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# Steady-State Performance Evaluation of an $M^{[\mathbb{X}]}/G(a,b)/1$ Queue with Low-Batch Service, Multiple Vacations and Uninterruptible Server Renovation

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**Abstract.** This study examines the behaviour of uninterruptible server breakdowns in a bulk service queueing system with multiple vacations and low-batch service (LBS). The system consists of a single server that operates under two service modes: the General Bulk Service Rule (GBSR) and low-batch service (LBS). When each time bulk service is completed, the server may break down with probability $\varphi$, in this case it is immediately sent for repair. Upon repair completion, or if no breakdown occurs (with probability $1-\varphi$), or at the moment of LBS completion, the server checks the queue length. If no customers are waiting, the server initiates multiple random-length vacations. Upon returning from a vacation, if fewer than '$a$' customers are present, the server takes another vacation, repeating this process until at least '$a$' customers are in the queue. Once this threshold is met, the server starts bulk service. After each service completion, the server start bulk service or low-batch service based on the queue length. This paper primarily analyses how LBS implementation influences key performance metrics and the breakdown probability ratio. Numerical results demonstrate the model's effectiveness, and a cost analysis reveals that adopting LBS significantly reduces overall operational expenses.

**2020 Mathematics Subject Classifications**: 60K25, 60K30, 90B22

**Key Words and Phrases**: Bulk queue, Multiple vacation, Renovation, Low-batch service

## 1. Introduction

Bulk queueing systems are common and necessary for the effective and efficient use of resources in many situations in the real world. In server vacation models, the server uses the unutilized periods for various activities. Applications of the server vacation model include maintaining files, process control, managing memory, shared resource maintenance,

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networking systems, chemical manufacturing industries and production systems. The term "server renovation works" refers to the process of determining the state of the service station's breakdown, repairing it and maintaining other resources. In real life, maintenance of the service station may be necessary in many circumstances because of breakdowns. The system is impacted by these breakdowns, particularly with regard to the length of the queue and waiting times for customers. In manufacturing, low-batch service is crucial for maintaining and improving the efficiency of bulk services. It offers the flexibility needed to adapt to changing demands, particularly when production volumes are insufficient for bulk processing. Low-batch service eliminates delays and maintains smooth workflows by creating smaller batches that are customized to meet particular demands. This support enhances manufacturers competitiveness by improving inventory management, conserving storage space and enabling quick responses to changes in demand. Combining bulk and low-batch services maintains client happiness and ensures efficiency.

An example of a bulk service queueing system in the manufacturing industry is a beverage bottling factory. At this plant, bottles are filled using a bulk service system, which has a minimum fill amount of 5,000 bottles and a maximum fill capacity of 20,000 bottles each shift. When the service capacity drops below the minimum, it enters a low-batch service mode. The factory fills and caps bottles on a production line with efficiency when demand is high since it operates at maximum capacity. The system automatically transitions to a low-batch service if the bottle supply drops below the minimum capacity, allowing the factory to run continuously. Suppose that a filling machine fails because of a defective sense during a busy manufacturing run, the operators quickly implement alternative procedures, such as transferring to a backup filling line or employing manual filling processes, in order to guarantee that the current batch of beverages is finished rather than stopping production totally. By using this method, the breakdown's effects are reduced while the plant can continue to operate. When the bulk or low-batch lines are empty of bottles, the plant may enter the vacation mode. Maintenance teams perform necessary tasks, such as cleaning machinery, preventative maintenance and updating equipment during this period. This proactive measure assumed that the machinery is in optimal working condition when manufacturing resumes. Once the filling machine is repaired and the defective sensor replaced, the bottling facility reopens for the next production session. With strict supervision, the operators make sure that everything runs well so that the facility can resume producing 20,000 bottles per shift, which is its maximum capacity. This example shows how manufacturers can adjust to equipment breakdowns and ensure efficient resource management while sustaining continuous operations to successfully manage production processes.

Another real-time example is a cloud computing bulk service queueing system. A cloud-based data center provides services for bulk data processing, operating with a bulk service system that has both a maximum and a minimum processing capacity for handling requests. When the incoming request rate drops below the minimum threshold, the system switches to a low-batch service mode to ensure continued operation. During periods of high demand, the data center can effectively handle requests related to tasks like data storage, retrieval and analysis to maintaining seamless service. The automatic switch to low-batch mode guarantees uninterrupted operations when request volumes are low. Suppose that during a high-traffic period, a critical network switch experiences a breakdown, causing some requests to be delayed. Instead of halting all operations, the data center employs alternative routing methods, such as redirecting traffic through backup switches or utilizing load balancers to distribute requests across available servers. This approach minimizes the impact of the breakdown while allowing the data center to continue processing as many requests as possible.

If there are no requests waiting in the bulk or low-batch service queues, the data center may enter a vacation mode. During this time, IT staff performs essential maintenance tasks, such as updating software, conducting security checks, or upgrading hardware. This proactive maintenance ensures that when traffic resumes, the data center operates efficiently and securely. Once the network switch has been repaired and rendered fully operational, the data center resumes normal processing levels in the ensuing session. The data center resumes processing the maximum number of requests every second, while the network engineers keep close watch on everything to make sure everything runs well. This example demonstrates how networking operations manage service delivery, respond to equipment failures, and ensure continuous operations while optimizing resource utilization and maintenance schedules. A queueing system is represented by the notation $M^{[\mathbb{X}]}/G(a,b)/1$ after those two examples featuring bulk service, low-batch service, multiple vacations, and uninterruptible server renovation.

## 2. Literature survey

Several authors have examined queueing systems (bulk services) using various types of combinations. Many authors in the queueing literature have investigated queues with multiple vacations. However, only a few have addressed the required renovations or repairs caused by service station breakdowns. Doshi [1] has given the best studies of queueing systems and its applications with server vacations. Bulk queues with Poisson input were first introduced by Neuts [2]. Lee et al. [3] conducted a detailed analysis of the $M^{[\mathbb{X}]}/G/1$ queueing model with N-policy and multiple vacations. Their findings offer valuable insights pertinent to this article. Cox [4] provided a foundational study on non-Markovian stochastic processes by introducing supplementary variables, which has influenced many subsequent queueing models. Moreover, Lee [5] examined steady-state probabilities for a server vacation model with group arrivals and a control operation policy, offering important results that complement the existing body of work on vacation queues.

Krishna Reddy et al. [6] performed an analysis of a batch arrival queueing model with N-policy, multiple vacations, along with set-up periods. This analysis contributes to a better understanding of queue management strategies. Arumuganathan and Jeyakumar [7] performed a steady-state assessment of a batch queueing system including N-policy, extended vacations, preparation times and shutdown phases. Jeyakumar and Senthilnathan [8] analyzed the behavior of the $M^{[\mathbb{X}]}/G(a,b)/1$ queueing model, focusing on uninterrupted server failures, along with extended vacations and downtime.

Ramaswami and Jeyakumar [9] employed a simulation-based approach to analyze a non-Markovian bulk service queueing system where the arrival process is state-dependent, and the server may take multiple vacations. Haridass and Arumuganathan [10] conducted an examination of a single-server batch arrival retrial queueing model featuring adjusted vacation periods and N-policy. Ayyappan and Karpagam [11] examined a bulk service queueing system that encompasses a standby server, multiple vacations, and a policy for managing re-service requests and server breakdown and repair. Ayyappan and Nirmala [12] modeled the $M^{[X]}/G(a,b)/1$ queueing model, incorporating breakdowns along with a two-stage repair mechanism involving delay periods, and multiple vacations. Their work enhances the understanding of queue dynamics under repair and vacation conditions.

Niranjan et al. [13] studied a batch arrival queueing model with vacation break-off that had a phase-dependent breakdown. They created a two-phase service method, known as the first and second vital services for the number of breakdowns varies between the first and second critical service periods. In a related study, Karthick and Suvitha [14] analyzed a heterogeneous two-server queueing system featuring multiple working vacations and server breakdowns, offering valuable performance measures. Furthermore, Karthick and Suvitha [15] investigated the time-dependent behavior of an $M/M/3$ heterogeneous server queueing system subject to system disasters and multiple vacation policies, extending and understanding of transient characteristics in complex service environments. Chakravarthy and Kulshrestha [16] examined a queueing model including server breakdowns, repair processes, vacation periods and the use of a backup server. An analytical simulation of the batch queueing model was conducted by Nithya and Haridass [17]. This study includes an investigation of performance metrics and simulation modeling for a bulk service queueing model applied in the textile sector. An analysis of both transient and steady-state behaviors of the $M/M^{([b])}/1$ queueing model with an additional discretionary service was carried out by Laxmi and George [18]. Their research provides insight on how the system behaves in various scenarios.

Jain et al. [19] explored the application of the auxiliary variable approach to analyze a non-memoryless single-server queueing system experiencing service disruptions (QSI). Begum and Choudhury [20] analyzed a batch arrival N-policy queueing system featuring dual service mechanisms, including breakdowns, delayed repairs and Bernoulli vacations with a repeated service policy. Dudin et al. [21] investigated single customer abandonment and server breakdowns in queueing systems, focusing on how customers make probabilistic decisions to stay or leave during a breakdown. The threshold-based repair system with threshold recovery strategy, intermittent servers and staged repairs was examined by Kumar et al. [22].The optimal management of a two-phase heterogeneous service retrial queueing model characterized by collisions and postponed vacations was studied by Xu et al. [23]. He and Tang et al. [24] examined the optimization and performance of a queueing model featuring N-policy and delayed uninterrupted multiple vacations.

Several researchers have studied bulk service systems with multiple vacations. Notably, S. Jeyakumar and B. Senthilnathan introduced the concept of server breakdown without interruption in their work titled "A Study on the Behaviour of the Server Breakdown Without Interruption in an $M^{[\mathbb{X}]}/G(a,b)/1$ Queueing System with Multiple Vacations and Closedown Time." In their model, the system enters vacation mode when the number of customers falls below a certain threshold. In contrast, our work introduces the concept of low-batch service, wherein the system remains active and continues to serve customers even when the number falls below the threshold. This flexible service approach is distinct from the traditional vacation model. Importantly, the integration of both bulk service and low-batch service has not yet been addressed in the existing literature. This section provides an overview of the current literature, highlighting areas that require further investigation due to existing gaps and inconsistencies.

The structure of this present work is as follows: The first two sections provide the "Introduction and the Literature survey". Section 3 explained the "Model description" outlines necessary assumptions for formulating the model. Section 4 develops the "Queue size distribution" and Section 5 described the "Probability generating function". In Section 6 "Performance indices", the performance metrics are obtained. In section 7, "Cost model" has been developed. To verify the findings of the analysis, "Numerical representation" can be found in Section 8. Finally, Section 9 is "Conclusion and future work".

## 3. Model description

The study examines the behavior of uninterruptible server breakdowns in a bulk service queueing system incorporating multiple vacations and low-batch service (LBS) mechanism. Customer arrivals follow a compound Poisson process under which customers arrive in batches. Service times are generally distributed for both bulk and low-batch services. The system consists of a single server that operates under two service modes: the General Bulk Service Rule (GBSR) and the Low-Batch Service Rule (LBSR). The service mode applied depends on the number of customers in the queue at a service completion epoch. The service selection rules under LBSR are as follows:

(i) $1 \leq Q \leq a-1$; all the customers are taken for service

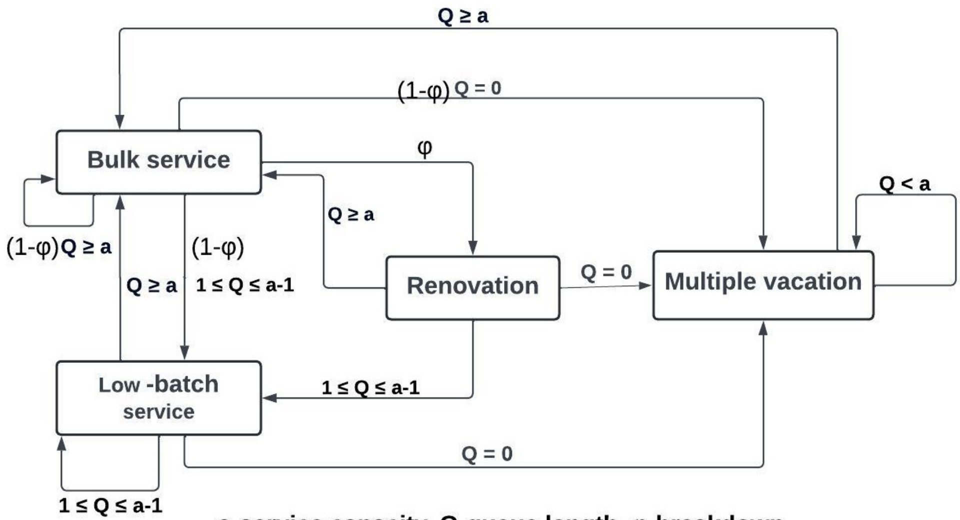
(ii) $Q \geq a$ ; the server switches to bulk service

(iii) $Q$=0; then the server is in idle

Here, $Q$ denotes the number of customers in the queue, and '$a$' is a predefined threshold that distinguishes between low-batch and bulk service initiation. Each time bulk service is completed, the server may break down with probability $\varphi$, in this case it is immediately sent for repair. Upon repair completion, or if no breakdown occurs (with probability $1-\varphi$), or at the moment of LBS completion, the server checks the queue length. If no customers are waiting, the server initiates multiple random-length vacations. Upon returning from a vacation, if fewer than '$a$' customers are present, the server takes another vacation, repeating this process until at least '$a$' customers are in the queue. Once this threshold is met, the server starts bulk service. After each service completion, the server start bulk service or low-batch service based on the queue length.

### 3.1. Notations

The following notations are used:
$\mathbb{X}$ - A random variable representing the group size.




**a-service capacity, Q-queue length, φ-breakdown**

$g_\kappa$- $\Pr\{\mathbb{X} = \kappa\}$.
$\lambda$ - Arrival rate.
$\mathbb{X}(z)$ - The PGF of the random variable $\mathbb{X}$.
$S(.)$ - Function of cumulative distribution for bulk service times.
$R(.)$ - Function of cumulative distribution for renovation times.
$C(.)$ - Function of cumulative distribution for low-batch service times.
$V(.)$ - Function of cumulative distribution for vacation times.
$s(\hbar)$ - The density function of probability for S.
$r(\hbar)$ - The density function of probability for R.
$c(\hbar)$ - The density function of probability for C.
$v(\hbar)$ - The density function of probability for V.
$\tilde{S}(\vartheta)$ - Laplace Transformation of Stieljies for S.
$\tilde{R}(\vartheta)$ - Laplace Transformation of Stieljies for R.
$\tilde{C}(\vartheta)$ - Laplace Transformation of Stieljies for C.
$\tilde{V}(\vartheta)$ - Laplace Transformation of Stieljies for V.
$S^0(t)$ - The remaining time for bulk services.
$R^0(t)$ - The remaining time for renovation.
$C^0(t)$ - The remaining time for low-batch servicing.
$V^0(t)$ - The remaining time for vacation.
$\mathcal{N}_s(t)$ - The number of customers in service at that time $t$.
$\mathcal{N}_q(t)$ - The number of customers in queue at that time $t$.

Define,

$$\varrho(t) = \begin{cases} 1, \text{ denotes that bulk service is occupying the server} \\ 0, \text{ denotes that the server is on vacation} \\ 3, \text{ denotes a server renovation} \\ 2, \text{ denotes that low-batch service is occupying the server} \end{cases}$$

$\wp(t) = \mathtt{f}$, if the server is on $\mathtt{f}^{th}$ vacation.

State probabilities are defined as follows:

$B_{\iota,\mathtt{f}}(\hbar,t)dt = Pr\{\mathcal{N}_s(t) = \iota, \mathcal{N}_q(t) = \mathtt{f}, \hbar \le S^0(t) \le \hbar + dt, \varrho(t) = 1\}, a \le \iota \le b, \mathtt{f} \ge 0,$
$L_{\iota,\mathtt{f}}(\hbar,t)dt = Pr\{\mathcal{N}_s(t) = \iota, \mathcal{N}_q(t) = \mathtt{f}, \hbar \le C^0(t) \le \hbar + dt, \varrho(t) = 2\}, 1 \le \iota \le a-1, \mathtt{f} \ge 0,$
$Q_{\mathtt{f},n}(\hbar,t)dt = Pr\{\mathcal{N}_q(t) = n, \hbar \le V^0(t) \le \hbar + dt, \wp(t) = \mathtt{f}, \varrho(t) = 0\}, \mathtt{f} \ge 1, n \ge 0,$
$R_n(\hbar,t)dt = Pr\{\mathcal{N}_q(t) = n, \hbar \le R^0(t) \le \hbar + dt, \varrho(t) = 3\}, n \ge 0.$

## 4. Queue size distribution

Queueing models have been effectively solved by using the technique supplementary variable, first introduced by Cox in 1955 and then improved by Lee in 1991. Applying this method to the proposed model at various epochs, the following steady-state equations are as follows:

$$-B'_{\iota,0}(\hbar) = -\lambda B_{\iota,0}(\hbar) + (1-\varphi)\sum_{m=a}^{b} B_{m,\iota}(0)s(\hbar) + \sum_{m=1}^{a-1} L_{m,\iota}(0)s(\hbar) + \sum_{l=1}^{\infty} Q_{l,\iota}(0)s(\hbar) + R_\iota(0)s(\hbar), \quad a \le \iota \le b, \tag{1}$$

$$-B'_{\iota,\mathtt{f}}(\hbar) = -\lambda B_{\iota,\mathtt{f}}(\hbar) + \sum_{\kappa=1}^{\mathtt{f}} B_{\iota,\mathtt{f}-\kappa}(\hbar)\lambda g_\kappa, \quad a \le \iota \le b-1, \mathtt{f} \ge 1, \tag{2}$$

$$-B'_{b,\mathtt{f}}(\hbar) = -\lambda B_{b,\mathtt{f}}(\hbar) + (1-\varphi)\sum_{m=a}^{b} B_{m,b+\mathtt{f}}(0)s(\hbar) + \sum_{\kappa=1}^{\mathtt{f}} B_{b,\mathtt{f}-\kappa}(\hbar)\lambda g_\kappa + \sum_{m=1}^{a-1} L_{m,b+\mathtt{f}}(0)s(\hbar) + \sum_{l=1}^{\infty} Q_{l,b+\mathtt{f}}(0)s(\hbar) + R_{b+\mathtt{f}}(0)s(\hbar), \quad \mathtt{f} \ge 1, \tag{3}$$

$$-L'_{\iota,0}(\hbar) = -\lambda L_{\iota,0}(\hbar) + \sum_{m=1}^{a-1} L_{m,i}(0)c(\hbar) + (1-\varphi)\sum_{m=a}^{b} B_{m,\iota}(0)c(\hbar) + R_\iota(0)c(\hbar), \quad 1 \le \iota \le a-1, \tag{4}$$

$$-L'_{\iota,\mathtt{f}}(\hbar) = -\lambda L_{\iota,\mathtt{f}}(\hbar) + \sum_{\kappa=1}^{\mathtt{f}} L_{\iota,\mathtt{f}-\kappa}(\hbar)\lambda g_\kappa, \quad 1 \le \iota \le a-1, \mathtt{f} \ge 1, \tag{5}$$

$$-R_0^{'}(\hbar)=-\lambda R_0(\hbar)+\varphi\sum_{m=a}^{b}B_{m,0}(0)r(\hbar), \tag{6}$$

$$-R_n^{'}(\hbar)=-\lambda R_n(\hbar)+\sum_{\kappa=1}^{n}R_{n-\kappa}(\hbar)\lambda g_\kappa+\varphi\sum_{m=a}^{b}B_{m,n}(0)r(\hbar),\quad n\geq 1, \tag{7}$$

$$-Q_{1,0}^{'}(\hbar)=-\lambda Q_{1,0}(\hbar)+(1-\varphi)\sum_{m=a}^{b}B_{m,0}(0)v(\hbar)+\sum_{m=1}^{a-1}L_{m,0}(0)v(\hbar)+R_0(0)v(\hbar), \tag{8}$$

$$-Q_{1,n}^{'}(\hbar)=-\lambda Q_{1,n}(\hbar)+\sum_{\kappa=1}^{n}Q_{1,n-\kappa}(\hbar)\lambda g_\kappa,\quad n\geq 1, \tag{9}$$

$$-Q_{\mathtt{f},0}^{'}(\hbar)=-\lambda Q_{\mathtt{f},0}(\hbar)+Q_{\mathtt{f}-1,0}(0)v(\hbar),\quad \mathtt{f}\geq 2, \tag{10}$$

$$-Q_{\mathtt{f},n}^{'}(\hbar)=-\lambda Q_{\mathtt{f},n}(\hbar)+Q_{\mathtt{f}-1,n}(0)v(\hbar)+\sum_{\kappa=1}^{n}Q_{\mathtt{f},n-\kappa}(\hbar)\lambda g_\kappa,$$
$$\mathtt{f}\geq 2,\quad 1\leq n\leq a-1, \tag{11}$$

$$-Q_{\mathtt{f},n}^{'}(\hbar)=-\lambda Q_{\mathtt{f},n}(\hbar)+\sum_{\kappa=1}^{n}Q_{\mathtt{f},n-\kappa}(\hbar)\lambda g_\kappa,\quad \mathtt{f}\geq 2,\quad n\geq a. \tag{12}$$

Equations (1)–(12) are treated on both sides by the Laplace-Stieltjes transform, and we obtain

$$\vartheta\tilde{B}_{\iota,0}(\vartheta)-B_{\iota,0}(0)=\lambda\tilde{B}_{\iota,0}(\vartheta)-(1-\varphi)\sum_{m=a}^{b}B_{m,\iota}(0)\tilde{S}(\vartheta)-\sum_{m=1}^{a-1}L_{m,\iota}(0)\tilde{S}(\vartheta)$$
$$-\sum_{l=1}^{\infty}Q_{l,\iota}(0)\tilde{S}(\vartheta)-R_\iota(0)\tilde{S}(\vartheta),\quad a\leq\iota\leq b, \tag{13}$$

$$\vartheta\tilde{B}_{\iota,\mathtt{f}}(\vartheta)-B_{\iota,\mathtt{f}}(0)=\lambda\tilde{B}_{\iota,\mathtt{f}}(\vartheta)-\sum_{\kappa=1}^{\mathtt{f}}\tilde{B}_{\iota,\mathtt{f}-\kappa}(\vartheta)\lambda g_\kappa,\ a\leq\iota\leq b-1,\ \ \mathtt{f}\geq 1, \tag{14}$$

$$\vartheta\tilde{B}_{b,\mathtt{f}}(\vartheta)-B_{b,\mathtt{f}}(0)=\lambda\tilde{B}_{b,\mathtt{f}}(\vartheta)-(1-\varphi)\sum_{m=a}^{b}B_{m,b+\mathtt{f}}(0)\tilde{S}(\vartheta)-\sum_{\kappa=1}^{\mathtt{f}}\tilde{B}_{b,\mathtt{f}-\kappa}(\vartheta)\lambda g_\kappa$$
$$-\sum_{m=1}^{a-1}L_{m,b+\mathtt{f}}(0)\tilde{S}(\vartheta)-\sum_{l=1}^{\infty}Q_{l,b+\mathtt{f}}(0)\tilde{S}(\vartheta)-R_{b+\mathtt{f}}(0)\tilde{S}(\vartheta),\quad \mathtt{f}\geq 1, \tag{15}$$

$$\vartheta\tilde{L}_{\iota,0}(\vartheta)-L_{\iota,0}(0)=\lambda\tilde{L}_{\iota,0}(\vartheta)-\sum_{m=1}^{a-1}L_{m,\iota}(0)\tilde{C}(\vartheta)-(1-\varphi)\sum_{m=a}^{b}B_{m,\iota}(0)\tilde{C}(\vartheta)$$
$$-R_\iota(0)\tilde{C}(\vartheta),\quad 1\leq\iota\leq a-1, \tag{16}$$

$$\vartheta\tilde{L}_{\iota,\mathtt{f}}(\vartheta)-L_{\iota,\mathtt{f}}(0)=\lambda\tilde{L}_{\iota,\mathtt{f}}(\vartheta)-\sum_{\kappa=1}^{\mathtt{f}}\tilde{L}_{\iota,\mathtt{f}-\kappa}(\vartheta)\lambda g_\kappa,\quad 1\leq\iota\leq a-1,\mathtt{f}\geq 1, \tag{17}$$

$$\vartheta\tilde{R}_0(\vartheta) - R_0(0) = \lambda\tilde{R}_0(\vartheta) - \varphi\sum_{m=a}^{b} B_{m,0}(0)\tilde{R}(\vartheta), \tag{18}$$

$$\vartheta\tilde{R}_n(\vartheta) - R_n(0) = \lambda\tilde{R}_n(\vartheta) - \sum_{\kappa=1}^{n}\tilde{R}_{n-\kappa}(\vartheta)\lambda g_\kappa - \varphi\sum_{m=a}^{b} B_{m,n}(0)\tilde{R}(\vartheta), \quad n \geq 1, \tag{19}$$

$$\begin{aligned}\vartheta\tilde{Q}_{1,0}(\vartheta) - Q_{1,0}(0) = \lambda\tilde{Q}_{1,0}(\vartheta) - (1-\varphi)\sum_{m=a}^{b} B_{m,0}(0)\tilde{V}(\vartheta) - \sum_{m=1}^{a-1} L_{m,0}(0)\tilde{V}(\vartheta)\\ - R_0(0)\tilde{V}(\vartheta),\end{aligned} \tag{20}$$

$$\vartheta\tilde{Q}_{1,n}(\vartheta) - Q_{1,n}(0) = \lambda\tilde{Q}_{1,n}(\vartheta) - \sum_{\kappa=1}^{n}\tilde{Q}_{1,n-\kappa}(\vartheta)\lambda g_\kappa, \quad n \geq 1, \tag{21}$$

$$\vartheta\tilde{Q}_{\mathtt{f},0}(\vartheta) - Q_{\mathtt{f},0}(0) = \lambda\tilde{Q}_{\mathtt{f},0}(\vartheta) - Q_{\mathtt{f}-1,0}(0)\tilde{V}(\vartheta), \quad \mathtt{f} \geq 2, \tag{22}$$

$$\begin{aligned}\vartheta\tilde{Q}_{\mathtt{f},n}(\vartheta) - Q_{\mathtt{f},n}(0) = \lambda\tilde{Q}_{\mathtt{f},n}(\vartheta) - Q_{\mathtt{f}-1,n}(0)\tilde{V}(\vartheta) - \sum_{\kappa=1}^{n}\tilde{Q}_{\mathtt{f},n-\kappa}(\vartheta)\lambda g_\kappa,\\ \mathtt{f} \geq 2, \quad 1 \leq n \leq a-1,\end{aligned} \tag{23}$$

$$\vartheta\tilde{Q}_{\mathtt{f},n}(\vartheta) - Q_{\mathtt{f},n}(0) = \lambda\tilde{Q}_{\mathtt{f},n}(\vartheta) - \sum_{\kappa=1}^{n}\tilde{Q}_{\mathtt{f},n-\kappa}(\vartheta)\lambda g_\kappa, \quad \mathtt{f} \geq 2, \quad n \geq a. \tag{24}$$

## 5. PGF of the queue size

The following probability-generating functions (PGFs) are established to derive the PGF for the queue size:

$$\begin{aligned}
&\tilde{B}_\iota(z,\vartheta) = \sum_{\mathtt{f}=0}^{\infty}\tilde{B}_{\iota,\mathtt{f}}(\vartheta)z^{\mathtt{f}}, \quad B_\iota(z,0) = \sum_{\mathtt{f}=0}^{\infty}B_{\iota,\mathtt{f}}(0)z^{\mathtt{f}}, \quad a \leq \iota \leq b,\\
&\tilde{L}_\iota(z,\vartheta) = \sum_{\mathtt{f}=0}^{\infty}\tilde{L}_{\iota,\mathtt{f}}(\vartheta)z^{\mathtt{f}}, \quad L_\iota(z,0) = \sum_{\mathtt{f}=0}^{\infty}L_{\iota,\mathtt{f}}(0)z^{\mathtt{f}}, \quad 1 \leq \iota \leq a-1,\\
&\tilde{Q}_{\mathtt{f}}(z,\vartheta) = \sum_{n=0}^{\infty}\tilde{Q}_{\mathtt{f},n}(\vartheta)z^{n}, \quad Q_{\mathtt{f}}(z,0) = \sum_{n=0}^{\infty}Q_{\mathtt{f},n}(0)z^{n}, \quad \mathtt{f} \geq 1,\\
&\tilde{R}(z,\vartheta) = \sum_{n=0}^{\infty}\tilde{R}_{n}(\vartheta)z^{n}, \quad R(z,0) = \sum_{n=0}^{\infty}R_{n}(0)z^{n}.
\end{aligned} \tag{25}$$

Eqs (25) is used to multiply equations (13) to (24) by $z^n$ and take $\lambda - \lambda\mathbb{X}(z) = \nu(z)$ :

$$\begin{aligned}(\vartheta - \nu(z))\tilde{Q}_1(z,\vartheta) = Q_1(z,0) - \tilde{V}(\vartheta)\Big[(1-\varphi)\sum_{m=a}^{b} B_{m,0}(0)\\ + \sum_{m=1}^{a-1} L_{m,0}(0) + R_0(0)\Big],\end{aligned} \tag{26}$$

$$(\vartheta-\nu(z))\tilde{Q}_{\mathtt{f}}(z,\vartheta)=Q_{\mathtt{f}}(z,0)-\tilde{V}(\vartheta)\sum_{n=0}^{a-1}Q_{\mathtt{f}-1,n}(0)z^{n},\qquad \mathtt{f}\geq 2, \tag{27}$$

$$(\vartheta-\nu(z))\tilde{B}_{\iota}(z,\vartheta)=B_{\iota}(z,0)-\tilde{S}(\vartheta)\Big[(1-\varphi)\sum_{m=a}^{b}B_{m,\iota}(0)+\sum_{m=1}^{a-1}L_{m,\iota}(0)+\sum_{l=1}^{\infty}Q_{l,\iota}(0)+R_{\iota}(0)\Big],\quad a\leq\iota\leq b-1, \tag{28}$$

$$(\vartheta-\nu(z))\tilde{L}_{\iota}(z,\vartheta)=L_{\iota}(z,0)-\tilde{C}(\vartheta)\Big[\sum_{m=1}^{a-1}L_{m,\iota}(0)+(1-\varphi)\sum_{m=a}^{b}B_{m,\iota}(0)+R_{\iota}(0)\Big],\quad 1\leq\iota\leq a-1, \tag{29}$$

$$\begin{aligned}z^{b}(\vartheta-\nu(z))\tilde{B}_{b}(z,\vartheta)&=z^{b}B_{b}(z,0)\\&-\tilde{S}(\vartheta)\Big\{(1-\varphi)\sum_{m=a}^{b}\Big[B_{m}(z,0)-\sum_{\mathtt{f}=0}^{b-1}B_{m,\mathtt{f}}(0)z^{\mathtt{f}}\Big]\Big\}\\&-\tilde{S}(\vartheta)\Big\{\sum_{m=1}^{a-1}\Big[L_{m}(z,0)-\sum_{\mathtt{f}=0}^{b-1}L_{m,\mathtt{f}}(0)z^{\mathtt{f}}\Big]\Big\}\\&-\tilde{S}(\vartheta)\Big\{\sum_{l=1}^{\infty}\Big[Q_{l}(z,0)-\sum_{\mathtt{f}=0}^{b-1}Q_{l,\mathtt{f}}(0)z^{\mathtt{f}}\Big]+\Big[R(z,0)-\sum_{\mathtt{f}=0}^{b-1}R_{\mathtt{f}}(0)z^{\mathtt{f}}\Big]\Big\}\end{aligned} \tag{30}$$

$$(\vartheta-\nu(z))\tilde{R}(z,\vartheta)=R(z,0)-\varphi\sum_{m=a}^{b}B_{m}(z,0)\tilde{R}(\vartheta). \tag{31}$$

Sustituting $\vartheta=\nu(z)$ in equations (26)-(31), we obtain:

$$Q_{1}(z,0)=\tilde{V}(\nu(z))\Big[(1-\varphi)\sum_{m=a}^{b}B_{m,0}(0)+\sum_{m=1}^{a-1}L_{m,0}(0)+R_{0}(0)\Big], \tag{32}$$

$$Q_{\mathtt{f}}(z,0)=\tilde{V}(\nu(z))\sum_{n=0}^{a-1}Q_{\mathtt{f}-1,n}(0)z^{n},\qquad \mathtt{f}\geq 2, \tag{33}$$

$$R(z,0)=\varphi\sum_{m=a}^{b}B_{m}(z,0)\tilde{R}(\nu(z)), \tag{34}$$

$$L_{\iota}(z,0)=\tilde{C}(\nu(z))\Big[\sum_{m=1}^{a-1}L_{m,\iota}(0)+(1-\varphi)\sum_{m=a}^{b}B_{m,\iota}(0)+R_{\iota}(0)\Big],1\leq\iota\leq a-1, \tag{35}$$

$$B_{\iota}(z,0)=\tilde{S}(\nu(z))\Big[(1-\varphi)\sum_{m=a}^{b}B_{m,\iota}(0)+\sum_{m=1}^{a-1}L_{m,\iota}(0)+\sum_{l=1}^{\infty}Q_{l,\iota}(0)+R_{\iota}(0)\Big],\quad a\leq\iota\leq b-1, \tag{36}$$

$$z^b B_b(z,0) = \tilde{S}(\nu(z))\Bigg\{(1-\varphi)\sum_{m=a}^{b}\Big[B_m(z,0) - \sum_{\mathtt{f}=0}^{b-1} B_{m,\mathtt{f}}(0)z^{\mathtt{f}}\Big] + \sum_{m=1}^{a-1}\Big[L_m(z,0) - \sum_{\mathtt{f}=0}^{b-1} L_{m,\mathtt{f}}(0)z^{\mathtt{f}}\Big] + \sum_{l=1}^{\infty}\Big[Q_l(z,0) - \sum_{\mathtt{f}=0}^{b-1} Q_{l,\mathtt{f}}(0)z^{\mathtt{f}}\Big] + \Big[R(z,0) - \sum_{\mathtt{f}=0}^{b-1} R_{\mathtt{f}}(0)z^{\mathtt{f}}\Big]\Bigg\},$$

When $B_b(z,0)$ is solved for, the following equation is obtained

$$B_b(z,0) = \frac{\tilde{S}(\nu(z))f(z)}{z^b - \epsilon(z)}. \tag{37}$$

where

$$\begin{aligned}
\epsilon(z) &= (1-\varphi)\tilde{S}(\nu(z)) + \varphi\tilde{S}(\nu(z))\tilde{R}(\nu(z)),\\
f(z) &= \Big[(1-\varphi) + \varphi\tilde{R}(\nu(z))\Big]\sum_{m=a}^{b-1} B_m(z,0) + \sum_{m=1}^{a-1} L_m(z,0) + \sum_{l=1}^{\infty} Q_l(z,0)\\
&\qquad - \Big[\sum_{\iota=0}^{b-1} d_\iota z^\iota + \sum_{i=0}^{b-1} l_\iota z^\iota + \sum_{\iota=0}^{b-1} q_\iota z^\iota\Big],\\
p_\iota &= \sum_{m=a}^{b} B_{m,\iota}(0), l_\iota = \sum_{m=1}^{a-1} L_{m,\iota}(0),\\
q_\iota &= \sum_{l=1}^{\infty} Q_{l,\iota}(0), R_\iota = R_\iota(0),\\
d_\iota &= (1-\varphi)p_\iota + R_\iota.
\end{aligned} \tag{38}$$

Using the equations (32)-(37) in Eqs.(26)-(31), after simplication we obtain

$$\tilde{Q}_1(z,\vartheta) = \frac{1}{\vartheta - (\nu(z))}\Bigg\{\Big[\tilde{V}(\nu(z)) - \tilde{V}(\vartheta)\Big]\Big[(1-\varphi)\sum_{m=a}^{b} B_{m,0}(0) + \sum_{m=1}^{a-1} L_{m,0}(0) + R_0(0)\Big]\Bigg\}, \tag{39}$$

$$\tilde{Q}_{\mathtt{f}}(z,\vartheta) = \frac{1}{\vartheta - (\nu(z))}\Bigg\{\Big[\tilde{V}(\nu(z)) - \tilde{V}(\vartheta)\Big]\sum_{n=0}^{a-1} Q_{\mathtt{f}-1,n}(0)z^n\Bigg\}, \quad \mathtt{f} \geq 2, \tag{40}$$

$$\tilde{R}(z,\vartheta) = \frac{1}{\vartheta - (\nu(z))}\Bigg\{\Big[\tilde{R}(\nu(z)) - \tilde{R}(\vartheta)\Big]\varphi\sum_{m=a}^{b} B_m(z,0)\Bigg\}, \tag{41}$$

$$\tilde{L}_\iota(z,\vartheta) = \frac{1}{\vartheta-(\nu(z))}\Bigg\{\Big[\tilde{C}(\nu(z))-\tilde{C}(\vartheta)\Big]\Big[(1-\varphi)\sum_{m=a}^{b}B_{m,\iota}(0) + \sum_{m=1}^{a-1}L_{m,\iota}(0)+R_\iota(0)\Big]\Bigg\}, \quad 1\le\iota\le a-1, \tag{42}$$

$$\tilde{B}_\iota(z,\vartheta) = \frac{1}{\vartheta-(\nu(z))}\Bigg\{\Big[\tilde{S}(\nu(z))-\tilde{S}(\vartheta)\Big]\Big[(1-\varphi)\sum_{m=a}^{b}B_{m,\iota}(0) + \sum_{m=1}^{a-1}L_{m,\iota}(0)+\sum_{l=1}^{\infty}Q_{l,\iota}(0)+R_\iota(0)\Big]\Bigg\}, \quad a\le\iota\le b-1, \tag{43}$$

$$\tilde{B}_b(z,\vartheta) = \frac{\Big[\tilde{S}(\nu(z))-\tilde{S}(\vartheta)\Big]f(z)}{(\vartheta-(\nu(z)))(z^b-\epsilon(z))} \tag{44}$$

In summary, the PGF associated with the size of the queue is

$$P(z) = \sum_{\iota=a}^{b-1}\tilde{B}_\iota(z,0)+\tilde{B}_b(z,0)+\sum_{\iota=1}^{a-1}\tilde{L}_\iota(z,0)+\sum_{\mathtt{f}=1}^{\infty}\tilde{Q}_{\mathtt{f}}(z,0)+\tilde{R}(z,0) \tag{45}$$

Substituting $\vartheta = 0$ on the Eqs. $(39)-(44)$ then the Eqn.(45) as follows:

$$\begin{aligned}
P(z) = &\frac{\Big[\tilde{S}(v(z))-1\Big]\sum_{\iota=a}^{b-1}\Big[(1-\varphi)\sum_{m=a}^{b}B_{m,\iota}(0)+\sum_{m=1}^{a-1}L_{m,\iota}(0)+\sum_{l=1}^{\infty}Q_{l,\iota}(0)+R_\iota(0)\Big]}{-(\nu(z))}\\
&+\frac{\Big[\tilde{C}(\nu(z))-1\Big]\sum_{\iota=1}^{a-1}\Big[(1-\varphi)\sum_{m=a}^{b}B_{m,\iota}(0)+\sum_{m=1}^{a-1}L_{m,\iota}(0)+R_\iota(0)\Big]}{-(\nu(z))}\\
&+\frac{\Big[\tilde{V}(\nu(z))-1\Big]\sum_{\mathtt{f}=1}^{\infty}\Big[(1-\varphi)\sum_{m=a}^{b}B_{m,0}(0)+\sum_{m=1}^{a-1}L_{m,0}(0)+\sum_{n=0}^{a-1}q_n(0)z^n+R_0(0)\Big]}{-(\nu(z))}\\
&+\frac{\Big[\tilde{R}(\nu(z))-1\Big]\varphi\sum_{m=a}^{b}B_m(z,0)}{-(\nu(z))}\\
&+\frac{\Big[\tilde{S}(\nu(z))-1\Big]f(z)}{-(\nu(z))(z^b-\epsilon(z))}.
\end{aligned}$$

Using Eqn.38 in the above equation, the result is:

$$P(z)=\frac{\left\{\begin{array}{l}\Big[(1-\varphi)\tilde{S}(\nu(z))+\varphi\tilde{S}(\nu(z))\tilde{R}(\nu(z))-1\Big]\sum\limits_{\iota=a}^{b-1}[z^b-z^\iota]c_\iota+\\ \Big[\tilde{S}(\nu(z))-z^b+\tilde{C}(\nu(z))(z^b-1)+\varphi\tilde{S}(\nu(z))\\ (\tilde{R}(\nu(z))-1)\Big]\sum\limits_{\iota=1}^{a-1}f_\iota+\Big[\tilde{V}(\nu(z))-1\Big](z^b-1)[f_0+\sum\limits_{n=0}^{a-1}q_nz^n]-\\ \Big[\tilde{S}(\nu(z))(1-\varphi)+\varphi\tilde{S}(\nu(z))\tilde{R}(\nu(z))-1\Big]\sum\limits_{\iota=1}^{a-1}f_\iota z^\iota\end{array}\right\}}{-(\nu(z))(z^b-\epsilon(z))} \tag{46}$$

where $c_\iota=(1-\varphi)p_\iota+l_\iota+q_\iota+R_\iota$ and $f_\iota=(1-\varphi)p_\iota+l_\iota+R_\iota$. The above equation represents the number of customers in the queue's PGF.

## 5.1. Steady-state condition

It is necessary for the probability-generating function $P(z)$ to have the value $P(1)=1$. L'Hospital's rule employed to evaluate the equation $\lim\limits_{z\to 1}P(z)$, yielding a value of 1. If it is possible to meets the condition, it follows that $\rho<1$ is necessary for the model under consideration to have a steady state. Then

$$\rho=\frac{\lambda E(\mathbb{X})[E(S)+\varphi E(R)]}{b} \tag{47}$$

## 5.2. Computation features

Equation (46) has '$b+a$' unknowns $q_0,q_1,...,q_{a-1},f_0,f_1,f_2,...,f_{a-1},c_a,c_{a+1},...,c_{b-1}$. Theorem 1 shows that $q_i$ can be written as a function of $f_0$ with the numerator containing only '$b$' constants.The equation (46) serves as the probability-generating function of the number of customers involving '$b$' unknowns. By Rouche's theorem, $z^b-\epsilon(z)$ has one on the unit circle $|z|=1$ and $b-1$ zeros inside. Since $P(z)$ is analytic inside and on the unit circle, it yields '$b$' equations and '$b$' unknowns, necessitating the numerator to vanish at these points. Gauss elimination method is employed to solve these equations.

**Theorem 1.**

$$q_n=\frac{\left(\beta_nf_0+\sum\limits_{\kappa=0}^{n-1}q_\kappa\beta_{n-\kappa}\right)}{1-\beta_0},n=1,2,...,a-1,q_0=\frac{\beta_0}{1-\beta_0}f_0,$$

*where $f_0=(1-\varphi)p_0+l_0+R_0$ and '$\kappa$' customers probability of arriving while on vacation is denoted by $\beta_\kappa$.*

*Proof. From equations* (32) *and* (33), *we have*

$$\sum_{n=0}^{\infty} q_n z^n = \tilde{V}(\nu(z))\Big[(1-\varphi)\sum_{m=a}^{b} B_{m,0}(0) + \sum_{m=1}^{a-1} L_{m,0}(0) + R_0(0) + \sum_{\mathtt{f}=2}^{\infty}\sum_{n=0}^{a-1} Q_{\mathtt{f}-1,n}(0) z^n\Big]$$
$$= \sum_{n=0}^{\infty} \beta_n z^n \Big[(1-\varphi)p_0 + l_0 + R_0 + \sum_{n=0}^{a-1} q_n z^n\Big]$$
$$= \sum_{n=0}^{\infty} \beta_n z^n \Big[f_0 + \sum_{n=0}^{a-1} q_n z^n\Big]$$

*Equating the coefficient of* $z^0$, *we get*

$$q_0 = \frac{\beta_0}{1-\beta_0} f_0,$$

*Equating the coefficient of* $z^1$, *we get*

$$q_1 = \frac{\Big(\beta_1 f_0 + q_0 \beta_1\Big)}{1-\beta_0},$$

*Proceeding like this, we get*

$$q_n = \frac{\Big(\beta_n f_0 + \sum_{\kappa=0}^{n-1} q_\kappa \beta_{n-\kappa}\Big)}{1-\beta_0}, \quad n = 1, 2, ..., a-1$$

Hence the theorem.

## 5.3. Particular cases

Case I: If $\tilde{S} = \tilde{C}$, $a = 1$, $b = 1$, and there is no server breakdown. The equation (46) becomes

$$P(z) = \frac{\Big[\tilde{V}(\nu(z)) - 1\Big][z-1]\Big[f_0 + \sum_{n=0}^{a-1} q_n z^n\Big]}{-(\nu(z))[z - \tilde{S}(\nu(z))]}. \tag{48}$$

which coincide with the findings of Lee et al.[1994] and $N = a$.

Case II: When there is no low-batch service and no breakdown, the equation (46) becomes

$$P(z) = \frac{\Big[\tilde{S}(\nu(z)) - 1\Big]\sum_{\iota=1}^{b-1}\Big[z^b - z^\iota\Big] f_\iota + \Big[\tilde{V}(\nu(z)) - 1\Big]\Big[z^b - 1\Big]\Big[f_0 + \sum_{n=0}^{a-1} q_n z^n\Big]}{-(\nu(z))\Big[z^b - \tilde{S}(\nu(z))\Big]}. \tag{49}$$

which coincide with the findings of Senthilnathan et al.'s [2012] and without closedown.

### 5.4. PGF of Queue Size at Various Epochs

- Queue size PGF in the epoch for bulk service completion

  Using (43) and (44) equations, we obtain

$$B(z) = \frac{\left[\tilde{S}(\nu(z)) - 1\right]\left[\sum_{\iota=a}^{b-1}[z^b - z^\iota]c_\iota + \tilde{C}(\nu(z))\sum_{\iota=1}^{a-1} f_\iota - \sum_{\iota=1}^{a-1} f_\iota z^\iota + \left[\tilde{V}(\nu(z)) - 1\right][f_0 + \sum_{n=0}^{a-1} q_n z^n]\right]}{-(\nu(z))(z^b - \epsilon(z))} \tag{50}$$

- Queue size PGF during the low-batch service completion epoch

  Using Eq. (42), we obtain

$$L(z) = \frac{\left[\tilde{C}(\nu(z)) - 1\right]\sum_{\iota=1}^{a-1} f_\iota}{-(\nu(z))}. \tag{51}$$

- Queue size PGF during the vacation completion epoch

  Using (39) and (40) equations, we obtain

$$V(z) = \frac{\left[\tilde{V}(\nu(z)) - 1\right]\left[f_0 + \sum_{n=0}^{a-1} q_n z^n\right]}{-(\nu(z))}. \tag{52}$$

- Queue size PGF during the renovation completion epoch

  Using (36), (37) and (41) equations, we obtain

$$R(z) = \frac{\left[\varphi\tilde{S}(\nu(z))\tilde{R}(\nu(z)) - \varphi\tilde{S}((\nu(z))\right]\left[\sum_{\iota=a}^{b-1}[z^b - z^\iota]c_\iota + \tilde{C}(\nu(z))\sum_{\iota=1}^{a-1} f_\iota - \sum_{\iota=1}^{a-1} f_\iota z^\iota + \left[\tilde{V}(\nu(z)) - 1\right][f_0 + \sum_{n=0}^{a-1} q_n z^n]\right]}{-(\nu(z))(z^b - \epsilon(z))} \tag{53}$$

# 6. Performance indices

## 6.1. Expected queue length

Differentiating $P(z)$ concerning $z$ and assessing the result at $z = 1$ the expected queue size $E(Q)$ at any given time. This is stated as follows:

$$E(Q) = \frac{\left\{ \begin{array}{l} \sum_{\iota=a}^{b-1} c_\iota[b(b-1) - \iota(\iota-1)]\mathbb{F}_1 + \sum_{\iota=a}^{b-1} c_\iota(b-\iota)\mathbb{F}_2 + \sum_{\iota=1}^{a-1} f_\iota \mathbb{F}_3 \\ + [f_0 + \sum_{\iota=0}^{a-1} q_\iota]\mathbb{F}_4 + \sum_{\iota=0}^{a-1} \iota q_\iota \mathbb{F}_5 - \sum_{\iota=1}^{a-1} f_\iota \mathbb{F}_6 \end{array} \right\}}{2\Big[(\lambda E(\mathbb{X}))(b - \mathbb{S}_1 - \varphi R_1)\Big]^2} \tag{54}$$

where

$$\begin{aligned}
&\mathbb{S}_1 = E(\mathbb{X})E(S)\lambda; \mathbb{S}_2 = E(\mathbb{X}^2)E(S)\lambda + E^2(X)E(S^2)\lambda^2;\\
&\mathbb{C}_1 = E(\mathbb{X})E(C)\lambda; \mathbb{C}_2 = E(\mathbb{X}^2)E(C)\lambda + E^2(\mathbb{X})E(C^2)\lambda^2;\\
&\mathbb{V}_1 = E(V)\lambda E(\mathbb{X}); \mathbb{V}_2 = E(V)\lambda E(\mathbb{X}^2) + E(V^2)\lambda^2 E^2(\mathbb{X});\\
&R1 = \lambda E(\mathbb{X})E(R);\\
&R2 = \lambda E(\mathbb{X}^2)E(R) + \lambda^2 E^2(\mathbb{X})E(R^2);\\
&T1 = (b - \mathbb{S}_1 - \varphi R1)\lambda E(\mathbb{X});\\
&T2 = (b - \mathbb{S}_1 - \varphi R1)\lambda E(\mathbb{X}^2) + \lambda E(\mathbb{X})(b(b-1) - \mathbb{S}_2 - \varphi R2 - 2\varphi \mathbb{S}_1 R1);\\
&T3 = \mathbb{S}_1 + \varphi R1;\\
&T4 = \mathbb{S}_2 + \varphi R2 + 2\varphi \mathbb{S}_1 R1;\\
&\mathbb{F}_1 = T3T1;\\
&\mathbb{F}_2 = T4T1 - T3T2;\\
&\mathbb{F}_3 = b\mathbb{C}_2 T1 + b(b-1)\mathbb{C}_1 T1 - b\mathbb{C}_1 T2;\\
&\mathbb{F}_4 = b\mathbb{V}_2 T1 + b(b-1)\mathbb{V}_1 T1 - b\mathbb{V}_1 T2;\\
&\mathbb{F}_5 = 2b\mathbb{V}_1 T1;\\
&\mathbb{F}_6 = \iota\, T4T1 + \iota(\iota-1)T3T1 - \iota\, T3T2;
\end{aligned}$$

## 6.2. Expected length of idle period

Considering $I$ to represent the random variable for the idle period, we can find the average idle period using $E(I)$.

Construct a random variable $\mathbb{U}_1$ as

$$\mathbb{U}_1 = \begin{cases} 0, & \text{if the server has at least '}a\text{' customers after the first vacation.} \\ 1, & \text{if the server finds fewer than '}a\text{' customers after the first vacation.} \end{cases}$$

Due to multiple vacations, the expected idle period duration $E(I)$ is expressed as

$$\begin{aligned} E(I) &= E(I/\mathbb{U}_1 = 0)P(\mathbb{U}_1 = 0) + E(I/\mathbb{U}_1 = 1)P(\mathbb{U}_1 = 1) \\ &= E(V)P(\mathbb{U}_1 = 0) + [E(V) + E(I)]P(\mathbb{U}_1 = 1) \end{aligned}$$

Once solved, we obtain

$$E(I) = \frac{E(V)}{P(\mathbb{U}_1 = 0)} = \frac{E(V)}{1 - P(\mathbb{U}_1 = 1)} \tag{55}$$

From Eq.(32), $Q_{1n}(0)$ =coefficient of $z^n$in $Q_1(z, 0)$

$$P(\mathbb{U}_1 = 0) = 1 - \sum_{n=0}^{a-1} Q_{1n}(0) = 1 - \sum_{n=0}^{a-1}\sum_{\iota=0}^{n} \beta_\iota f_0$$

Substitute in (55),we get

$$E(I) = \frac{E(V)}{1-\sum_{n=0}^{a-1}\sum_{\iota=0}^{n} \beta_\iota f_0}$$

## 6.3. Expected length of busy period

**Theorem 2.** *Let $\mathcal{B}$ be the random variable during the busy period. The expected duration of the busy time is then*

$$E(\mathcal{B}) = \frac{E(T)}{f_0}. \tag{56}$$

*where $E(T) = E(S) + E(C) + \varphi E(R)$.*

*Proof. Let $T$ represent the residence time that the server utilizes for renovation or low-batch or bulk services.*

$$E(T) = E(S) + E(C) + \varphi E(R).$$

*Construct a random variable $\mathbb{U}_2$ as*

$$\mathbb{U}_2 = \begin{cases} \text{1, if beyond the residency time, the server finds at least one customer.} \\ \text{0, if no customer is found by the server after the residence time.} \end{cases}$$

*Due to multiple vacations, the expected busy period duration $E(\mathcal{B})$ is expressed as*

$$\begin{aligned} E(\mathcal{B}) &= E(T/\mathbb{U}_2 = 0)P(\mathbb{U}_2 = 0) + E(T/\mathbb{U}_2 = 1)P(\mathbb{U}_2 = 1) \\ &= E(T)P(\mathbb{U}_2 = 0) + [E(T) + E(\mathcal{B})]P(\mathbb{U}_2 = 1) \end{aligned}$$

*The mean service time is indicated by $E(T)$, and $E(\mathcal{B})$ can be found by solving for it.*

$$E(\mathcal{B}) = \frac{E(T)}{P(\mathbb{U}_2=0)} = \frac{E(T)}{f_0}.$$

### 6.4. Expected waiting time

Little's formula allows for the following calculation of the expected waiting time:

$$E(W) = \frac{E(Q)}{\lambda E(\mathbb{X})}, \tag{57}$$

In Equation. (54), $E(Q)$ is defined.

## 7. Cost model

In all practical situations, cost analysis represents the important factor across all tiers. Costs include those for holding, operating, renovating, starting up and rewarding (if appropriate). It makes sense that the system's administration would want to keep the average cost as low as possible.

Under the following assumptions, the formula for determining the total average cost is developed.

- $C_s$ : Cost of startup.
- $C_h$ : Holding expenses for each customer.
- $C_o$ : Operational costs per unit of time.
- $C_v$ : Renovation costs per unit of time.
- $C_r$ : Time-per-unit reward cost.

Since the duration of a cycle equals the total of its idle and busy periods, as given by equations (55) and (56), the expected length of the cycle $E(T_c)$ is equal to $E(I) + E(\mathcal{B})$.

Total average cost = Starting costs for each cycle
+ Renovation costs for each cycle
+ Holding cost of customer waiting in line for each unit of time
+ Operating costs for each unit of time ($\rho$)
− The cost of the rewards for each cycle of vacation.

$$TAC = \left[C_s + \varphi C_v E(R) - C_r \frac{E(V)}{P(\mathbb{U}_1 = 0)}\right] \frac{1}{E(T_c)} + C_o\rho + C_h E(Q) \tag{58}$$

where

$$\rho = \frac{\lambda E(\mathbb{X})[E(S) + \varphi E(R)]}{b}$$

## 8. Numerical representation

An illustrative example for the prescribed model is investigated in the context of a specific scenario under the subsequent presumptions:

- The arrival batch size follows a geometric distribution with a mean of 2.
- The Erlang-2 service time distribution is used for both the main (bulk) server and the low-batch server.
- The parameters of the exponential distribution are $\alpha = 5$ and $\beta = 5$, respectively, for vacation and renovation rates.
- For the main (bulk) server and low-batch server, let $\mu_1 = 15$ and $\mu_2 = 10$ represent the respective service rates.
- $\varphi$ (Breakdown rate)= 0.1
- Cost of startup: Rs = 4.00.
  Per-customer holding costs: Rs = 0.50.
  Operating costs per unit of time: Rs = 5.00.
  Reward cost per unit of time: Rs = 2.00.
  Per unit hour, the cost of renovation is Rs = 0.4.

Applying numerical methods, the queue size distribution's unknown probabilities are calculated. With MATLAB, simultaneous equations are solved and the function's zeros are determined. For different arrival rates and service rates, the expected length of the line, expected waiting time, expected busy period, expected idle period and overall average cost are computed and presented.

Table 1: Arrival Rate versus Performance Metrics. $\mu_1 = 15$, $\mu_2 = 10$, $\varphi = 0.1$, $a = 5$, $b = 8$, $\alpha = 5$, $\beta = 5$

| $\lambda$ | $\rho$ | $E(Q)$ | $E(W)$ | $E(B)$ | $E(I)$ | $TAC$ |
|---|---|---|---|---|---|---|
| 11.0 | 0.421667 | 6.3713 | 0.2896 | 0.3949 | 0.4614 | 8.8966 |
| 12.0 | 0.460000 | 7.1611 | 0.2984 | 0.4126 | 0.4188 | 9.6939 |
| 13.0 | 0.498333 | 8.0541 | 0.3098 | 0.4359 | 0.3824 | 10.4821 |
| 14.0 | 0.536667 | 9.0923 | 0.3247 | 0.4646 | 0.3521 | 11.2743 |
| 15.0 | 0.575000 | 10.2994 | 0.3433 | 0.5006 | 0.3266 | 12.0804 |
| 16.0 | 0.613333 | 11.7297 | 0.3666 | 0.5455 | 0.3049 | 12.9272 |
| 17.0 | 0.651667 | 13.4571 | 0.3958 | 0.6018 | 0.2866 | 13.8533 |
| 18.0 | 0.690000 | 15.5923 | 0.4331 | 0.6735 | 0.2709 | 14.9167 |
| 19.0 | 0.728333 | 18.3100 | 0.4818 | 0.7667 | 0.2574 | 16.2078 |
| 20.0 | 0.766667 | 21.8943 | 0.5474 | 0.8919 | 0.2457 | 17.8716 |
| 21.0 | 0.805000 | 26.8578 | 0.6395 | 0.9718 | 0.2356 | 20.1676 |

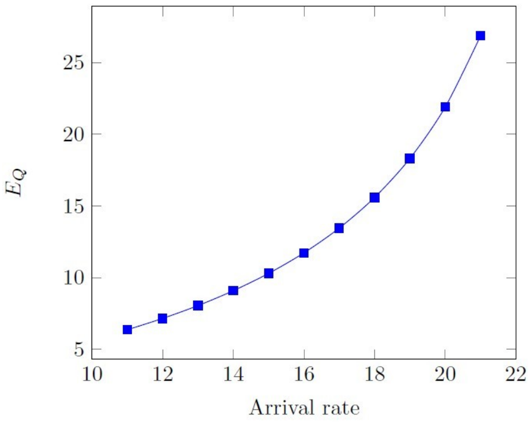



Figure 2: Arrival rate vs E(Q)

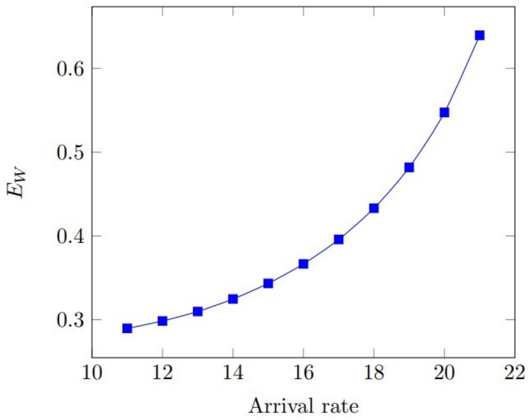



Figure 3: Arrival rate vs E(W)

As the arrival rate rises, this system becomes busier, leading to longer queues, extended waiting times, and more prolonged busy periods. Consequently, the idle period of the system decreases.

Table 1 illustrates that the average busy period, queue duration, and waiting times all rise, while the average idle period declines as the arrival rate increases, maintaining fixed bulk service and low-batch service rates.

Figures 2 and 3 clearly show that as the arrival rate increases, both the average queue length and waiting time also increase.

Table 2: Service Rate versus Performance Metrics. $\lambda = 20$, $\mu_2 = 0$, $\varphi = 0.1$, $a = 5$, $b = 8$, $\alpha = 5$, $\beta = 5$

| $\mu_1$ | $\rho$ | $E(Q)$ | $E(W)$ | $E(B)$ | $E(I)$ | $TAC$ |
|---|---|---|---|---|---|---|
| 11.5 | 0.969565 | 175.5064 | 4.3877 | 2.2346 | 0.2085 | 94.0709 |
| 12.0 | 0.933333 | 79.6504 | 1.9913 | 0.9826 | 0.2196 | 47.4605 |
| 12.5 | 0.900000 | 52.9566 | 1.3239 | 0.6317 | 0.2309 | 35.0891 |
| 13.0 | 0.869231 | 40.4427 | 1.0111 | 0.4670 | 0.2424 | 29.5339 |
| 13.5 | 0.840741 | 33.2469 | 0.8312 | 0.3710 | 0.2541 | 26.4260 |
| 14.0 | 0.814286 | 28.5686 | 0.7142 | 0.3084 | 0.2660 | 24.4080 |
| 14.5 | 0.789655 | 25.3236 | 0.6331 | 0.2640 | 0.2782 | 22.9767 |
| 15.0 | 0.766667 | 22.9223 | 0.5731 | 0.2312 | 0.2905 | 21.8636 |
| 15.5 | 0.745161 | 21.0880 | 0.5272 | 0.2058 | 0.3031 | 20.9545 |
| 16.0 | 0.725000 | 19.6460 | 0.4912 | 0.1857 | 0.3159 | 20.1799 |
| 16.5 | 0.706061 | 18.4864 | 0.4622 | 0.1692 | 0.3289 | 19.4988 |
| 17.0 | 0.688235 | 17.5350 | 0.4384 | 0.1556 | 0.3422 | 18.8855 |
| 17.5 | 0.671429 | 16.7377 | 0.4184 | 0.1441 | 0.3557 | 18.3222 |
| 18.0 | 0.655556 | 16.0593 | 0.4015 | 0.1343 | 0.3693 | 17.7991 |

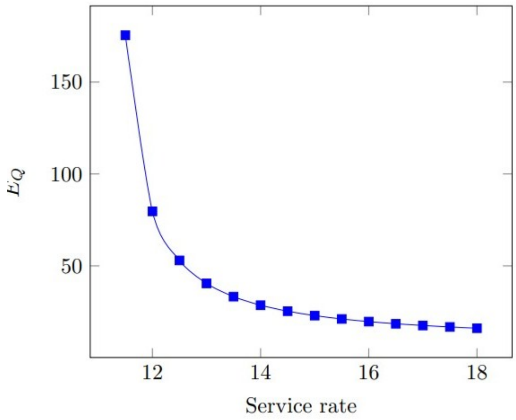



Figure 4: Service rate vs E(Q)

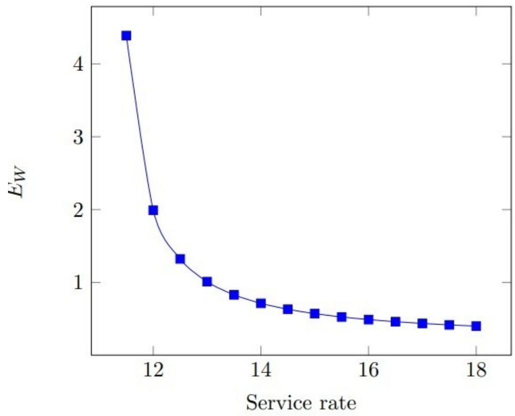



Figure 5: Service rate vs E(W)

The average busy period, waiting time, and length of the queue all decrease while the average idle period increases when the bulk service rate rises, but the low-batch service rate stays at zero, as Table 2 illustrates.

Figures 4 and 5 illustrate that as the service rate increases, both the average queue length and waiting time decrease.

Table 3: Service Rate versus Expected Queue Length. $\lambda = 20$, $\varphi = 0.1$, $a = 5$, $b = 8$, $\alpha = 5$, $\beta = 5$

| $\mu_1$ | $\rho$ | $\mu_2 = 8$ | $\mu_2 = 9$ | $\mu_2 = 10$ | $\mu_2 = 11$ |
|---|---|---|---|---|---|
| 11.5 | 0.969565 | 174.9996 | 174.6549 | 174.4410 | 174.3050 |
| 12.0 | 0.933333 | 79.1337 | 78.7948 | 78.5815 | 78.4497 |
| 12.5 | 0.900000 | 52.4373 | 52.1010 | 51.8897 | 51.7619 |
| 13.0 | 0.869231 | 39.9400 | 39.6041 | 39.3941 | 39.2658 |
| 13.5 | 0.840741 | 32.7405 | 32.4073 | 32.1993 | 32.0722 |
| 14.0 | 0.814286 | 28.0711 | 27.7400 | 27.5323 | 27.4062 |
| 14.5 | 0.789655 | 24.8219 | 24.4932 | 24.2873 | 24.1624 |
| 15.0 | 0.766667 | 22.4263 | 22.0992 | 21.8943 | 21.7695 |
| 15.5 | 0.745161 | 20.5967 | 20.2708 | 20.0672 | 19.9431 |
| 16.0 | 0.725000 | 19.1582 | 18.8343 | 18.6315 | 18.5083 |
| 16.5 | 0.706061 | 18.0012 | 17.6786 | 17.4771 | 17.3542 |
| 17.0 | 0.688235 | 17.0520 | 16.7309 | 16.5304 | 16.4081 |
| 17.5 | 0.671429 | 16.2562 | 15.9367 | 15.7372 | 15.6156 |
| 18.0 | 0.655556 | 15.5856 | 15.2819 | 15.0681 | 14.9470 |

The comparison of Tables 2 and 3 shows that, whereas a zero low-batch service rate (as in Table 2 ) results in a higher expected queue length, providing an increasing rate in low-batch service causes the queue length to gradually reduce (as in Table 3 ).

Table 4: Service Rate versus Expected Waiting Time. $\lambda = 20$, $\varphi = 0.1$, $a = 5$, $b = 8$, $\alpha = 5$, $\beta = 5$

| $\mu_1$ | $\rho$ | $\mu_2 = 8$ | $\mu_2 = 9$ | $\mu_2 = 10$ | $\mu_2 = 11$ |
|---|---|---|---|---|---|
| 11.5 | 0.969565 | 4.3750 | 4.3664 | 4.3610 | 4.3576 |
| 12.0 | 0.933333 | 1.9783 | 1.9699 | 1.9645 | 1.9612 |
| 12.5 | 0.900000 | 1.3109 | 1.3025 | 1.2972 | 1.2940 |
| 13.0 | 0.869231 | 0.9985 | 0.9901 | 0.9849 | 0.9816 |
| 13.5 | 0.840741 | 0.8185 | 0.8102 | 0.8050 | 0.8018 |
| 14.0 | 0.814286 | 0.7018 | 0.6935 | 0.6883 | 0.6852 |
| 14.5 | 0.789655 | 0.6205 | 0.6123 | 0.6072 | 0.6041 |
| 15.0 | 0.766667 | 0.5607 | 0.5525 | 0.5474 | 0.5442 |
| 15.5 | 0.745161 | 0.5149 | 0.5068 | 0.5017 | 0.4986 |
| 16.0 | 0.725000 | 0.4790 | 0.4709 | 0.4658 | 0.4627 |
| 16.5 | 0.706061 | 0.4500 | 0.4420 | 0.4369 | 0.4339 |
| 17.0 | 0.688235 | 0.4263 | 0.4183 | 0.4133 | 0.4102 |
| 17.5 | 0.671429 | 0.4064 | 0.3984 | 0.3934 | 0.3904 |
| 18.0 | 0.655556 | 0.3896 | 0.3820 | 0.3767 | 0.3737 |

One can compare the findings of Tables 2 and 4 to observe that while the expected waiting time is increased at zero low-batch service rate (Table 2), an increase in low-batch service rates results in a gradual decrease in the waiting time (Table 4).

Table 5: Breakdown Probability Rate versus Performance Metrics. $\lambda = 11$, $\mu_1 = 15$, $\mu_2 = 10$, $a = 5$, $b = 8$, $\alpha = 5$, $\beta = 5$

| $\varphi$ | $\rho$ | $E(Q)$ | $E(W)$ | $E(B)$ | $E(I)$ | TAC |
|---|---|---|---|---|---|---|
| 0.1 | 0.421667 | 06.3713 | 0.2896 | 0.3949 | 0.4614 | 08.8966 |
| 0.2 | 0.476667 | 07.2082 | 0.3276 | 0.4635 | 0.4082 | 09.6579 |
| 0.3 | 0.531667 | 09.1862 | 0.4176 | 0.4884 | 0.4082 | 10.5541 |
| 0.4 | 0.586667 | 09.5339 | 0.4334 | 0.6572 | 0.3324 | 11.1029 |
| 0.5 | 0.641667 | 11.2208 | 0.5100 | 0.7999 | 0.3044 | 11.9256 |
| 0.6 | 0.696667 | 13.5104 | 0.6141 | 0.9954 | 0.2811 | 12.9693 |
| 0.7 | 0.751667 | 20.4674 | 0.9303 | 1.0394 | 0.2811 | 16.3631 |
| 0.8 | 0.806667 | 21.9431 | 0.9974 | 1.7272 | 0.2442 | 16.8187 |
| 0.9 | 0.861667 | 31.1468 | 1.4158 | 2.5354 | 0.2294 | 21.1886 |
| 1.0 | 0.916667 | 52.4404 | 2.3837 | 4.4194 | 0.2165 | 31.5902 |

Table 5 clearly shows that the server's busy period increases and its idle time decreases when the probability of a breakdown increases.

Table 5 and Figures 6 and 7 shows that as the breakdown probability increases,the average waiting time and queue length also increase proportionally.

Tables 6 and 7 demonstrate that, for a fixed renovation rate, an increase in the breakdown probability ratio results in higher expected queue lengths and longer waiting times. In contrast, increasing the renovation rate leads to a comparative reduction in both expected queue length and waiting time.

Table 8 and Figures 8 and 9 demonstrate that the anticipated queue length and waiting duration with server breakdown exceed those without server breakdown as the threshold value increases.

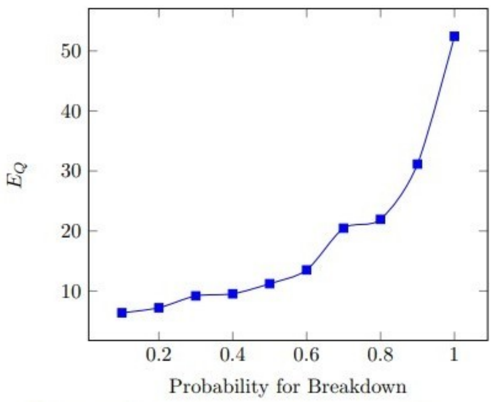



Figure 6: Breakdown rate vs E(Q)

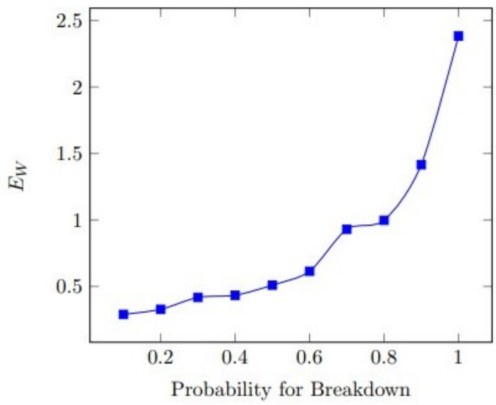



Figure 7: Breakdown rate vs E(W)

Table 6: Breakdown Probability Rate versus EQ for varying $\beta$. $\lambda = 11$, , $\mu_1 = 15$, $\mu_2 = 5$, $a = 5$, $b = 8$

| $\varphi$ | $\beta = 5$ | $\beta = 6$ | $\beta = 7$ | $\beta = 8$ | $\beta = 9$ |
|---|---|---|---|---|---|
| 0.1 | 10.3594 | 10.1619 | 10.0467 | 9.9622 | 9.9110 |
| 0.2 | 11.2193 | 10.7665 | 10.4978 | 10.3251 | 10.2045 |
| 0.3 | 13.6112 | 12.3454 | 11.6393 | 11.2012 | 10.9077 |
| 0.4 | 14.0458 | 12.7828 | 12.0496 | 11.5798 | 11.2542 |
| 0.5 | 15.2110 | 13.4009 | 12.3655 | 11.7385 | 11.3271 |
| 0.6 | 17.7013 | 14.7254 | 13.2323 | 12.3634 | 11.8045 |
| 0.7 | 26.3668 | 18.7205 | 15.6070 | 13.9735 | 12.9863 |
| 0.8 | 26.3741 | 19.1998 | 16.1029 | 14.4297 | 13.3998 |
| 0.9 | 35.8388 | 21.9549 | 17.2923 | 15.0341 | 13.7313 |
| 1.0 | 57.7410 | 26.8644 | 19.5223 | 16.3457 | 14.6126 |

Table 7: Breakdown Probability Rate versus $E(W)$ for varying $\beta$. $\lambda = 11$, , $\mu_1 = 15$, $\mu_2 = 5$, $a = 5$, $b = 8$

| $\varphi$ | $\beta = 5$ | $\beta = 6$ | $\beta = 7$ | $\beta = 8$ | $\beta = 9$ |
|---|---|---|---|---|---|
| 0.1 | 0.4709 | 0.4619 | 0.4567 | 0.4528 | 0.4505 |
| 0.2 | 0.5100 | 0.4894 | 0.4772 | 0.4693 | 0.4638 |
| 0.3 | 0.6187 | 0.5612 | 0.5291 | 0.5091 | 0.4958 |
| 0.4 | 0.6384 | 0.5810 | 0.5477 | 0.5264 | 0.5116 |
| 0.5 | 0.6914 | 0.6091 | 0.5621 | 0.5336 | 0.5149 |
| 0.6 | 0.8046 | 0.6693 | 0.6015 | 0.5620 | 0.5366 |
| 0.7 | 1.1985 | 0.8509 | 0.7094 | 0.6352 | 0.5903 |
| 0.8 | 1.1988 | 0.8727 | 0.7319 | 0.6559 | 0.6091 |
| 0.9 | 1.6290 | 0.9979 | 0.7860 | 0.6834 | 0.6241 |
| 1.0 | 2.6246 | 1.2211 | 0.8874 | 0.7430 | 0.6642 |

Table 8: Threshold Value versus Performance Metrics. $\lambda = 20$, $\mu_1 = 15$, $\mu_2 = 10$, $\varphi = 0.1$, $b = 8$, $\alpha = 5$, $\beta = 5$

| $a$ | Server Breakdown | | | Without Server Breakdown | | |
|---|---|---|---|---|---|---|
| | $E(Q)$ | $E(W)$ | TAC | $E(Q)$ | $E(W)$ | TAC |
| 2 | 21.4424 | 0.5361 | 18.2406 | 13.3444 | 0.3336 | 14.8714 |
| 3 | 21.5845 | 0.5396 | 18.0931 | 13.5769 | 0.3394 | 14.6671 |
| 4 | 21.7315 | 0.5433 | 17.9695 | 13.8003 | 0.3450 | 14.5024 |
| 5 | 21.8943 | 0.5474 | 17.8716 | 14.0167 | 0.3504 | 14.3661 |
| 6 | 22.0600 | 0.5515 | 17.7881 | 14.2275 | 0.3557 | 14.2503 |
| 7 | 22.2278 | 0.5557 | 17.7160 | 14.4394 | 0.3610 | 14.1533 |

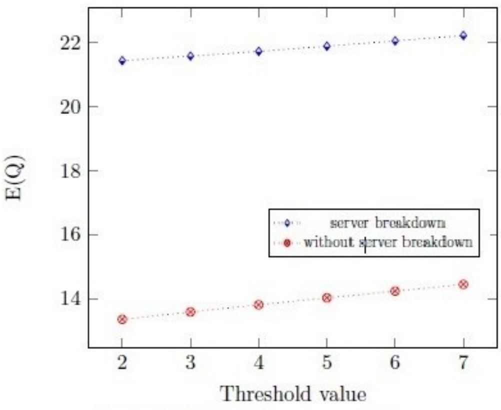



Figure 8: Threshold value vs E(Q)

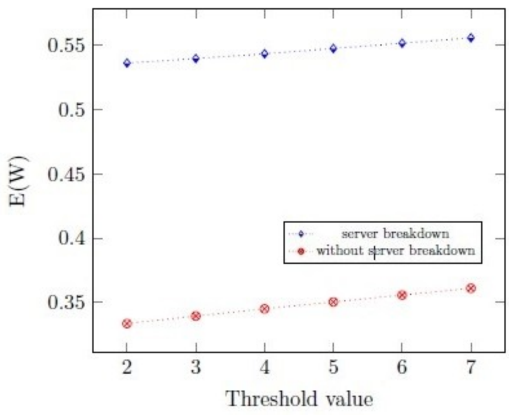



Figure 9: Threshold value vs E(W)

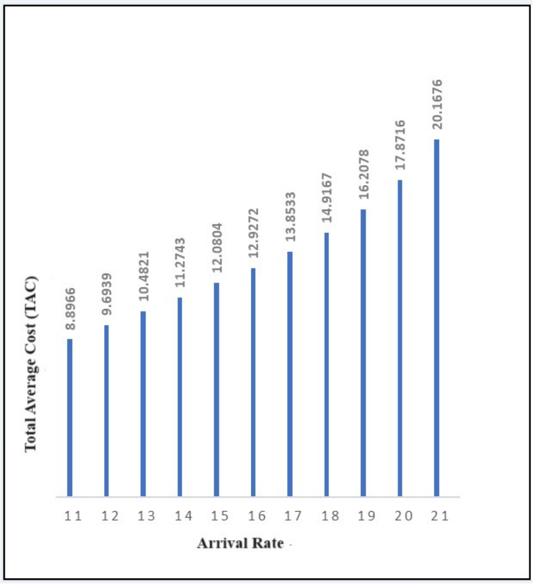



Figure 10: Arrival rate vs TAC

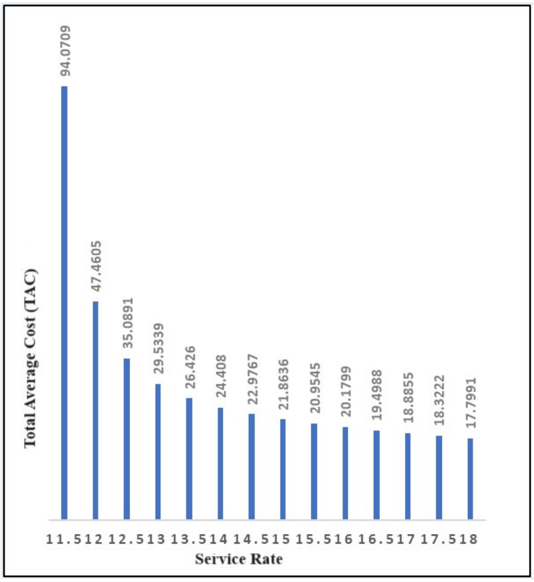



Figure 11: Service rate vs TAC

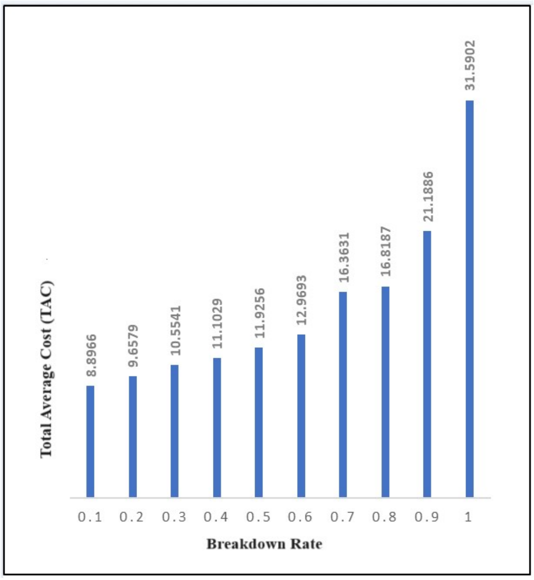



Figure 12: Breakdown rate vs TAC

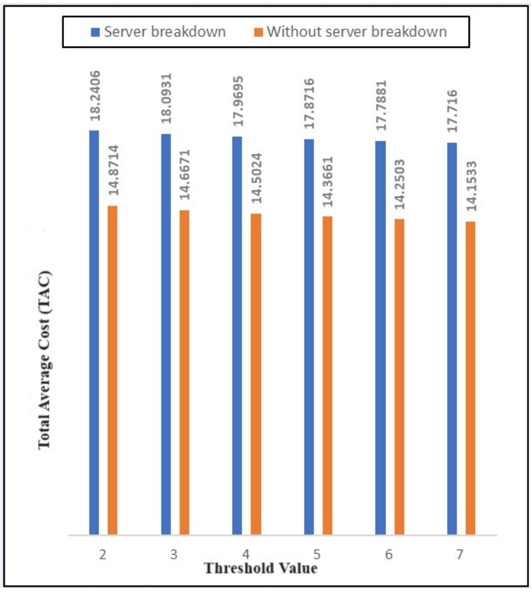



Figure 13 :Threshold value vs TAC

The overall average expenses for bulk service are calculated quantitatively for a range of arrivals and service rates, with the support of low-batch service.
Based on Tables 1 and 2 as well as Figures 10 and 11, the following results are noted:
i. The total average cost rises in response to an increase in the arrival rate.
ii. The total average cost decreases overall when the service rate is increased.
Table 5 and Figure 12 demonstrate that a rise in the Breakdown Probability ratio corresponds to an increase in the overall cost.
Table 8 and Figure 13 show that when a breakdown happens, the overall average cost rises.

## 9. Conclusion and Future Work

This paper evaluates a batch arrival queueing framework referred to as $M^{[\mathbb{X}]}/G(a,b)/1$, which includes low-batch service, uninterruptible server breakdown, multiple vacations and renovation. Applying the supplementary variable technique, the probability-generating function for queue size at each time epoch is obtained. Performance metrics are derived, and specific cases of the model are formulated. Numerical examples are provided for developing a cost model. Numerical results show that as the arrival rate increases, the expected waiting time and queue length is increase. When the bulk service rate increases while the low-batch service rate is zero, the expected waiting time and queue length decrease. As the low-batch service rate increases steadily, the expected waiting time and queue length decrease further compared to when the low-batch service rate is zero. "*The main conclusion is that bulk service with low-batch support significantly reduces the expected waiting time and queue length in real-world situations.*"

This research may be extended in the future to incorporate a queueing model that accounts for breakdowns affecting both bulk service and low-batch service. Further improving efficiency and responsiveness inside the system would be the implementation of an interrupted vacation system with a set-up and closedown time. Furthermore, this work employs advanced techniques such as ANFIS (Adaptive Neuro-Fuzzy Inference System) to enhance the accuracy of performance evaluation through extensive comparison of numerical results.

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