paracompact (or* μ_E*-paracompact subspace) if* (E,μ_E) *is* μ_E*-paracompact as a subspace.*

Definition 5. *Let* (S_1,μ_1) *and* (S_2,μ_2) *be* GTS*. A mapping* $\psi : (S_1,\mu_1) \to (S_2,\mu_2)$ *is called:*

(i) (μ_1,μ_2)*-continuous [4] if* $\psi^{-1}(H) \in \mu_1$ *for each* $H \in \mu_2$*.*

(ii) (μ_1,μ_2)*-open [12] if* $\psi(G) \in \mu_2$ *for each* $G \in \mu_1$*.*

(iii) (μ_1,μ_2)*-closed [13] if* $\psi(M)$ *is* μ_2*-closed in* S_2 *for each* μ*-closed set* M *of* S_1*.*

Proposition 1. *[8] A function* $\psi : (S_1,\mu_1) \to (S_2,\mu_2)$ *is* (μ_1,μ_2)*-closed iff for each* $e \in S_2$ *and* $G \in \mu_1$ *with* $\psi^{-1}(e) \subseteq G$*, there is* $H \in \mu_2(e)$ *with* $\psi^{-1}(H) \subseteq G$*.*

3. α-$g\mu$-paracompact and β-$g\mu$-paracompact sets

In this section, the concepts of α-$g\mu$-paracompact and β-$g\mu$-paracompact subsets are illustrated and the relationship between them with some of their properties are investigated.

Proposition 2. *Let* (S,μ) *be a* GTS *with* $E \subseteq S$ *and* $\mathcal{G} = \{G_\alpha : \alpha \in \Delta,\, G_\alpha \subseteq E\}$*. Then:*

(i) $\mathcal{G}$ *is* $g\mu{-}LF_{(E,\mu_E)}$ *if it is* $g\mu{-}LF_{(S,\mu)}$*.*

(ii) $\mathcal{G}$ *is* $g\mu{-}LF_{(S,\mu)}$ *if it is* $g\mu{-}LF_{(E,\mu_E)}$ *provided that* E *is* μ*-closed.*

Proof. (i) It follows from Definition 3.
(ii) Let $\mathcal{G}$ be $g\mu{-}LF_{(E,\mu_E)}$. If $s \in S$, then either $s \in E$ or $s \notin E$. If $s \in E$, then there is $H \in \mu_E^*(s)$ with the set $\{\eta : H \cap G_\eta \neq \phi\}$ is finite. Now $H = W \cap E$ for some $W \in \mu^*(s)$. Since $\mathcal{G}$ is a collection of subsets of E, then $\{\eta : W \cap G_\eta \neq \phi\}$ is finite. If $s \notin E$ then $S - E \in \mu^*(s)$ which intersects no member of $\mathcal{G}$.

Example 1. *Let (S,μ) be a GTS where $S = \mathbb{R}$ and $\mu = \{G : 0 \notin G\}$. Put $E = \mathbb{Q} - \{0\}$. Then the collection $\{\{e\} : e \in E\}$ is $g\mu{-}LF_{(E,\mu_E)}$ while it is not $g\mu{-}LF_{(S,\mu)}$.*

Definition 6. *Let (S,μ) be a GTS and $E \subseteq S$. Then:*

- *(i) E is called α-$g\mu$-paracompact in (S,μ) (simply, α-$g\mu$-paracompact) if each (S,μ)-cover of E has a $g\mu{-}LF_{(S,\mu)}$ (S,μ)-refinement.*
- *(ii) E is called β-$g\mu$-paracompact in (S,μ) (simply β-$g\mu$-paracompact) if each (E,μ_E)-cover of E has a $g\mu{-}LF_{(E,\mu_E)}$ (E,μ_E)-refinement.*

Proposition 3. *Each α-$g\mu$-paracompact is β-$g\mu$-paracompact.*

Proof. It follows from Definition 3 and Proposition 2.

Note that the converse of Proposition 3 is not true in general. In Example 1, E is β-$g\mu$-paracompact, since each (E,μ_E)-cover of E has a $g\mu{-}LF_{(E,\mu_E)}$ (E,μ_E)-refinement (that is $\{\{e\} : e \in E\}$). On the other hand, $\mathcal{G} = \{\{e\} : e \in E\}$ is an (S,μ)-cover of E and it has no $g\mu{-}LF_{(S,\mu)}(S,\mu)$-refinement since $\mu^*(0) = \phi$. Therefore, E is not α-$g\mu$-paracompact.

Theorem 2. *Let (S,μ) be a $g\mu$-paracompact GTS and $E \subseteq S$. Then E is α-$g\mu$-paracompact if one of the following holds:*

- *(i) E is μg-closed in (S,μ).*
- *(ii) E is μ-closed in (S,μ).*

Proof. (i) Let $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ be an (S,μ)-cover of E. Since $c_\mu(E) \subseteq \cup_{\alpha\in\Delta} G_\alpha$, then $\mathcal{G}_1 = \mathcal{G} \cup \{S - c_\mu(E)\}$ is an (S,μ)-cover of S. So $\mathcal{G}_1$ has a $g\mu{-}LF_{(S,\mu)}$ (S,μ)-refinement, say $\mathcal{H} = \{H_\beta : \beta \in \Lambda\}$. Therefore, the collection $\mathcal{H}_1 = \{H_\beta \in \mathcal{H} : H_\beta \subseteq G_\alpha$ for some $G_\alpha \in \mathcal{G}, \alpha \in \Delta$ and $\beta \in \Lambda\}$ is $g\mu{-}LF_{(S,\mu)}$ (S,μ)-refinement for $\mathcal{G}$.
(ii) The proof is obvious since every μ-closed set is μg-closed.

Corollary 1. *Let (S,μ) be a $g\mu$-paracompact GTS and $E \subseteq S$. Then E is β-$g\mu$-paracompact if one of the following holds:*

- *(i) E is μg-closed in (S,μ).*
- *(ii) E is μ-closed in (S,μ).*

Proof. It follows from Proposition 3 and Theorem 2.

Theorem 3. *Let (S,μ) be a GTS. If each μ-open subset of (S,μ) is α-$g\mu$-paracompact, then each subset E of S is β-$g\mu$-paracompact.*

Proof. Let $\mathcal{G}_E = \{G_\alpha \cap E : G_\alpha \in \mu, \alpha \in \Delta\}$ be (E,μ_E)-cover of E. Then $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ is an (S,μ)-cover of $\cup_{\alpha\in\Delta} G_\alpha$ and so $\mathcal{G}$ has a $g\mu{-}LF_{(S,\mu)}$ (S,μ)-refinement, say $\mathcal{H} = \{H_\beta : \beta \in \Lambda\}$. Define $\mathcal{H}_E = \{H_\beta \cap E : \beta \in \Lambda\}$. Then $\mathcal{H}_E$ is $g\mu{-}LF_{(E,\mu_E)}(E,\mu_E)$-refinement for $\mathcal{G}_E$. Note that if $s \in E$ there is $G \in \mu^*(s)$ with the set $\{\eta : G \cap H_\eta \neq \phi\}$ is finite which implies that the set $\{\eta : (G \cap E) \cap (H_\eta \cap E) \neq \phi\}$ is finite. Finally, since for each $H_\beta \cap E \in \mathcal{H}_E$, there is some $G_\alpha \in \mathcal{G}$ with $H_\beta \subseteq G_\alpha$ so we obtain $H_\beta \cap E \subseteq G_\alpha \cap E$. Therefore, E is β-$g\mu$-paracompact.

Theorem 4. *Let (S,μ) be a GTS and $E \subseteq S$. If for each μ-open set G containing E, there is β-$g\mu$-paracompact Z with $E \subseteq Z \subseteq G$, then E is β-$g\mu$-paracompact .*

Proof. Let $\mathcal{G}_E = \{E \cap H_\alpha : H_\alpha \in \mu, \alpha \in \Delta\}$ be an (E,μ_E)-cover of E. Then there is β-$g\mu$-paracompact Z with $E \subset Z \subset \cup_{\alpha\in\Delta} H_\alpha$. Since $\mathcal{G}_Z = \{H_\alpha \cap Z : \alpha \in \Delta\}$ is a (Z,μ_Z)-cover of Z, then $\mathcal{G}_Z$ has a $g\mu{-}LF_{(Z,\mu_Z)}$ (Z,μ_Z)-refinement, say $\mathcal{H}_Z = \{H_\beta \cap Z : H_\beta \in \mu, \beta \in \Lambda\}$. Put $\mathcal{H}_E = \{H_\beta \cap E : \beta \in \Lambda\}$. Then $\mathcal{H}_E$ is $g\mu{-}LF_{(E,\mu_E)}(E,\mu_E)$-refinement for $\mathcal{G}_E$, since for $s \in E$ there is $H \cap Z \in \mu_Z^*(s)$ with $H \in \mu^*(s)$ and the set $\{\eta : (H \cap Z) \cap (H_\eta \cap Z) \neq \phi\}$ is finite which implies that the set $\{\eta : [(H \cap Z) \cap (H_\eta \cap Z)] \cap E \neq \phi\} = \{\eta : (H \cap E) \cap (H_\eta \cap E) \neq \phi\}$ is finite. Now, for each $H_\beta \cap E \in \mathcal{H}_E$ there is $H_\alpha \cap Z \in \mathcal{G}_Z$ with $H_\beta \cap Z \subseteq H_\alpha \cap Z$ and so $H_\beta \cap E \subseteq H_\alpha \cap E$. Therefore, E is β-$g\mu$-paracompact.

Theorem 5. *Let (S,μ) be a GTS and $E \subseteq Z \subseteq S$. If E is α-$g\mu$-paracompact in (S,μ), then E is α-$g\mu$-paracompact in (Z,μ_Z).*

Proof. Let $\mathcal{G}_Z = \{Z \cap H_\alpha : H_\alpha \in \mu, \alpha \in \Delta\}$ be a (Z,μ_Z)-cover of E. Then $\mathcal{G}_1 = \{H_\alpha : \alpha \in \Delta\}$ is an (S,μ)-cover of E and so it has a $g\mu{-}LF_{(S,\mu)}$ (S,μ)-refinement, say $\mathcal{H} = \{H_\beta : \beta \in \Lambda\}$. Put $\mathcal{H}_1 = \{H_\beta \cap Z : \beta \in \Lambda\}$. As in the proof of Theorem 3, we can show $\mathcal{H}_1$ is a $g\mu{-}LF_{(Z,\mu_Z)}(Z,\mu_Z)$-refinement of $\mathcal{G}_Z$. Therefore, E is α-$g\mu$-paracompact in (Z,μ_Z).

Example 2. *Let (S,μ) be a GTS where $S = \mathbb{R}$ and $\mu = \{G : \mathbb{Q} \subseteq G\} \cup \{\phi\}$. Put $E = Z = \mathbb{R} - \mathbb{Q}$. Then $\mu_Z = \mathcal{P}(Z)$ and Z is μ-closed in (S,μ). Note that, E is α-$g\mu$-paracompact in (Z,μ_Z). On the other hand, E is not α-$g\mu$-paracompact in (S,μ) since $\{\mathbb{Q} \cup \{x\} : x \in E\}$ is (S,μ)-cover of E has no $g\mu{-}LF_{(S,\mu)}$ (S,μ)-refinement.*

Theorem 6. *Let (S,μ) be a GTS and $E \subseteq S$ with E is α-$g\mu$-paracompact in a μ-closed subspace (Z,μ_Z). If there is a μ^*-open set G with $E \subseteq G \subseteq Z$, then each (S,μ)-cover of E has a $g\mu{-}LF_{(S,\mu)}$ (S,μ^*)-refinement.*

Proof. Let $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ be an (S,μ)-cover of E. Then, the collection $\{Z \cap G_\alpha : \alpha \in \Delta\}$ is a (Z,μ_Z)-cover of E and so it has a $g\mu - LF_{(Z,\mu_Z)}$ (Z,μ_Z)-refinement, say $\mathcal{H}$. Now, for each $s \in E$ there is $H_s \in \mathcal{H}$ and $W_s \in \mu^*(s)$ with $s \in H_s = W_s \cap Z$. Since G is μ^*-open, then $\mathcal{W}_E = \{W_s \cap G : s \in E\}$ is a $g\mu$-$LF_{(S,\mu)}$ (S,μ^*)-refinement of $\mathcal{G}$. At first, since Z is μ-closed and for all $s \in E$, $W_s \cap G \subseteq W_s \cap Z = H_s$ and the collection $\{W_s \cap Z : s \in E\}$ is $g\mu - LF_{(Z,\mu_Z)}$ and so, by Proposition 2, $\mathcal{W}_E$ is $g\mu - LF_{(S,\mu)}$. Moreover, for each $s \in E$, there is $G_{\alpha(s)} \in \mathcal{G}$ with $s \in W_s \cap G \subseteq W_s \cap Z \subseteq G_{\alpha(s)} \cap Z \subseteq G_{\alpha(s)}$ and hence $\mathcal{W}_E$ is (S, μ^*)-refinement of $\mathcal{G}$.

Proposition 4. *Let (S,μ) be a GTS and $E \subseteq Z \subseteq S$. Then E is β-$g\mu$-paracompact in (S,μ) iff E is β-$g\mu$-paracompact in (Z,μ_Z).*

Proof. Note that, $(\mu_Z)_E = \{E \cap G : G \in \mu_Z\} = \{E \cap Z \cap H : H \in \mu\} = \{E \cap H : H \in \mu\} = \mu_E$. Also, for each $s \in E$, $(\mu_Z)^*_E(s) = \mu^*_E(s)$. Then, the result becomes obvious.

Lemma 1. *Let (S,μ) be a GTS and $E \subseteq S$. If E is a μ^*-open set, then there is $H \in \mu$ with $E \subseteq H$.*

Proof. For each $s \in E$, there is $G_s \in \mu^*(s)$ with $E = \bigcup_{s \in E} G_s$. Now $G_s = \bigcap_{i=1}^{n_s} H_{i(s)}$ where $H_{i(s)} \in \mu(s)$ for each $1 \leq i \leq n_s$. Finally, for each $s \in E$, choose $1 \leq i \leq n_s$ with $s \in H_{i(s)}$. Therefore, $E \subseteq \bigcup_{i=1}^{n_s} H_{i(s)} = H$ and $H \in \mu$.

Theorem 7. *Let (S,μ) be an GTS and $E, Z \subseteq S$. Then:*

(i) $E \cap Z$ is α-$g\mu$-paracompact if E is μ^*-closed in (S,μ) and Z is α-$g\mu$-paracompact.

(ii) $E \cap Z$ is β-$g\mu$-paracompact if E is μ^*-closed in (S,μ) and Z is β-$g\mu$-paracompact.

(iii) $E \cap Z$ is α-$g\mu$-paracompact (resp., β-$g\mu$-paracompact) if E is μ-closed in (S,μ) and Z is α-$g\mu$-paracompact (resp., β-$g\mu$-paracompact).

Proof. (i) Let $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ be an (S,μ)-cover of $E \cap Z$. Since $S - E$ is μ^*-open, by Lemma 1, there is $W \in \mu$ with $S - E \subseteq W$and hence $\mathcal{G}_1 = \{G_\alpha : \alpha \in \Delta\} \cup \{W\}$ is an (S,μ)-cover of Z. So $\mathcal{G}_1$ has a $g\mu{-}LF_{(S,\mu)}$ (S,μ)-refinement, say $\mathcal{H} = \{H_\beta : \beta \in \Lambda\}$. Hence the family $\mathcal{H}_1 = \{H_\beta \in \mathcal{H} : H_\beta \subseteq G_\alpha$ for some $G_\alpha \in \mathcal{G}, \alpha \in \Delta$ and $\beta \in \Lambda\}$ is $g\mu{-}LF_{(S,\mu)}$ (S,μ)-refinement of $\mathcal{G}$. Therefore, $E \cap Z$ is α-$g\mu$-paracompact.
(ii) Let $\mathcal{G} = \{G_\alpha \cap (E \cap Z) : G_\alpha \in \mu, \alpha \in \Delta\}$ be an $(E \cap Z, \mu_{E \cap Z})$-cover of $E \cap Z$. Then $\mathcal{G}_1 = \mathcal{G}^* \cup \{W \cap Z\}$ is a (Z,μ_Z)-cover of Z, where $\mathcal{G}^* = \{G_\alpha \cap Z : \alpha \in \Delta\}$ and W is an μ-open set with $S - E \subseteq W$. So $\mathcal{G}_1$ has a $g\mu{-}LF_{(Z,\mu_Z)}$ (Z,μ_Z)-refinement, say $\mathcal{H} = \{H_\beta \cap Z : H_\beta \in \mu, \beta \in \Lambda\}$. Hence, the family $\mathcal{H}_1 = \{H_\beta \cap (E \cap Z) : H_\beta \cap Z \subseteq G_\alpha \cap Z$ for some $G_\alpha \cap Z \in \mathcal{G}^*, \alpha \in \Delta$ and $\beta \in \Lambda\}$ is $g\mu{-}LF_{(E \cap Z, \mu_{E \cap Z})}$ $(E \cap Z, \mu_{E \cap Z})$-refinement of $\mathcal{G}$. Therefore, $E \cap Z$ is β-$g\mu$-paracompact.
(iii) Follows from (i) and (ii).

Theorem 8. *Let (S,μ) be γ_μ-regular GTS and $E \subseteq S$. If E is α-$g\mu$-paracompact, then $\gamma_\mu(E)$ is α-$g\mu$-paracompact.*

Proof. Let $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ be (S, μ)-cover of $\gamma_\mu(E)$. Since $\mathcal{G}$ is (S, μ)-cover of E, then $\mathcal{G}$ has a $g\mu$-$LF_{(S,\mu)}(S, \mu)$-refinement, say $\mathcal{H} = \{H_\beta : \beta \in \Lambda\}$. To show that $\mathcal{H}$ is a cover for $\gamma_\mu(E)$, let $H_\beta \in \mathcal{H}$. Since (S, μ) is a γ_μ-regular GTS, then for each $s \in H_\beta$ there is $W_{\beta_s} \in \mu(s)$ with $\gamma_\mu(W_{\beta_s}) \subseteq H_\beta$. Now, $\mathcal{W} = \{W_{\beta_s} : \beta \in \Lambda, s \in H_\beta\}$ is an (S, μ)-cover of E and so it has a $g\mu$-$LF_{(S,\mu)}(S, \mu)$-refinement, say $T = \{T_\lambda : \lambda \in \theta\}$. By Theorem 1, we have $\gamma_\mu(E) \subseteq \gamma_\mu(\cup T_\lambda) = \cup\gamma_\mu(T_\lambda) \subseteq \cup\gamma_\mu(W_{\beta_s}) \subseteq \cup H_\beta$. Therefore, $\gamma_\mu(E)$ is α-$g\mu$-paracompact.

Theorem 9. *Let (S, μ) be γ_μ-regular GTS and $E \subseteq S$. If E is α-$g\mu$-paracompact, then each (S, μ)-cover of E has a μ^*-closed $g\mu$-$LF_{(S,\mu)}$ refinement.*

Proof. Let $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ be (S, μ)-cover of E. For each $s \in E$ pick $G_s \in \mathcal{G}$ with $G_s \in \mu(s)$. Since (S, μ) is γ_μ-regular, then there is $H_s \in \mu(s)$ with $s \in H_s \subseteq \gamma_\mu(H_s) \subseteq G_s$. Then, the collection $\mathcal{H} = \{H_s : s \in E\}$ is (S, μ)-cover of E and so it has a $g\mu$-$LF_{(S,\mu)}(S, \mu)$-refinement, say $\mathcal{W} = \{W_\beta : \beta \in \Lambda\}$. Therefore, by Theorem 1, $\gamma_\mu(\mathcal{W}) = \{\gamma_\mu(W_\beta) : \beta \in \Lambda\}$ is μ^*-closed $g\mu$-$LF_{(S,\mu)}$ refinement.

Theorem 10. *Let (S, μ) be a GTS and $E \subseteq S$. If E is α-$g\mu$-paracompact subset of a μ-T_2-space, then E is μ^*-closed.*

Proof. Let $s \notin E$. Since (S, μ) is μ-T_2-space, then for each $e \in E$ there is $G_e \in \mu(e)$ and $s \notin c_\mu(G_e)$. Therefore, $\mathcal{G} = \{G_e : e \in E\}$ is an (S, μ)-cover of E and hence it has a $g\mu{-}LF_{(S,\mu_S)}$ (S, μ)-refinement, say $\mathcal{W}$. Put $H = \cup\{W : W \in \mathcal{W}\}$, then $\gamma_\mu(H) = \cup\{\gamma_\mu(W) : W \in \mathcal{W}\}$. Finally, take $H^* = S - \gamma_\mu(H)$. Since H^* is μ^*-open with $s \in H^*$ and $H^* \cap E = \phi$, then $s \notin \gamma_\mu(E)$ and hence E is μ^*-closed.

The converse of Theorem 10 is not true in general (see Example 20.11, page 148 of [17]). Then (S, τ) is a T_2-space such that (S, τ) is not paracompact and S is μ^*-closed $(\mu = \tau)$ while S it is not α-$g\mu$-paracompact.

Theorem 11. *Let (S, μ) be a GTS. If $\{E_\alpha : \alpha \in \Delta\}$ is a $g\mu$-$LF_{(S,\mu)}$ collection such that E_α is α-$g\mu$-paracompact of (S, μ) for each $\alpha \in \Delta$, then $E = \cup_{\alpha\in\Delta}E_\alpha$ is α-$g\mu$-paracompact of (S, μ).*

Proof. Let $\mathcal{G}$ be an (S, μ)-cover of E. For each $\alpha \in \Delta$, $\mathcal{G}$ is an (S, μ)-cover of E_α and hence it has a $g\mu$-$LF_{(S,\mu)}$ (S, μ)-refinement, say $\mathcal{H}_\alpha = \{H_\beta : \beta \in \Lambda_\alpha\}$. It is clear that the collection $\mathcal{H}$=$\{H_\beta : \beta \in \Lambda_\alpha, \alpha \in \Delta\}$ is an (S, μ)-refinement of $\mathcal{G}$. To show that $\mathcal{H}$ is $g\mu$-$LF_{(S,\mu)}$, let $s \in S$. Then there is $G_s \in \mu^*(s)$ and a finite subset Δ_s of Δ with $G_s \cap E_\alpha = \phi$ for each $\alpha \in \Delta - \Delta_s$. Now, for each $\alpha \in \Delta_s$, there is $W_{\alpha(s)} \in \mu^*(s)$ that intersects at most finitely many members of $\mathcal{H}_\alpha$. Define $T_s = G_s \cap (\cap_{\alpha\in\Delta_s}W_{\alpha(s)})$. Then $T_s \in \mu^*(s)$ which intersects at most finitely many members of $\mathcal{H}$.

Definition 7. *Let (S, μ) be a GTS. If each μ^*-open cover of S has a finite subcover, then (S, μ) is called μ^*-compact.*

Lemma 2. *Let $\psi : (S_1, \mu_1) \to (S_2, \mu_2)$ be a (μ_1, μ_2)-continuous function. If $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ is a $g\mu{-}LF_{(S_2,\mu_2)}$, then $\psi^{-1}(\mathcal{G}) = \{\psi^{-1}(G_\alpha) : \alpha \in \Delta\}$ is $g\mu{-}LF_{(S_1,\mu_1)}$.*

Proof. Let $s \in S_1$ with $e = \psi(s)$. Then there is $H \in \mu_2^*(e)$ with the set $\{\eta : H \cap G_\eta \neq \phi\}$ is finite. Since $H = \cap_{i=1}^n W_i$ where $W_i \in \mu_2(e)$, then $\psi^{-1}(H) = \cap_{i=1}^n \psi^{-1}(W_i)$ where $\psi^{-1}(W_i) \in \mu_1(s)$. Therefore, $\psi^{-1}(H) \in \mu_1^*(s)$ with the set $\{\eta : \psi^{-1}(H) \cap \psi^{-1}(G_\eta) \neq \phi\}$ is finite. Hence $\psi^{-1}(\mathcal{G})$ is $g\mu{-}LF_{(S_1,\mu_1)}$.

Lemma 3. *Let $\psi : (S_1, \mu_1) \to (S_2, \mu_2)$ be a surjection (μ_1, μ_2)-closed function with $\psi^{-1}(e)$ is μ_1^*-compact for each $e \in S_2$. If $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ is a $g\mu{-}LF_{(S_1,\mu_1)}$, then $\psi(\mathcal{G}) = \{\psi(G_\alpha) : \alpha \in \Delta\}$ is $g\mu{-}LF_{(S_2,\mu_2)}$.*

Proof. Let $e \in S_2$. For each $s \in \psi^{-1}(e)$ choose $H_s \in \mu_1^*(s)$ with the set $\{\eta : H_s \cap G_\eta \neq \phi\}$ is finite. Therefore, the collection $\{H_s : s \in \psi^{-1}(e)\}$ is a μ_1^*-open cover of the μ_1^*-compact subset $\psi^{-1}(e)$ and so there is a finite number of points $s_1, s_2, ...s_n$ in $\psi^{-1}(e)$ with $\psi^{-1}(e) \subseteq \cup_{i=1}^n H_{s_i}$. Note that, for each $1 \leq i \leq n$, H_{s_i} is a finite intersection of members of $\mu_1(s_i)$. Therefore, $\cup_{i=1}^n H_{s_i} = \cap_{j=1}^m K_j$ where K_j is a finite union of μ_1-open sets and so K_j is μ_1-open for each $1 \leq j \leq m$. By Proposition 1, there is $W_{e_{(j)}} \in \mu_2(e)$ with $\psi^{-1}(W_{e_{(j)}}) \subseteq K_j$. Put $W = \bigcap\limits_{j=1}^{m} W_{e_{(j)}}$. Then $W \in \mu_2^*(e)$ with the set $\{\eta : W \cap \psi(G_\eta) \neq \phi\}$ is finite. Since if $t \in W \cap \psi(G_\alpha)$ then there is $r \in S_1$ with $r \in \psi^{-1}(t) \subseteq \psi^{-1}(W) = \cap_{j=1}^m \psi^{-1}(W_{e_{(j)}}) \subseteq \cap_{j=1}^m K_j = K_s$, this means $H_{s_i} \cap G_\alpha \neq \phi$ for some i.

Theorem 12. *Let $\psi : (S_1, \mu_1) \to (S_2, \mu_2)$ be a (μ_1, μ_2)-continuous mapping, (μ_1, μ_2)-open and (μ_1, μ_2)-closed surjective with $\psi^{-1}(e)$ is μ_1^*-compact for each $e \in S_2$. If E is α-$g\mu$-paracompact in (S_1, μ_1), then $\psi(E)$ is α-$g\mu$-paracompact in (S_2, μ_2).*

Proof. Let $\mathcal{H} = \{H_\alpha : \alpha \in \Delta\}$ be (S_2, μ_2)-cover of $\psi(E)$. Since ψ is a (μ_1, μ_2)-continuous mapping, then the collection $\mathcal{G} = \{\psi^{-1}(H_\alpha) : \alpha \in \Delta\}$ is an (S_1, μ_1)-cover of E and so $\mathcal{G}$ has a $g\mu{-}LF_{(S_1,\mu_1)}$ (S_1, μ_1)-refinement, say $\mathcal{W} = \{W_\beta : \beta \in \Lambda\}$. Therefore, by Lemma 3, $\psi(\mathcal{W}) = \{\psi(W_\beta) : \beta \in \Lambda\}$ is $g\mu{-}LF_{(S_2,\mu_2)}$ (S_2, μ_2)-refinement of $\mathcal{H}$ in (S_2, μ_2).

4. Some application on α-$g\mu$-paracompact sets

In this section, we introduce the notion of *co-α-$g\mu$*-paracompact set as an application of α-$g\mu$-paracompact and study some of its properties.

Definition 8. *Let (S, μ) be a GTS. A subset $E \subseteq S$ is called co-α -$g\mu$-paracompact if for each $s \in E$ there is a pair (H, G) with $H \in \mu$ and G is α-$g\mu$-paracompact such that $s \in H - G \subseteq E$. The collection of all co-α-$g\mu$-paracompact sets will be denoted by $\mu^{\alpha g\mu}$.*

Theorem 13. *Let (S, μ) be a GTS. Then:*

(i) $(S, \mu^{\alpha g\mu})$ *is a GTS with* $\mu \subseteq \mu^{\alpha g\mu}$.

(ii) $\mathcal{B}(\mu^{\alpha g\mu}) = \{H - G : H \in \mu$ *and G is α-$g\mu$-paracompact* $\}$ *generate a base for* $\mu^{\alpha g\mu}$.

Proof. (i) Let $\mathcal{E} = \{E_\alpha : \alpha \in \Delta\}$ be a collection of $\mu^{\alpha g\mu}$ subset. If $s \in \underset{\alpha\in\Delta}{\cup} E_\alpha$, then there is $\alpha_\circ \in \Delta$ and a pair (H, G) with $H \in \mu$ and G is α -$g\mu$-paracompact such that $s \in H - G \subseteq E_{\alpha_\circ} \subseteq \underset{\alpha\in\Delta}{\cup} E_\alpha$. Since ϕ is α -$g\mu$-paracompact then $\mu^{\alpha gm}$ is a GTS on S. Moreover, if $H \in \mu$ then $H = H - \phi \in \mu^{\alpha gm}$.
(ii) It follows from Definition 8.

The following example will show that the reverse inclusion of Theorem 13 is not true in general.

Example 3. *Consider* $S = (0, 1)$ *and* $\mathcal{B} = \{\phi\} \cup \{(0, a), (a, 1) : a \in (0, 1)\}$. *Assume that* $(S, \mu(\mathcal{B}))$ *is the* GTS *generated on* S *by the base* $\mathcal{B}$. *Then* $S \in \mu(\mathcal{B})$ *and hence* $S - \{\frac{1}{3}, \frac{1}{2}\} \in \mu^{\alpha gm} - \mu(\mathcal{B})$.

Theorem 14. *Let* (S, μ) *be a* GTS. *Then the following are equivalent:*

(i) $\mu = \{H - G : H \in \mu$ *and* G *is* α-$g\mu$-*paracompact* $\}$;

(ii) $\mu_{E^c} \subseteq \mu$ *for each* E *is* α-$g\mu$-*paracompact;*

(iii) $\mu = \mu^{\alpha g\mu}$.

Proof. $(i \Rightarrow ii)$ Let $H \in \mu_{E^c}$. Then $H = G \cap E^c = G - E$ with $G \in \mu$ and E is α-$g\mu$-paracompact. By part (i), $H \in \mu$.

$(ii \Rightarrow iii)$ Let $E \in \mu^{\alpha g\mu}$. Then, for each $s \in E$ there is a pair (H, G) with $H \in \mu$ and G is α-$g\mu$-paracompact such that $s \in H - G \subseteq E$. Since $H - G \in \mu_{G^c} \subseteq \mu$, then $E \in \mu$.

$(iii \Rightarrow i)$ From Definition 8, the collection $\{H-G : H \in \mu$ and G is α-$g\mu$-paracompact$\}\subseteq \mu^{\alpha g\mu} = \mu$. Now, let $E \in \mu$, then $E - \phi \in \{H - G : H \in \mu$ and G is α-$g\mu$-paracompact$\}$ and hence the result follows.

Theorem 15. *Let* (S, μ) *be a* GTS. *If* E *is* μ^* *closed, then* $(\mu^{\alpha g\mu})_E \subseteq (\mu_E)^{\alpha g\mu}$.

Proof. Let $H \in (\mu^{\alpha g\mu})_E$ with $s \in H$. Then $H = G \cap E$ with $G \in \mu^{\alpha g\mu}$. Since there is a pair (Z, W) with $Z \in \mu$ and W is α-$g\mu$-paracompact such that $s \in Z - W \subseteq G$, then $s \in (Z \cap E) - (W \cap E) \subseteq H$. Now $Z \cap E \in \mu_E$ and by Theorems 5 and 7, $W \cap E$ is α -$g\mu$-paracompact in (E, μ_E). Therefore, $H \in (\mu_E)^{\alpha g\mu}$.

Proposition 5. *Let* $\psi : (S_1, \mu_1) \rightarrow (S_2, \mu_2)$ *be* (μ_1, μ_2)-*homeomorphism with* $\psi^{-1}(e)$ *is* μ^*-*compact for each* $e \in S_2$. *Then* $\psi : (S_1, \mu_1^{\alpha g\mu_1}) \rightarrow (S_2, \mu_2^{\alpha g\mu_2})$ *is open mapping.*

Proof. Let $E \in \mathcal{B}(\mu_1^{\alpha g\mu_1})$. Then by Theorem 12, $\psi(E) \in \mu_2^{\alpha g\mu_2}$ and hence ψ is open.

Question: Let (S, μ) be a GTS. What are the conditions to become $\mu^{\alpha g\mu} = (\mu^{\alpha g\mu})^{\alpha g\mu}$?

The following consequence is a partial answer to this question.

Recall that a subset E is called α-paracompact of (S, μ) [3] if each (S, μ)-cover of E has a $\mu{-}LF_{(S,\mu)}$ (S, μ)-refinement.

Theorem 16. *Let (S, μ) be a T_2-topological space. Then* $\mu^{\alpha g\mu} = (\mu^{\alpha g\mu})^{\alpha g\mu}$.

Proof. At first, note that a subset E of (S, μ) is α-$g\mu$-paracompact iff it is α-paracompact in (S, μ) since $\mu = \mu^*$. As in the proof of Theorem 10 each α-paracompact set in (S, μ) is closed and so by Theorem 14, $\mu = \mu^{\alpha g\mu}$. Therefore, $(\mu^{\alpha g\mu})^{\alpha g\mu} = \mu^{\alpha g\mu} = \mu$.

5. Conclusion

One major area of study in topological studies is the exploration of topological notions and topics through extensions of classical topology. Generalized topology is one of the recent extensions of topology and hence we investigate the definition of $g\mu$-paracompact space that is defined in [6], to study the main characteristics of two types of $g\mu$-paracompact subsets, namely, α-$g\mu$-paracompact and β-$g\mu$-paracompact and we examine the relationship between them. In future work, we intend to study $g\mu$-paracompact spaces in other structures such as supra and infra-topological spaces. Furthermore, we can study and define other forms of $g\mu$-paracompact spaces by using μ-semi-open or μ-preopen sets which are defined in [5].

Acknowledgements

The publication of this paper was supported by the Yarmouk University Research Council.

Availability of data and material: No data were used to support this study.

Conflicts of interest: The authors declare no conflict of interest.

References

[1] K. Y. Al-Zoubi, B. Al-Nashef, I-Lindelöf spaces, *Int. J. Math. Math. Sci.*, **vol. 2004** , Article ID 173213, 7 pages, 2004.

[2] K.Y. Al-Zoubi, On I-Lindelöf sets, *Acta Math. Hungar.*, **118** (2008), 75-83.

[3] C. E. Aull, Paracompact subsets, *Proc. of the Second Prague Topological Symposium*, Prague (1966), 45-51.

[4] Á. Császár, Generalized topology, generalized continuity, *Acta Math. Hungar.*, **96** (2002), 351-357.

[5] Á. Császár, Generalized open sets in generalized topologies, *Acta Math. Hungar.*, **106** (1-2) (2005), 53-66.

[6] A. Deb Ray and R. Bhowmick, μ-paracompact and $g\mu$-paracompact generalized topological spaces, *Hacettepe J. Math. Stat.*, **45**(2) (2016), 447-453.

[7] J. Dieudonné, Une generalization des espaces compacts, *J. Math. Pures Appl.*, **23** (1944), 65-76.

[8] X. Ge, J. Gong and I. Reilly, Some characterizations of mappings on generalized topological spaces, *New Zealand J. Math.*, **46** (2016), 73-81.

[9] S. Kowalczyk, M. Turowska, On continuity in generalized topology, *Topol. Appl.*, **297** (2021), 107702. doi: 10.1016/j.topol.2021.107702.

[10] E. Michael, A note on paracompact spaces, *Proc. Amer. Math. Soc.*, **4**(5) (1953), 831-838.

[11] A. Qahis and T. Noiri, μ-paracompactness via hereditary classes, *Missouri J. of Math. Sci.*, **32**(1) (2020), 21-31.

[12] B. Roy, A note on weakly (μ, λ)-closed function, *Math. Bohemica*, **138**(4) (2013), 397-405.

[13] B. Roy, On a type of generalized open sets, *Appl. Gen. Topology*, **12** (2011), 163-173.

[14] M. S. Sarsak, Weak separation axioms in generalized topological spaces, *Acta Math. Hungar.*, **131** (2011), 110-121.

[15] R. H. Sorgenfrey, On the topological product of paracompact spaces, *Bull. Amer. Math. Soc.*, **53** (1947), 631-632.

[16] A. H. Stone, Paracompactness and product spaces, *Bull. Amer. Math. Soc.*, **54** (1948), 977-982.

[17] S. Willard, General Topology, Addition Wesley (1970).