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Stieltjes Limits in Continuous Function Spaces

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Abstract. Sir Isaac Newton and Gottfried Wilhelm Leibniz first discovered the concept of limits in calculus in the late 17^{th} century. Both developed calculus with different but complementary approaches and notations. The Stieltjes limit is a generalization of the standard limit by replacing x approaching x_0 with $f(x)$ approaching $f(x_0)$. In this paper, we define the limit and the Stieltjes limit on function-valued continuous operators, specifically, on bounded operators that take values in continuous functions. To support this definition, we first introduce the concepts of neighborhoods, continuous function operators, increasing operators, strictly increasing operators, decreasing operators, strictly decreasing operators, and their respective limits in function-valued operators. The results reveal the necessary conditions and properties of limits and Stieltjes limits on function-valued operators, which share similarities with limits and Stieltjes limits in real-valued functions. These findings broaden our understanding of limit concepts in a more general context and provide new insights into operator analysis.

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1. Introduction

Mathematics continuously evolves, with integral theory being one of its fundamental branches, which undergoes ongoing research and development. Integral theory plays a crucial role in various mathematical applications, particularly in solving engineering problems [1]. For instance, in financial engineering, integrals are utilized in investment calculations [2]; in mechanical engineering, integrals are applied in fluid mechanics [3]; and in electrical engineering, integral analysis helps determine the current, voltage, and power in electrical circuits [4].

Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) independently explored the fundamental concept of integrals. Their work laid the foundation for differentiation and integration, leading to the continuous advancement of integral theory,

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including its modern applications [5]. One significant development in integral theory is its extension to function spaces, particularly for continuous functions [6].

In the 19^{th} century, German mathematician Georg Friedrich Bernhard Riemann (1826 – 1866) formally defined the concept of definite integrals, even though Newton and Leibniz had already established the fundamental theorem of calculus. Due to his contributions to integral theory, the definite integral is often referred to as the Riemann integral [7]. This definition serves as the foundation for solving various mathematical problems in elementary calculus [8].

Further advancements in integral theory led to the introduction of a generalized form of the Riemann integral, known as the Riemann-Stieltjes integral, which addresses functions that are not Riemann-integrable [9]. This integral was first introduced by Thomas Joannes Stieltjes in 1894 [10]. The Riemann-Stieltjes integral extends the Riemann integral by replacing the variable of integration x with a function $r(x)$, thereby generalizing the concept to accommodate a broader class of functions [10].

Several studies have investigated the Riemann-Stieltjes integral with real-number boundaries, including works by Gregory Convertito and David Cruz-Uribe [11], as well as Joong Kwoen Lee and Han Ju Lee [12]. However, this study defines the Riemann-Stieltjes integral with integration limits defined by continuous functions on $[a, b]$. Using the ordering relation $f \preceq g$, which implies $f(x) \leq g(x)$ for all x in $[a, b]$, this research aims to establish the properties of the Riemann-Stieltjes integral under these integration limits. By extending the integral boundaries to continuous functions, we uncover new properties and insights that further enrich the integral theory landscape.

2. Preliminaries

This study focuses on the concept of the limit of operator-valued continuous functions and the Stieltjes limit for operator-valued functions. Therefore, a deep understanding of the Stieltjes limit and continuous functions in the space $\mathcal{C}[a, b]$ is required.

Definition 1. *[13] The set $\mathcal{C}[a, b]$ consists of all real-valued continuous functions defined on $[a, b] \subset \mathbb{R}$.*

In defining and proving theorems, a unit function in $\mathcal{C}[a, b]$ is required to ensure that the results remain continuous functions. The unit function is defined as follows:

Definition 2. *[6] The function e is the unit element in $\mathcal{C}[a, b]$, where $e(x) = 1$ for all x in the interval $[a, b]$.*

The definition of the Stieltjes limit for function-valued operators involves operations on continuous functions. Therefore, we present some properties of continuous functions in $\mathcal{C}[a, b]$ as follows:

Theorem 1. *[6] For any $f, g \in \mathcal{C}[a, b]$ and any $\alpha, \beta \in \mathbb{R}$, the following holds:*

(i) $\alpha f + \beta g \in \mathcal{C}[a, b]$

(ii) $fg \in \mathcal{C}[a, b]$

(iii) $\frac{f}{g} \in \mathcal{C}[a, b]$ *if* $g(x) \neq 0$ *for all* $x \in [a, b]$

In addition to unit function, the ordering properties of continuous functions in $\mathcal{C}[a, b]$ are also necessary for defining the Stieltjes limit. The ordering properties of continuous functions in $\mathcal{C}[a, b]$ is as follows:

Definition 3. *[6] For any* f *and* g *in* $\mathcal{C}[a, b]$*, we define:*

(i) $f = g \Leftrightarrow f(x) = g(x)$ *for every* $x \in [a, b]$.

(ii) $f \preceq g \Leftrightarrow f(x) \leq g(x)$ *for every* $x \in [a, b]$.

(iii) $f \prec g \Leftrightarrow f(x) < g(x)$ *for every* $x \in [a, b]$.

(iv) $f \succeq g \Leftrightarrow f(x) \geq g(x)$ *for every* $x \in [a, b]$.

(v) $f \succ g \Leftrightarrow f(x) > g(x)$ *for every* $x \in [a, b]$.

Theorem 2. *[14] For any* $f, g,$ *and* h *in* $\mathcal{C}[a, b]$*, the following holds:*

(i) $f \prec g \Rightarrow (f + h \prec g + h)$ *for every* $h \in \mathcal{C}[a, b]$.

(ii) $f \prec g \Rightarrow (\alpha f \prec \alpha g)$ *for every* α *preceded by* $\mathbf{0}$.

To compute the Stieltjes limit for function-valued operators, we define open and closed intervals, which serve as the domain for the limit process, as follows:

Definition 4. *[13] For any* f *and* g *in* $\mathcal{C}[a, b]$ *such that* $f \prec g$*, we define:*

(i) $(f, g) = \{h \in \mathcal{C}[a, b] : f \prec h \prec g\}$ *as the open interval in* $\mathcal{C}[a, b]$.

(ii) $[f, g] = \{h \in \mathcal{C}[a, b] : f \preceq h \preceq g\}$ *as the closed interval in* $\mathcal{C}[a, b]$.

Definition 5. *[15] Let* $A \subseteq \mathcal{C}[a, b]$ *be a non-empty set.*

(i) The set A *is said to be bounded above if there exists* $m \in \mathcal{C}[a, b]$ *such that* $m \succeq f$ *for every* $f \in A$*. In this case,* m *is called an upper bound of* A*.*

(ii) The set A *is said to be bounded below if there exists* $n \in \mathcal{C}[a, b]$ *such that* $n \preceq f$ *for every* $f \in A$*. In this case,* n *is called a lower bound of* A*.*

(iii) The set A *is said to be bounded if it is both bounded above and below.*

(iv) If A *is bounded above, a function* $u \in \mathcal{C}[a, b]$ *is called the least upper bound or supremum of* A *if* u *is an upper bound of* A *and for every upper bound* m *of* A*, we have* $u \preceq m$*, denoted as:*

$$u = \sup(A).$$

(v) If A *is bounded below, a function* $l \in \mathcal{C}[a, b]$ *is called the greatest lower bound or infimum of* A *if* l *is a lower bound of* A *and for every lower bound* n *of* A*, we have* $l \succeq n$*, denoted as:*

$$l = \inf(A).$$

3. Main Theorem

This section addresses the limit of operators on $C[a, b]$ and the Stieltjes limit on $C[a, b]$. The proof of the main theorem is established through a rigorous analysis, following a structured approach that builds upon the results and concepts developed in the preceding subsections, which serve as the theoretical foundation for the argument.

3.1. Limit of Operators on $\mathcal{C}[a, b]$

In this subsection, we define the concept of the limit point of a set of continuous functions and the limit of an operator whose values are continuous functions. This definition serves as the foundation for defining the Stieltjes limit on operators that take continuous function values.

Definition 6. *Given $f \in \mathcal{C}[a, b]$ and a positive real number ε, the ε-neighborhood of the function f is defined as the set:*

$$V_\varepsilon(f) = \{g \in \mathcal{C}[a, b] : |g - f| \prec \varepsilon e\}.$$

For example, if we consider $\mathbf{0} \in \mathcal{C}[0, 1]$, the 3-neighborhood of $\mathbf{0}$ is given by

$$V_3(\mathbf{0}) = \{g \in \mathcal{C}[0, 1] : |g - \mathbf{0}| \prec 3e\} = \{g \in \mathcal{C}[0, 1] : |g| \prec 3e\}.$$

Definition 7. *Given $A \subset \mathcal{C}[a, b]$, a function $f_0 \in \mathcal{C}[a, b]$ is called a limit point of the set A if, for any real number $\varepsilon > 0$, the following condition holds:*

$$(V_\varepsilon(f_0) \cap A) \setminus \{f_0\} \neq \emptyset.$$

The function f_0 may or may not be contained in A. Explicitly, there must exist a function in $(V_\varepsilon(f_0) \cap A)$ that is different from f_0 for f_0 to be a limit point of the set A. As an illustration, consider the following example.

Example 1. Given $A = (g_1, g_2) \cup \{g_3\} \subset \mathcal{C}[1, 2]$ with $g_1(x) = x$, $g_2(x) = 3x$, and $g_3(x) = 5x$, each defined on the interval $[1, 2]$, we will show that the function $h \in A$ with $h(x) = 2x$ and $g_1 \notin A$ is a limit point of A, while $g_3 \in A$ is not a limit point of A.

The function $h(t)$ is a limit point of the set A because for any real number $\varepsilon > 0$, there exists:

$$f(x) = \begin{cases} h + \frac{1}{2}\varepsilon e, & \text{if } \varepsilon < 1, \\ \frac{3}{2}h, & \text{if } \varepsilon > 1, \end{cases}$$

such that $f \in (V_\varepsilon(h) \cap A)$ and $f \neq h$, or equivalently, $(V_\varepsilon(h) \cap A) \setminus \{h\} \neq \emptyset$.

Similarly, the function $g_1 \notin A$ is a limit point of the set A because for any real number $\varepsilon > 0$, there exists:

$$f(x) = \begin{cases} g + \frac{1}{2}\varepsilon e, & \text{if } \varepsilon < 2, \\ 2g_1, & \text{if } \varepsilon > 2, \end{cases}$$

such that $f \in (V_\varepsilon(g_1) \cap A)$ and $f \neq g_1$, or equivalently, $(V_\varepsilon(g_1) \cap A) \setminus \{g_1\} \neq \emptyset$.

Next, we show that $g_3 \in A$ is not a limit point of the set A. The function $g_3 \in A$ is not a limit point of A because there exists a real number $\varepsilon = 1$ such that

$$(V_\varepsilon(g_3) \cap A) \setminus \{g_3\} = \{g_3\},$$

which simplifies to

$$(V_\varepsilon(g_3) \cap A) \setminus \{g_3\} = \emptyset.$$

Definition 8. *Given $A \subseteq \mathcal{C}[a,b]$ and a function f_0 as a limit point of the set A, an operator $F : A \to \mathcal{C}[a,b]$ has a limit operator L at f_0 if for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for every $f \in A$ satisfying $0 \prec |f - f_0| \prec \delta e$, the following holds:*

$$|F(f) - L| \prec \varepsilon e.$$

The inequality $0 \prec |f - f_0|$ is equivalent to the statement $f \neq f_0$. If L is the *limit operator* of F at f_0, then it can be written as:

$$L = \lim_{f \to f_0} F(f).$$

Example 2. It will be shown that:

$$\lim_{f \to 2e} (2f + e) = 5e.$$

For any positive real number ε, we choose a positive real number $\delta = \frac{\varepsilon}{2}$ such that for every $f \in A$ satisfying $0 \prec |f - 2e| \prec \delta e$, we obtain:

$$\begin{aligned} &|(2f + e) - (5e)| \\ &\prec |2f - 4e| \\ &= |2(f - 2e)|. \end{aligned}$$

$$|\ 2(f - 2e)| = |2|\ |f - 2e| = 2|f - 2e| \prec 2\delta e = 2\delta e.$$

By substituting $\delta = \frac{\varepsilon}{2}$, we obtain:

$$2\delta e = 2\left(\frac{\varepsilon}{2}\right) e = \varepsilon e.$$

Thus:

$$\lim_{f \to 2e} (2f + e) = 5e.$$

Example 3. Consider $F : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ defined as:

$$F(f) = \begin{cases} e, & \text{if } f \preceq \mathbf{0}, \\ 2e, & \text{if } f \succ \mathbf{0}. \end{cases}$$

It will be shown that $F(\mathbf{0}) = e$ is not a *limit operator* of F at $\mathbf{0}$. The continuous function $F(\mathbf{0}) = e$ is not a *limit operator* of F at $\mathbf{0}$ because there exists a positive real number $\varepsilon = \frac{1}{2}$ and any real number $\delta > 0$ such that there exists $f_0 = (\mathbf{0} + \frac{1}{2}\delta e) \in A$ satisfying:

$$0 \prec |f_0 - \mathbf{0}| \prec \delta e,$$

but:

$$|F(f_0) - F(\mathbf{0})| = |2e - e| = |e| = e \succ \frac{1}{2}e = \varepsilon e.$$

Thus, $F(\mathbf{0}) = e$ is not a *limit operator* of F at $\mathbf{0}$.

Before discussing the properties of *limit operators* on $\mathcal{C}[a,b]$, we define the addition, subtraction, multiplication, and division of operators on $\mathcal{C}[a,b]$. This definition is crucial for the subsequent discussion.

Definition 9. *Given $A \subseteq \mathcal{C}[a,b]$ and operators F and G, each mapping A to $\mathcal{C}[a,b]$, the definitions of addition $F+G$, subtraction $F-G$, and multiplication FG, each from A to $\mathcal{C}[a,b]$, are as follows:*

$$(F+G)(f) = F(f) + G(f),$$
$$(F-G)(f) = F(f) - G(f),$$
$$(FG)(f) = F(f)G(f),$$

For every $f \in A$.

In addtition, if $\gamma \in \mathcal{C}[a,b]$, we define γF as follows:

$$(\gamma F)(f) = \gamma F(f),$$

For every $f \in A$.

Finally, if $G(f)(x) \neq 0$ for all $x \in [a,b]$, we define $\frac{F}{G}$ as follows:

$$\left(\frac{F}{G}\right)(f) = \frac{F(f)}{G(f)},$$

for every $f \in A$.

Theorem 3. *If $f_0 \in \mathcal{C}[a,b]$ such that $\mathbf{0} \preceq f_0 \prec \varepsilon e$ for every $\varepsilon > 0$, then $f_0 = \mathbf{0}$.*

Proof. Suppose $f_0 \succ \mathbf{0}$. Choosing a positive real number $\varepsilon_0 < \min\{\frac{1}{2}f_0(x) | x \in [a,b]\}$, we obtain:

$$\mathbf{0} \prec \varepsilon_0 e \preceq \frac{1}{2}f_0 \prec f_0 \quad \text{or equivalently,} \quad \varepsilon_0 e \prec f_0.$$

This contradicts $f_0 \prec \varepsilon e$ for every positive real number ε. Thus, it must be that $f_0 = \mathbf{0}$.

From Definition 4.1.7 and Theorem 4.1.4, several properties of the limit operator on $\mathcal{C}[a,b]$ can be established as follows:

Theorem 4. *Given $A \subseteq \mathcal{C}[a,b]$, an operator F mapping from A to $\mathcal{C}[a,b]$, and f_0 as a limit point of A, if:*

$$\lim_{f \to f_0} F(f) = L \quad \textit{and} \quad \lim_{f \to f_0} F(f) = M,$$

then $L = M$.

Proof. Given that $\lim_{f\to f_0} F(f) = L$, it follows that for every real number $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |f - f_0| \prec \delta_1 e$, then:

$$|F(f) - L| \prec \frac{\varepsilon}{2} e.$$

Similarly, given $\lim_{f\to f_0} F = M$, for every $\varepsilon > 0$, there exists $\delta_2 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |f - f_0| \prec \delta_2 e$, then:

$$|F(f) - M| \prec \frac{\varepsilon}{2} e.$$

Choosing $\delta = \min(\delta_1, \delta_2)$, if $\mathbf{0} \prec |f - f_0| \prec \delta e$, then:

$$|L - M| = |L - F(f) + F(f) - M| \preceq |L - F(f)| + |F(f) - M|.$$

By substitution, we obtain:

$$|L - M| \prec \frac{\varepsilon}{2} e + \frac{\varepsilon}{2} e = \varepsilon e.$$

This holds for every $\varepsilon > 0$, it follows that:

$$|L - M| = \mathbf{0} \quad \text{or} \quad L = M.$$

The implication of Theorem 4 is that if an operator F has a limit at f_0, then the limit is unique.

Theorem 5. *Given $A \subseteq \mathcal{C}[a,b]$, operators F and G, each mapping from A to $\mathcal{C}[a,b]$, and f_0 as a limit point of the set A, along with a function $\gamma \in \mathcal{C}[a,b]$. If $\lim_{f\to f_0} F = L$ and $\lim_{f\to f_0} G = M$, then:*

(i) $\lim_{f\to f_0} (F(f) + G(f)) = L + M$

(ii) $\lim_{f\to f_0} (F(f) - G(f)) = L - M$

(iii) $\lim_{f\to f_0} (F(f)G(f)) = LM$

(iv) $\lim_{f\to f_0} (\gamma F(f)) = \gamma L$

(v) $\lim_{f\to f_0} \left(\frac{F(f)}{G(f)}\right) = \frac{L}{M}$,
provided that $G(f)(x) \neq 0$ and $M(x) \neq 0$, for all $x \in [a,b]$.

Proof.

(i) Given that $\lim_{f \to f_0} F(f) = L$, it follows that for every real number $\varepsilon > 0$, there exists a real number $\delta_1 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |f - f_0| \prec \delta_1 e$, then:

$$|F(f) - L| \prec \frac{\varepsilon}{2} e.$$

Similarly, given that $\lim_{f \to f_0} G(f) = M$, it follows that for every real number $\varepsilon > 0$, there exists a real number $\delta_2 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |f - f_0| \prec \delta_2 e$, then:

$$|G(f) - M| \prec \frac{\varepsilon}{2} e.$$

By choosing $\delta = \min(\delta_1, \delta_2)$, we obtain that for every $f \in A$ and $\mathbf{0} \prec |f - f_0| \prec \delta e$, the following holds:

$$|[F(f) + G(f)] - [L + M]| \preceq |F(f) - L| + |G(f) - M| \prec \frac{\varepsilon}{2} e + \frac{\varepsilon}{2} e = \varepsilon e.$$

Thus,

$$\lim_{f \to f_0} (F(f) + G(f)) = L + M.$$

(ii) Given that $\lim_{f \to f_0} F(f) = L$, it follows that for every real number $\varepsilon > 0$, there exists a real number $\delta_1 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |f - f_0| \prec \delta_1 e$, then:

$$|F(f) - L| \prec \frac{\varepsilon}{2} e.$$

Similarly, given that $\lim_{f \to f_0} G(f) = M$, it follows that for every real number $\varepsilon > 0$, there exists a real number $\delta_2 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |f - f_0| \prec \delta_2 e$, then:

$$|G(f) - M| \prec \frac{\varepsilon}{2} e.$$

By choosing $\delta = \min(\delta_1, \delta_2)$, we obtain that for every $f \in A$ and $\mathbf{0} \prec |f - f_0| \prec \delta e$, the following holds:

$$|[F(f) - G(f)] - [L - M]| \preceq |F(f) - L| + |G(f) - M| \prec \frac{\varepsilon}{2} e + \frac{\varepsilon}{2} e = \varepsilon e.$$

Thus,

$$\lim_{f \to f_0} (F - G) = L - M.$$

(iii) Given that $\lim_{f \to f_0} F(f) = L$, it follows that for every real number $\varepsilon > 0$, there exists a real number $\delta_1 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |f - f_0| \prec \delta_1 e$, then:

$$|F(f) - L| \prec \varepsilon \left(\frac{e}{2|M| + e} \right).$$

Similarly, given that $\lim_{f \to f_0} G(f) = M$, it follows that for every real number $\varepsilon > 0$, there exists a real number $\delta_2 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |f - f_0| \prec \delta_2 e$, then:

$$|G(f) - M| \prec \varepsilon \left(\frac{e}{2(\sup |F(f)| + e)} \right),$$

where $\sup |F(f)|$ is the supremum of $|F(f)|$ in the neighborhood of f_0 where $f \neq f_0$. By choosing $\delta = \min(\delta_1, \delta_2)$, we obtain that for every $\mathbf{0} \prec |f - f_0| \prec \delta e$, the following holds:

$$\begin{aligned} |F(f)G(f) - LM| &= |F(f)G(f) - F(f)M + F(f)M - LM| \\ &= |F(f)[G(f) - M] + M[F(f) - L]| \\ &\preceq |F(f)|\ |G(f) - M| + |M|\ |F(f) - L|. \end{aligned}$$

From this, we conclude:

$$\lim_{f \to f_0} (F(f)G(f)) = LM.$$

(iv) Given that $\lim_{f \to f_0} F(f) = L$, it follows that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that if $f \in A$ and $\mathbf{0} \prec |f - f_0| \prec \delta e$, then:

$$|F(f) - L| \prec \frac{\varepsilon e}{|\gamma|}.$$

Since $\gamma \in \mathcal{C}[a, b]$, for every $\mathbf{0} \prec |f - f_0| \prec \delta e$, the following holds:

$$|\gamma F(f) - \gamma L| = |\gamma|\ |F(f) - L| \prec |\gamma|\ \frac{\varepsilon e}{|\gamma|} = \varepsilon e.$$

Thus,

$$\lim_{f \to f_0} (\gamma F(f)) = \gamma L.$$

Next, the concept of a one-sided limit of an operator will be defined in more detail. This definition will be used to understand the behavior of a function around a specific operator in a single direction. The following defines a one-sided limit of an operator on $\mathcal{C}[a, b]$.

Definition 10. *Consider $A \subseteq \mathcal{C}[a, b]$ and let f_0 be a limit point of the set A.*

(i) For an operator $F : A \to \mathcal{C}[a, b]$, the function L is called the right limit of the operator F at f_0 if, for every positive real number ε, there exists a positive real number δ such that for every function $f \in A \cap \{f \in \mathcal{C}[a, b] : f \succ f_0\}$ and $\mathbf{0} \prec |f - f_0| \prec \delta e$, the following holds:

$$|F(f) - F(f_0)| \prec \varepsilon e.$$

If L is the right limit of the operator F at f_0, it is written as

$$L = \lim_{f \to f_0^+} F(f).$$

(ii) *For an operator $F : A \to \mathcal{C}[a,b]$, the function L is called the left limit of the operator F at f_0 if, for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for every $f \in A \cap \{f \in \mathcal{C}[a,b] : f \prec f_0\}$ and $\mathbf{0} \prec |f - f_0| \prec \delta e$, the following holds:*

$$|F(f) - F(f_0)| \prec \varepsilon e.$$

If L is the left limit of the operator F at f_0, it is written as

$$L = \lim_{f \to f_0^-} F(f).$$

Example 4. *Consider $F : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ defined by:*

$$F(f) = \begin{cases} e, & \text{if } f \preceq \mathbf{0}, \\ 2e, & \text{if } f \succ \mathbf{0}. \end{cases}$$

We will show that

$$\lim_{f \to \mathbf{0}^+} F(f) = 2e \quad \text{and} \quad \lim_{f \to \mathbf{0}^-} F(f) = e.$$

(i) *Take any positive real number ε, and choose a positive real number $\delta = 5$ such that for every*

$$f \in \mathcal{C}[a,b] \cap \{f \in \mathcal{C}[a,b] : f \succ \mathbf{0}\}$$

and

$$\mathbf{0} \prec |f - \mathbf{0}| \prec \delta e,$$

We obtain:

$$|F(f) - 2e| = |2e - 2e| = |\mathbf{0}| = \mathbf{0} < \varepsilon e.$$

Thus:

$$\lim_{f \to \mathbf{0}^+} F(f) = 2e.$$

(ii) *Take any positive real number ε, and choose a positive real number $\delta = 5$ such that for every*

$$f \in \mathcal{C}[a,b] \cap \{f \in \mathcal{C}[a,b] : f \prec \mathbf{0}\}$$

and

$$\mathbf{0} \prec |f - \mathbf{0}| \prec \delta e,$$

We obtain:

$$|F(f) - e| = |e - e| = |\mathbf{0}| = \mathbf{0} < \varepsilon e.$$

Thus:

$$\lim_{f \to \mathbf{0}^-} F(f) = e.$$

Next, we discuss a theorem that establishes the relationship between the general definition of the limit of an operator and the definition of the one-sided limit of an operator (right-hand limit or left-hand limit), as described below.

Theorem 6. *Consider $F : A \subseteq \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ and let $f_0 \in A$ be a limit point of the set $A \cap \{f \in \mathcal{C}[a,b] : f \succ f_0\}$ and the set $A \cap \{f \in \mathcal{C}[a,b] : f \prec f_0\}$.*

We obtain $\lim_{f\to f_0} F(f) = L$ if and only if

$$\lim_{f\to f_0^+} F(f) = L = \lim_{f\to f_0^-} F(f).$$

Proof. Consider $F : A \subseteq \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ and let $f_0 \in A$ be a limit point of the set $A \cap \{f \in \mathcal{C}[a,b] : f \succ f_0\}$ and the set $A \cap \{f \in \mathcal{C}[a,b] : f \prec f_0\}$.

($\Rightarrow$) Suppose $\lim_{f\to f_0} F = L$, meaning that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for every $f \in A$ and $\mathbf{0} \prec |f - f_0| \prec \delta e$, we have:

$$|F(f) - L| \prec \varepsilon e.$$

Since $A \cap \{f \in \mathcal{C}[a,b] : f \succ f_0\} \subseteq A$ and $A \cap \{f \in \mathcal{C}[a,b] : f \prec f_0\} \subseteq A$, it follows that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for every $f \in A \cap \{f \in \mathcal{C}[a,b] : f \succ f_0\}$ and $\mathbf{0} \prec |f - f_0| \prec \delta e$, we obtain:

$$|F(f) - L| \prec \varepsilon e.$$

Similarly, for every $f \in A \cap \{f \in \mathcal{C}[a,b] : f \prec f_0\}$ and $\mathbf{0} \prec |f - f_0| \prec \delta e$, we have:

$$|F(f) - L| \prec \varepsilon e.$$

Thus, we conclude:

$$\lim_{f\to f_0^+} F(f) = L \quad \text{and} \quad \lim_{f\to f_0^-} F(f) = L.$$

($\Leftarrow$) Suppose $\lim_{f\to f_0^+} F(f) = L$, meaning that for every real number $\varepsilon > 0$, there exists a real number $\delta_1 > 0$ such that for every $f \in A \cap \{f \in \mathcal{C}[a,b] : f \succ f_0\}$ and $\mathbf{0} \prec |f - f_0| \prec \delta_1 e$, we obtain:

$$|F(f) - L| \prec \varepsilon e.$$

Similarly, since $\lim_{f\to f_0^-} F(f) = L$, for every real number $\varepsilon > 0$, there exists a real number $\delta_2 > 0$ such that for every $f \in A \cap \{f \in \mathcal{C}[a,b] : f \prec f_0\}$ and $\mathbf{0} \prec |f - f_0| \prec \delta_2 e$, we obtain:

$$|F(f) - L| \prec \varepsilon e.$$

Choosing $\delta = \min\{\delta_1, \delta_2\}$, we conclude that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for every $f \in A$ and $\mathbf{0} \prec |f - f_0| \prec \delta e$, the following holds:

$$|F(f) - L| < \varepsilon e.$$

Thus,

$$\lim_{f \to f_0} F(f) = L.$$

Next, we define the concept of operator continuity at $f_0 \in A \subseteq \mathcal{C}[a,b]$ or over $B \subseteq \mathcal{C}[a,b]$, which is frequently used in subsequent subsections.

Definition 11. *Given $A \subseteq \mathcal{C}[a,b]$ and an operator $R : A \to \mathcal{C}[a,b]$, the operator R is said to be continuous at f_0 if, for every real number $\varepsilon > 0$, there exists $\delta > 0$ such that for every $f \in A$ with $|f - f_0| \prec \delta e$, the following holds:*

$$|R(f) - R(f_0)| \prec \varepsilon e.$$

Furthermore, the operator R is said to be continuous on the set B if R is continuous at every $f \in B$.

3.2. Stieltjes Limit on $\mathcal{C}[a,b]$

Before defining the Stieltjes limit on $\mathcal{C}[a,b]$, it is important to first establish the definition of increasing, strictly increasing, decreasing, and strictly decreasing operators. These fundamental concepts serve as a crucial foundation for understanding the definition and properties of the Stieltjes limit on $\mathcal{C}[a,b]$. The following definitions describe increasing, strictly increasing, decreasing, and strictly decreasing operators:

Definition 12.

(i) An operator R is said to be increasing on A if for every $t, v \in A$ with $t \prec v$, then $R(t) \preceq R(v)$.

(ii) An operator R is said to be strictly increasing on A if for every $t, v \in A$ with $t \prec v$, then $R(t) \prec R(v)$.

(iii) An operator R is said to be decreasing on A if for every $t, v \in A$ with $t \prec v$, then $R(t) \succeq R(v)$.

(iv) An operator R is said to be strictly decreasing on A if for every $t, v \in A$ with $t \prec v$, then $R(t) \succ R(v)$.

Next, we define and discuss the Stieltjes limit Theorem for an operator R in the space of continuous functions $\mathcal{C}[a,b]$. The Stieltjes limit of an operator with continuous function values generalizes the concept of the limit of an operator with continuous function values.

Definition 13. *Given $A \subseteq \mathcal{C}[a,b]$ and a function f_0 as a limit point of the set A, an operator $F : A \to \mathcal{C}[a,b]$, and an operator $R : A \to \mathcal{C}[a,b]$ that is continuous and increasing, a function L is called the limit operator of F at f_0 with respect to R if, for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for every $f \in A$ satisfying $0 \prec |R(f) - R(f_0)| \prec \delta e$, the following holds:*

$$|F(f) - L| \prec \varepsilon e.$$

If L is the limit operator of F with respect to R at f_0, it is written as:

$$L = \lim_{R(f)\to R(f_0)} F(f).$$

Next, in this subsection, each operator F and R considered is an operator mapping from $A \subseteq \mathcal{C}[a,b]$ to $\mathcal{C}[a,b]$ that is defined on an open interval, with R being a continuous and increasing operator.

Example 5. *Given $R(f) = 2f$ and $F(f) = 2f + e$, we will show that:*

$$\lim_{R(f)\to R(e)} (2f + e) = 3e.$$

Taking any positive real number ε, we choose a positive real number $\delta = \varepsilon$ such that for every $f \in A$ satisfying $0 \prec |R(f) - R(e)| = |2f - 2e| \prec \delta e$, we obtain:

$$|(2f + e) - (3e)| \prec |2f - 2e| \prec \delta e = \varepsilon e.$$

Thus:

$$\lim_{f\to 2e} (2f + e) = 3e.$$

Theorem 7. *Given $A \subseteq \mathcal{C}[a,b]$, an operator F mapping from A to $\mathcal{C}[a,b]$, and f_0 as a limit point of the set A, if:*

$$\lim_{R(f)\to R(f_0)} F(f) = L \quad \textit{and} \quad \lim_{R(f)\to R(f_0)} F(f) = M,$$

then $L = M$.

Proof. Given $\lim_{R(f)\to R(f_0)} F(f) = L$, meaning that for every real number $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |R(f) - R(f_0)| \prec \delta_1 e$, then:

$$|F(f) - L| \prec \frac{\varepsilon}{2} e.$$

Similarly, given $\lim_{R(f)\to R(f_0)} F(f) = M$, meaning that for every $\varepsilon > 0$, there exists $\delta_2 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |R(f) - R(f_0)| \prec \delta_2 e$, then:

$$|F(f) - M| \prec \frac{\varepsilon}{2} e.$$

Choosing $\delta = \min(\delta_1, \delta_2)$, if $\mathbf{0} \prec |f - f_0| \prec \delta e$, then:

$$|L - M| = |L - F(f) + F(f) - M| \preceq |L - F(f)| + |F(f) - M|.$$

By substitution, we obtain:

$$|L - M| \prec \frac{\varepsilon}{2} e + \frac{\varepsilon}{2} e = \varepsilon e.$$

This holds for every $\varepsilon > 0$, by Theorem 4.1.11:

$$|L - M| = \mathbf{0} \quad \text{or} \quad L = M.$$

The significance of Theorem 4.5.4 is that if an operator F has a limit at f_0 with respect to R, then the limit is unique.

Theorem 8. *Given $A \subseteq \mathcal{C}[a,b]$, operators F, G, and R, each mapping from A to $\mathcal{C}[a,b]$, with R being an increasing operator, and f_0 as a limit point of the set A, along with a function $\gamma \in \mathcal{C}[a,b]$. If $\lim_{R(f)\to R(f_0)} F(f) = L$ and $\lim_{R(f)\to R(f_0)} G(f) = M$, then:*

(i) $\lim_{R(f)\to R(f_0)}(F+G)(f) = L+M$

(ii) $\lim_{R(f)\to R(f_0)}(F-G) = L-M$

(iii) $\lim_{R(f)\to R(f_0)}(FG)(f) = LM$

(iv) $\lim_{R(f)\to R(f_0)}(\gamma F(f)) = \gamma L$

(v) $\lim_{R(f)\to R(f_0)} \left(\frac{F}{G}\right)(f) = \frac{L}{M}$, *provided that $M(x) \neq 0$ for all $x \in [a,b]$.*

Proof.

(i) Given that $\lim_{R(f)\to R(f_0)} F = L$, meaning that for every real number $\varepsilon > 0$, there exists a real number $\delta_1 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |R(f) - R(f_0)| \prec \delta_1 e$, then:

$$|F(f) - L| \prec \frac{\varepsilon}{2}e.$$

Similarly, given that $\lim_{R(f)\to R(f_0)} G = M$, meaning that for every real number $\varepsilon > 0$, there exists a real number $\delta_2 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |R(f) - R(f_0)| \prec \delta_2 e$, then:

$$|G(f) - M| \prec \frac{\varepsilon}{2}e.$$

Choosing $\delta = \min(\delta_1, \delta_2)$, we obtain that for every $f \in A$ and $\mathbf{0} \prec |R(f) - R(f_0)| \prec \delta e$, the following holds:

$$|[F(f) + G(f)] - [L + M]| \preceq |F(f) - L| + |G(f) - M| \prec \frac{\varepsilon}{2}e + \frac{\varepsilon}{2}e = \varepsilon e.$$

Thus,

$$\lim_{R(f)\to R(f_0)} (F+G)(f) = L + M.$$

(ii) Given that $\lim_{R(f)\to R(f_0)} F(f) = L$, meaning that for every real number $\varepsilon > 0$, there exists a real number $\delta_1 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |R(f) - R(f_0)| \prec \delta_1 e$, then:

$$|F(f) - L| \prec \frac{\varepsilon}{2}e.$$

Similarly, given that $\lim_{R(f)\to R(f_0)} G(f) = M$, meaning that for every real number $\varepsilon > 0$, there exists a real number $\delta_2 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |R(f) - R(f_0)| \prec \delta_2 e$, then:

$$|G(f) - M| \prec \frac{\varepsilon}{2}e.$$

Choosing $\delta = \min(\delta_1, \delta_2)$, we obtain that for every $f \in A$ and $\mathbf{0} \prec |R(f) - R(f_0)| \prec \delta e$, the following holds:

$$|[F(f) - G(f)] - [L - M]| \preceq |F(f) - L| + |G(f) - M| \prec \frac{\varepsilon}{2}e + \frac{\varepsilon}{2}e = \varepsilon e.$$

Thus,

$$\lim_{R(f)\to R(f_0)} (F - G)(f) = L - M.$$

(iii) Given that $\lim_{R(f)\to R(f_0)} F(f) = L$, meaning that for every real number $\varepsilon > 0$, there exists a real number $\delta_1 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |R(f) - R(f_0)| \prec \delta_1 e$, then:

$$|F(f) - L| \prec \varepsilon \left(\frac{e}{2|M| + e} \right).$$

Similarly, given that $\lim_{R(f)\to R(f_0)} G(f) = M$, meaning that for every real number $\varepsilon > 0$, there exists a real number $\delta_2 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |R(f) - R(f_0)| \prec \delta_2 e$, then:

$$|G(f) - M| \prec \varepsilon \left(\frac{e}{2(\sup |F(f)| + e)} \right),$$

where $\sup |F(f)|$ is the supremum of $|F(f)|$ in the neighborhood of f_0 where $f \neq f_0$. Choosing $\delta = \min(\delta_1, \delta_2)$, we obtain that for every $\mathbf{0} \prec |R(f) - R(f_0)| \prec \delta e$, the following holds:

$$\begin{aligned} |F(f)G(f) - LM| &= |F(f)G(f) - F(f)M + F(f)M - LM| \\ &= |F(f)[G(f) - M] + M[F(f) - L]| \\ &\preceq |F(f)|\ |G(f) - M| + |M|\ |F(f) - L|. \end{aligned}$$

Thus,

$$\lim_{R(f)\to R(f_0)} (FG)(f) = LM.$$

(iv) Given that $\lim_{R(f)\to R(f_0)} F(f) = L$, meaning that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that if $f \in A$ and $\mathbf{0} \prec |R(f) - R(f_0)| \prec \delta e$, then:

$$|\gamma F(f) - \gamma L| = |\gamma|\ |F(f) - L| \prec |\gamma| \frac{\varepsilon e}{|\gamma|} = \varepsilon e.$$

Thus,

$$\lim_{R(f)\to R(f_0)} (\gamma F(f)) = \gamma L.$$

(v) Given that $\lim_{R(f)\to R(f_0)} F(f) = L$, meaning that for every real number $\varepsilon > 0$, there exists a real number $\delta_1 > 0$ such that if $f \in A$ and $\mathbf{0} \prec |R(f) - R(f_0)| \prec \delta_1 e$, then:

$$|F(f) - L| \prec \frac{|M|}{4} e.$$

Similarly, given $\lim_{R(f)\to R(f_0)} G = M$, the same holds, ensuring:

$$\lim_{R(f)\to R(f_0)} \left(\frac{F}{G}\right)(f) = \frac{L}{M}.$$

4. Conclusion

The findings of this study indicate that the characteristics of the limit and the Stieltjes limit of function-valued continuous functions are comparable to those of the limit and the Stieltjes limit of real-valued functions.

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References

[1] Umi Mahnuna Hanung and Ch Rini Indrati. INTEGRAL HENSTOCK-STIELTJES FUNGSI BERNILAI VEKTOR. *PYTHAGORAS Jurnal Pendidikan Matematika*, 5(2), 2009.

[2] Neva Satyahadewi Widyawati, Evy Sulistianingsih, et al. Penggunaan Model Black Scholes untuk Penentuan Harga Opsi Jual Tipe Eropa. *Bimaster: Buletin Ilmiah Matematika, Statistika dan Terapannya*, 2(1), 2013.

[3] Toto Supriyono. Mekanika Fluida Lanjut, 2021.

[4] S Pd Usman, M Patima, S Pd Puspapratiwi, et al. Buku Referensi Matematika Terapan, 2024.

[5] Miftahul Fikri, Samsurizal Samsurizal, and Andi Makkulau. Perbandingan Penyelesaian Integral Riemann, Lebesgue dan HK Berdasarkan Definisi. *Limits: Journal of Mathematics and Its Applications*, 18(2):169–186, 2021.

[6] Firdaus Ubaidillah, Soeparna Darmawijaya, and R Indrati. Integral Henstock-Kurzweil Fungsi Bernilai C [a, b]: Teorema Kekonvergenan Seragam. *Prosiding KNM XVII*, pages 1–7, 2014.

[7] Riva Yasin Nurandini, Encum Sumiaty, and Cece Kustiawan. Integral Perron dan Ekuivalensinya dengan Integral Denjoy. *Jurnal EurekaMatika*, 6(2):12–24, 2018.

[8] Develin O Omayan and Greig Bates C Flores. Some Fundamental Properties of Variational Kurzweil-Henstock-Stieltjes Integral on a Compact Interval in $\mathrm{R}^n. Asian Research Journal of Mathematics, 18(9) : 69 - -81, 2022.$

[9] Septian Pirade, Tohap Manurung, and Jullia Titaley. Integral Riemann-Stieltjes Pada Fungsi Bernilai Real. *d'CARTESIAN: Jurnal Matematika dan Aplikasi*, 6(1):1–7, 2017.

[10] Kalfin Muchtar, Jullia Titaley, and Mans Mananohas. Integral Baire-1 Stieltjes, Henstock-Stieltjes dan Riemann-Stieltjes. *d'Cartesian*, 5(1):7–12, 2016.

[11] Gregory Convertito and David Cruz-Uribe. *The Stieltjes Integral*. Chapman and Hall/CRC, 2023.

[12] Joong Kwoen Lee and Han Ju Lee. Riemann-Stieltjes Integrals and their Representing Measures. *The Pure and Applied Mathematics*, 31(4):453–476, 2024.

[13] Andrew Felix IV Suarez Cunanan and Julius Benitez. Simple Properties and Existence Theorem for the Henstock-Kurzweil-Stieltjes Integral of Functions Taking Values on C [a, b] Space-valued Functions. *European Journal of Pure and Applied Mathematics*, 13(1):130–143, 2020.

[14] Firdaus Ubaidillah, Soeparna Darmawijaya, and Ch I Rini. On the Henstock-Kurzweil Integral of C [a; b] Space-valued Functions. *Int. J. Math. An*, 9(37):1831–1846, 2015.

[15] Firdaus Ubaidillah, Soeparna Darmawijaya, and Rini Indrati. Kekonvergenan Barisan di Dalam Ruang Fungsi Kontinu C [a, b]. *CAUCHY: Jurnal Matematika Murni dan Aplikasi*, 2(4):184–188, 2013.