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Exploring Fixed Point Theory in Quasi-Partial Metric Spaces: A Unified Approach with Applications in Integral Equations and Computational Sciences

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Abstract. This paper explores the concept of $\mathcal{C}$-class functions, which encompass a wide range of contractive conditions and applies them to derive common fixed-point results for two pairs of self-mappings in quasi-partial metric spaces. These findings extend existing theories by introducing generalized contractive conditions and providing a broader framework for fixed-point analysis. A detailed example is constructed to validate the results, illustrating the practical application of the established theorems. Furthermore, the paper demonstrates the relevance of these results by applying them to solve a system of integral equations diffusion reaction, highlighting their utility in addressing real-world mathematical problems.

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Key Words and Phrases: Weakly compatible contraction, $\mathcal{C}$-class functions, quasi partial metric spaces

1. Introduction

The study of metric spaces has undergone significant evolution through various generalizations, such as partial metric spaces, metric-like spaces, $b-$metric spaces and quasi-metric spaces. Partial metric spaces were first introduced by Matthews in [1, 2], providing a foundational framework for fixed-point theory in settings where self-distance may not be zero. Several fixed-point results have since been established in such spaces, as detailed in works like [3–5]. Later, Künzi et al. [6] expanded this concept by introducing partial quasi-metric spaces, which relaxed the symmetry condition inherent in partial metric spaces. Karapinar et al. [7] contributed to this area by renaming partial quasi-metric spaces as quasi-partial metric spaces and presenting some initial fixed-point results and properties. Further advancements in this domain can be found in [8–19].

This research builds upon the foundational concept of $\mathcal{C}$-class functions, which provide a flexible and comprehensive framework for addressing a wide spectrum of contractive conditions. These functions have played a pivotal role in broadening the scope of fixed-point

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theory, offering generalized approaches to analyzing self-mappings in various mathematical structures, as explored in [20–23]. In this study, we extend these ideas by introducing a novel class of Ciric-type contractive conditions specifically designed for quasi-partial metric spaces. Our primary goal is to derive new results concerning coincidence and common fixed points within this setting. The findings presented herein not only unify and expand previous work but also refine existing theoretical models by incorporating more generalized contractive mechanisms. This advancement paves the way for broader applications in both mathematical analysis and applied sciences, particularly in the study of nonlinear equations and iterative algorithms.

To reinforce the theoretical developments established in this work, we present a concrete example that highlights the validity of our results. Furthermore, we demonstrate the practical significance of our findings by applying them to solve a system of Fredholm integral equations as well as a system modeling diffusion-reaction processes. These applications underscore the utility of quasi-partial metric spaces in tackling intricate mathematical challenges that arise in various scientific and engineering domains. By bridging abstract theoretical concepts with real-world problem-solving, our study showcases the adaptability and effectiveness of quasi-partial metric frameworks in analyzing nonlinear systems and iterative solution methods.

2. Preliminaries

In this section, we present fundamental definitions and essential properties related to quasi-partial metric spaces.

A partial metric space extends standard metric space concepts (when $p(x,x) = 0$ recovers standard metric) and allows modeling of objects with non-zero self-similarity. The mapping $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ forms a standard metric.

Definition 1. *[1, 2] Let X be a non-empty set. A function $p : X \times X \to \mathbb{R}^+$ is called a partial metric if it satisfies the following conditions for all $x, y, z \in X$:*

(i) Symmetry: $p(x,y) = p(y,x)$.

(ii) Non-negativity and Indistinguishability: If $p(x,x) = p(x,y) = p(y,y) = 0$, then $x = y$.

(iii) Small Self-Distance: $p(x,x) \leq p(x,y)$.

(iv) Modified Triangle Inequality: $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$.

A pair (X,p) satisfying these conditions is called a partial metric space.

Note that if $p(x,y) = 0$, then by conditions (1) and (2), we deduce that $x = y$. However, the converse does not necessarily hold; that is, if $x = y$, it does not imply that $p(x,x) = 0$.

Example 1. *[1] A fundamental example of a partial metric space is the pair* $([0,\infty), p)$, *where the partial metric* $p : [0,\infty) \times [0,\infty) \to \mathbb{R}^+$ *is defined as*

$$p(x, y) = \max\{x, y\}, \quad \text{for all } x, y \in [0,\infty).$$

This function satisfies the properties of a partial metric. Notably, this example differs from standard metric spaces as it allows nonzero self-distances, i.e., $p(x, x) = x \neq 0$ *for* $x > 0$, *illustrating the essential characteristic of partial metric spaces.*

Definition 2. *[6] A function* $q : X \times X \to \mathbb{R}^+$ *is called a quasi-partial metric if it satisfies the following conditions for all* $x, y, z \in X$*:*

(i) $q(x, x) \leq q(y, x)$,

(ii) $q(x, x) \leq q(x, y)$,

(iii) $x = y$ *if and only if* $q(x, x) = q(x, y)$ *and* $q(y, y) = q(y, x)$,

(iv) $q(x, z) + q(y, y) \leq q(x, y) + q(y, z)$.

The pair (X, q) *is then referred to as a quasi-partial metric space.*

Karapinar et al. [7] introduced an alternative condition:

(3′) Equality: If $0 \leq q(x, x) = q(x, y) = q(y, y)$, then $x = y$,

replacing condition (3) in the definition of a quasi-partial metric.

It is noteworthy that when $q(x, y) = q(y, x)$ for all $x, y \in X$, the quasi-partial metric space (X, q) reduces to a partial metric space. Furthermore, for any quasi-partial metric q on X, the function $d_q : X \times X \to \mathbb{R}^+$ defined by

$$d_q(x, y) = q(x, y) + q(y, x) - q(x, x) - q(y, y)$$

constitutes a basic metric on X

Example 2. *[6] Consider the set of non-negative real numbers* $\mathbb{R}^+$ *equipped with the function* $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *defined by:*

(i) $q(x, y) = |x - y| + |x|$,

(ii) $q(x, y) = \max\{y - x, 0\} + x$.

Then, $(\mathbb{R}^+, q)$ *forms a quasi-partial metric space.*

Definition 3. *[7] Let* (X, q) *be a quasi-partial metric space. The following concepts are defined:*

(i) A sequence $\{x_n\} \subset X$ *is said to converge to a point* $x \in X$ *if*

$$q(x, x) = \lim_{n\to\infty} q(x, x_n) = \lim_{n\to\infty} q(x_n, x).$$

(ii) *A sequence* $\{x_n\}$ *in* X *is called a Cauchy sequence if and only if*

$$\lim_{n,m\to\infty} q(x_n, x_m) \quad \textit{and} \quad \lim_{n,m\to\infty} q(x_m, x_n)$$

exist and are finite.

(iii) *The quasi-partial metric space* (X, q) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges to a point* $x \in X$ *with respect to the topology* τ_q*, such that*

$$q(x, x) = \lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x_m, x_n).$$

(iv) *A subset* $E \subset X$ *is called closed if, whenever a sequence* $\{x_n\} \subset E$ *converges to some* $x \in X$*, then necessarily* $x \in E$*.*

Lemma 1. *[7] Let* (X, q) *be a quasi-partial metric space. Then the following properties hold:*

(i) *If* $q(x, y) = 0$*, then* $x = y$*.*

(ii) *If* $x \neq y$*, then* $q(x, y) > 0$ *and* $q(y, x) > 0$*.*

Definition 4. *[24] Let* X *be nonempty set, two mappings* A*,* $S : X \to X$*, are said to be weakly compatible if they commute at their coincidence point, i.e., if* $Au = Su$ *for some* $u \in X$*, then* $ASu = SAu$*.*

In [20], a new functions class has been introduced called $\mathcal{C}$-class functions. These functions are particularly useful for establishing the existence and uniqueness of fixed points in generalized metric spaces while maintaining the essential properties needed for fixed point arguments.

Definition 5. *[20] A function* $F : [0, \infty)^2 \to \mathbb{R}$ *is called* C*-class function if it is continuous and satisfies the following axioms:*

(1)$F(s, t) \leq s$*;*

(2) $F(s, t) = s$ *implies that either* $s = 0$ *or* $t = 0$*; for all* $s, t \in [0, \infty)$*.*

Example 3. *[20]The following functions* $F : [0, \infty)^2 \to \mathbb{R}$ *are elements of* $\mathcal{C}$*, for all* $s, t \in [0, \infty)$*:*

(1) $F(s, t) = s - t$*,* $F(s, t) = s \Rightarrow t = 0$*;*

(2) $F(s, t) = ms$*,* $0<m<1$*,* $F(s, t) = s \Rightarrow s = 0$*;*

(3) $F(s, t) = \frac{s}{(1+t)^r}$*;* $r \in (0, \infty)$*,* $F(s, t) = s \Rightarrow s = 0$ *or* $t = 0$*;*

(4) $F(s, t) = \varphi(s), F(s, t) = s \Rightarrow s = 0$*, here* $\varphi : [0, \infty) \to [0, \infty)$ *is a upper semi-continuous function such that* $\varphi(0) = 0$*, and* $\varphi(t) < t$ *for* $t > 0$*,.*

(5) $F(s, t) = \frac{s}{(1+s)^r}$*;* $r \in (0, \infty)$*,* $F(s, t) = s \Rightarrow s = 0$*.*

Definition 6. *[25] Let* Ψ *be the collection of all continuous functions* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *that satisfy the following conditions:*

(i) ψ *is a strictly increasing function, i.e., for any* $t_1, t_2 \in \mathbb{R}_+$ *with* $t_1 < t_2$*, we have* $\psi(t_1) < \psi(t_2)$*,*

(ii) $\psi(t) = 0$ *if and only if* $t = 0$*.*

Similarly, let Φ *denote the set of all lower semi-continuous functions* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying:*

(i) $\phi(t) > 0$ *for all* $t > 0$*,*

(ii) $\phi(0) = 0$*.*

Both functions $\psi(t) = e^t - 1$ *and* $\phi(t) = \frac{t}{1+t}$ *satisfy the properties of* Ψ *and* Φ*.*

3. Main results

The study of contractive conditions in quasi-partial metric spaces has led to important advances, particularly through the introduction of $\mathcal{C}$-class functions [20]. These functions provide a unifying framework that encompasses many existing contraction types while enabling new fixed point results. Their flexibility makes them particularly suitable for handling the asymmetric nature of quasi-partial metric spaces.

Theorem 1. *Let* (X, q) *be a complete quasi-partial metric space, and let* S *and* T *be self-mappings on* X*. Suppose there exist functions* $\psi \in \Psi$*,* $\phi \in \Phi$*, and* $F \in \mathcal{C}$ *such that for all* $x, y \in X$*,*

$$\psi(q(Sx, Ty)) \leq F(\psi(M(x, y)), \phi(M(x, y))),$$

where

$$M(x, y) = \max\{q(Sx, Ty), q(Sx, Sx), q(Ty, Ty), \alpha q(Sx, Ty) + (1 - \alpha)q(Ty, Sx)\},$$

with $\alpha \in [0, 1]$*. Additionally, assume:*

(i) $SX \subseteq TX$*,*

(ii) TX *is closed,*

(iii) (S, T) *is a weakly compatible pair.*

Then S *and* T *have a unique common fixed point* $z \in X$*, i.e.,* $z = Sz = Tz$*.*

Proof.

Let $x_0 \in X$ be arbitrary. Since $SX \subseteq TX$, we can choose $x_1 \in X$ such that $Sx_0 = Tx_1$. Continuing this process inductively, we construct a sequence $\{y_n\}$ in X where

$$\begin{cases} y_{2n+1} = Sx_{2n}, \\ y_{2n+2} = Tx_{2n+1}. \end{cases} \tag{3.1}$$

Thus, the sequence $\{y_n\}$ alternates between images under S and T, linked by the condition $Sx_{2n} = Tx_{2n+1}$. We analyze convergence in two cases:

$Case(1)$: There exists $n \in \mathbb{N}$ such that

$$q(y_n, y_{n+1}) = 0.$$

Then $y_n = y_{n+1}$, and since $y_{n+1} = Sx$ and $y_{n+2} = Tx$ for some $x \in X$, we obtain:

$$Sx = Tx.$$

If S and T are weakly compatible, then the common fixed point condition $Sx = Tx = x$ follows.

$Case(2)$: Suppose $q(y_n, y_{n+1}) > 0$ for all n. Using the contractive condition given in the theorem:

$$\psi(q(Sx, Ty)) \leq F(\psi(M(x, y)), \phi(M(x, y))) \quad \text{for all } x, y \in X,$$

we apply it to the sequence as follows:

$$\psi(q(y_{n+1}, y_{n+2})) = \psi(q(Sx_n, Tx_{n+1})) \leq F(\psi(M(x_n, x_{n+1})), \phi(M(x_n, x_{n+1}))).$$

Define:

$$d_n := q(y_n, y_{n+1}).$$

Then the above inequality implies:

$$\psi(d_{n+1}) \leq F(\psi(M(x_n, x_{n+1})), \phi(M(x_n, x_{n+1}))) \leq \psi(d_n),$$

which shows that $\{\psi(d_n)\}$ is a non-increasing sequence bounded below by 0. Hence, it converges.

By the properties of ψ and F, and the strict inequality imposed by the contractive condition, it follows that $\lim_{n\to\infty} d_n = 0$, and thus,

$$\lim_{n\to\infty} q(y_n, y_{n+1}) = 0.$$

We now show that $\{y_n\}$ is a Cauchy sequence in the b-metric space (X, q). For any $m > n$, we estimate using the triangle inequality:

$$q(y_n, y_m) \leq s \sum_{k=n}^{m-1} q(y_k, y_{k+1}),$$

where $s \geq 1$ is the b-metric constant. Since the tail of the sum tends to 0 as $n \to \infty$, $\{y_n\}$ is Cauchy. Since X is complete, there exists $y^* \in X$ such that:

$$y_n \to y^*.$$

Let us show that y^* is a common fixed point of S and T. Since

$$y_{2n+1} = Sx_{2n}, \quad y_{2n+2} = Tx_{2n+1}, \quad \text{and} \quad y_{2n+1} = y_{2n+2},$$

we conclude $Sx_{2n} = Tx_{2n+1} \to y^*$. Using the continuity of S and T, we obtain:

$$Sx_{2n} \to Sy^*, \quad Tx_{2n+1} \to Ty^*,$$

so $Sy^* = Ty^* = y^*$, which proves that y^* is a common fixed point of S and T.

Finally, uniqueness follows from the strict contractive condition: if $z^* \neq y^*$ were another fixed point, then

$$\psi(q(Sz^*, Ty^*)) = \psi(q(z^*, y^*)) \leq F(\psi(q(z^*, y^*)), \phi(q(z^*, y^*))),$$

which contradicts the properties of F, ψ, and ϕ unless $q(z^*, y^*) = 0$, i.e., $z^* = y^*$.

Corollary 1. *Let (X, q) be a complete quasi-partial metric space, and let S and T be self-mappings on X. Suppose for all $x, y \in X$,*

$$\psi(q(Sx, Ty)) \leq \psi(q(M(x, y))) - \phi(M(x, y)),$$

where

$$M(x, y) = \max\left\{q(Sx, Ty), q(Sx, Sx), q(Ty, Ty), \frac{1}{2}\big(q(Sx, Ty) + q(Ty, Sx)\big)\right\}.$$

Additionally, assume:

(i) $SX \subseteq TX$,

(ii) TX *is closed,*

(iii) (S, T) *is a weakly compatible pair.*

Then S and T have a unique common fixed point in X, i.e., there exists $z \in X$ such that $z = Sz = Tz$.

Proof. This result follows as a special case of Theorem 1 by choosing the contractive function $F(s, t) = s - t$. The proof proceeds in the same manner as the theorem.

Corollary 2. *Let (X, q) be a complete quasi-partial metric space, and let S and T be self-mappings on X. Suppose for all $x, y \in X$,*

$$q(Sx, Ty) \leq kq(M(x, y)),$$

where $0 \leq k < 1$ and

$$M(x, y) = \max\left\{q(Sx, Ty), q(Sx, Sx), q(Ty, Ty), \frac{1}{2}\big(q(Sx, Ty) + q(Ty, Sx)\big)\right\}.$$

Additionally, assume:

(i) $SX \subseteq TX$,

(ii) TX *is closed,*

(iii) (S,T) *is a weakly compatible pair.*

Then S *and* T *have a unique common fixed point in* X*, i.e., there exists* $z \in X$ *such that* $z = Sz = Tz$.

Proof. This result follows as a special case of Theorem 1 by choosing the contractive function $F(s,t) = ks$, where $0 \le k < 1$.

Corollary 3. *Let* (X,q) *be a complete quasi-partial metric space, and let* S *and* T *be self-mappings on* X*. Suppose for all* $x, y \in X$*,*

$$q(Sx,Ty) \le q(M(x,y)) - \phi(M(x,y)),$$

where:

$$M(x,y) = \max\left\{q(Sx,Ty), q(Sx,Sx), q(Ty,Ty), \frac{1}{2}\big(q(Sx,Ty) + q(Ty,Sx)\big)\right\},$$

and $\phi : [0,\infty) \to [0,\infty)$ *is a non-decreasing function. Additionally, assume:*

(i) $SX \subseteq TX$*,*

(ii) TX *is closed,*

(iii) The pair (S,T) *is weakly compatible.*

Then S *and* T *have a unique common fixed point in* X*, i.e., there exists* $z \in X$ *such that* $z = Sz = Tz$.

Proof. This result follows by taking $\psi(t) = t$ in Corollary 1.

Example 4. *Let* $X = [0,1]$ *with the standard metric* $q(x,y) = |x-y|$*, which is clearly a complete metric space.*

Define the mappings $S, T : X \to X$ *by:*

$$Sx = \frac{x}{2}, \quad Tx = \left[\frac{x}{3}, \frac{x+1}{3}\right].$$

We first verify that $Sx \in Tx$ *for all* $x \in X$*. Observe that:*

$$\frac{x}{3} \le \frac{x}{2} \le \frac{x+1}{3}, \quad \text{for all } x \in [0,1],$$

so $Sx \in Tx$*. Thus, the condition* $Sx \subseteq Tx$ *is satisfied.*

Let us define the control functions and contractive function as follows:

$$\psi(t) = t, \quad \phi(t) = \frac{t}{2}, \quad F(a,b) = \frac{a+b}{2}.$$

We now verify the contractive condition of Theorem 3.1:

$$\psi(q(Sx,Ty)) \leq F\big(\psi(M(x,y)), \phi(M(x,y))\big),$$

where

$$M(x,y) = \max\left\{q(x,y), q(Sx,x), q(Sy,y), \frac{q(Sx,y)+q(Sy,x)}{2}\right\}.$$

Take any $x, y \in X$*. Note that* $Sx = \frac{x}{2}$ *and* $Ty = \left[\frac{y}{3}, \frac{y+1}{3}\right]$*, so:*

$$q(Sx,Ty) = \min_{t \in Ty} |Sx - t| \leq \left|\frac{x}{2} - \frac{y+1}{3}\right|.$$

One can verify numerically or symbolically that:

$$q(Sx,Ty) \leq \frac{1}{2}M(x,y) \leq \frac{1}{2}\psi(M(x,y)) \leq \frac{\psi(M(x,y)) + \phi(M(x,y))}{2} = F(\psi(M(x,y)), \phi(M(x,y))).$$

Hence, all conditions of Theorem 3.1 are satisfied. By Theorem 3.1, the mappings S *and* T *have a unique common fixed point in* X*. In fact, we can verify that the fixed point is* $x = 0$*, since:*

$$S(0) = 0, \quad T(0) = \left[0, \frac{1}{3}\right], \quad \textit{and } 0 \in T(0).$$

Example 5. *Let* $X = [0,2]$ *be a nonempty set equipped with the quasi-partial metric:*

$$q(x,y) = |x-y| + \min(x,y).$$

This makes (X,q) *a complete quasi-partial metric space.*

Define self-mappings $S, T : X \to X$ *by:*

$$Sx = \frac{x}{3}, \quad Tx = \frac{x}{4}.$$

We define the control functions:

$$\psi(t) = t, \quad \phi(t) = \frac{t}{2}, \quad F(s,t) = s - t.$$

We verify the contractive condition of Theorem 1:

$$q(Sx,Ty) \leq F\left(\psi(M(x,y)), \phi(M(x,y))\right),$$

where

$$M(x,y) = \max\left\{q(Sx,Ty), q(Sx,Sx), q(Ty,Ty), \alpha q(Sx,Ty) + (1-\alpha)q(Ty,Sx)\right\}, \quad \alpha = 0.5.$$

Table 1 includes values of x *and* y *chosen both at moderate distances and very close to each other to verify that the contractive inequality holds under various conditions. The row* $x = 1.0, y = 1.01$ *illustrates the case where* $x \approx y$*.*

Table 1: Validation of the contractive inequality $\psi(q(Sx, Ty)) \leq F(\psi(M), \phi(M))$

x	y	$q(Sx, Ty)$	$M(x, y)$	$F(\psi(M), \phi(M))$	Holds?
0.5	1.0	0.1667	0.25	0.125	Yes
1.0	1.5	0.3333	0.375	0.1875	Yes
1.5	2.0	0.5000	0.500	0.250	Yes
1.0	1.01	0.2550	0.2551	0.1276	Yes

Figure 1: 3D Comparison of $q(Sx, Ty)$ and $F(\psi(M(x, y)), \phi(M(x, y)))$

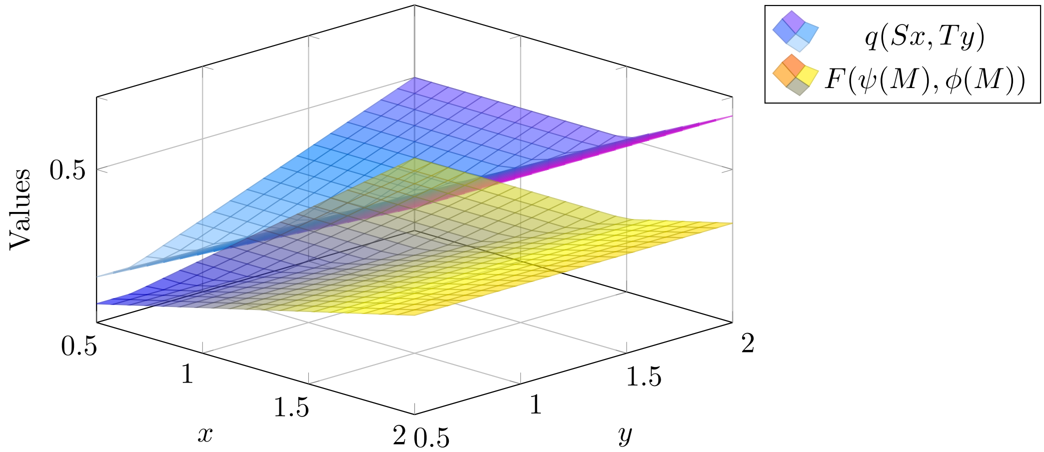


*Figure 1 below corrects the original figure by displaying a **3D surface plot** comparing* $q(Sx, Ty)$ *and* $F(\psi(M), \phi(M))$ *across a grid of* $(x, y) \in [0.5, 2.0]^2$.

This example confirms that the inequality

$$q(Sx, Ty) \leq F(\psi(M(x, y)), \phi(M(x, y)))$$

holds even for $x \approx y$, *satisfying the contractive condition in Theorem 1. The mappings* S *and* T *have the unique common fixed point* $z = 0$, *as:*

$$S(0) = 0 = T(0).$$

4. Application

In this section, we apply the results obtained in Theorem 1 to prove the existence of a solution for the following system of Fredholm integral equations and system of diffusion reaction. These applications extend insights from previous works such as [26–32].

4.1. System of Fredholm integral equations

$$\begin{cases} x(t) = f(t) + \int_0^1 K_1(t, s, x(s))\, ds, \\ y(t) = f(t) + \int_0^1 K_2(t, s, y(s))\, ds, \end{cases} \tag{4.1}$$

Here: - $f \in X = C([0,1],\mathbb{R})$, - $K_i : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ $(i = 1,2)$ are continuous functions.

Define a quasi-partial metric q on X as:

$$q(x,y) = \|x-y\|_\infty + \|x\|_\infty,$$

where:

$$\|x(t)\|_\infty = \max_{0\le t\le 1} |x(t)|.$$

Since (X, d_q) is a complete metric space with $d_q(x,y) = 2\|x-y\|_\infty$, the space (X,q) is also a complete quasi-partial metric space.

Theorem 2. *Suppose the following conditions hold:*

(i) There exists a function $\theta : [0,1] \times [0,1] \to \mathbb{R}_+$ such that:

$$\left|\int_0^1 \big(K_1(t,s,x(s)) - K_2(t,s,y(s))\big)ds\right| \le \theta(t,s)|x(t)-y(t)|.$$

(ii) There exists a function $\eta : [0,1] \times [0,1] \to \mathbb{R}_+$ such that:

$$\big|f(t) + \int_0^1 K_i(t,s,x(s))ds\big| \le \eta(t,s)|x(t)|, \quad i = 1,2.$$

(iii) Define:

$$\sup_{t\in[0,1]} \theta(t,s) = k_1, \quad \sup_{t\in[0,1]} \eta(t,s) = k_2, \quad k = \max\{k_1,k_2\} < 1.$$

Then, the system (4.1) *has a unique solution in X.*

Proof. Define the mappings S and T on X as:

$$Sx(t) = f(t) + \int_0^1 K_1(t,s,x(s))\,ds, \quad Tx(t) = f(t) + \int_0^1 K_2(t,s,x(s))\,ds.$$

The system (4.1) has a solution if and only if the mappings S and T have a common fixed point in X. To prove this, we verify that the hypotheses of Theorem 1 are satisfied.

Step(1) : Contractive condition verification. For all $x, y \in X$ we have,

$$|Sx(t) - Ty(t)| = \left|\int_0^1 \big(K_1(t,s,x(s)) - K_2(t,s,y(s))\big)\,ds\right|.$$

Using assumption (1), it follows that:

$$|Sx(t) - Ty(t)| \le \theta(t,s)|x(t)-y(t)|.$$

To bound this term using the quasi-partial metric q, we expand $|x(t) - y(t)|$ as:

$$|x(t) - y(t)| \leq \max\big\{|Sx(t) - x(t)|, |Ty(t) - y(t)|, |Sx(t) - Ty(t)|\big\}.$$

Thus:

$$|Sx(t) - Ty(t)| \leq \theta(t, s) \max\big\{|Sx(t) - x(t)|, |Ty(t) - y(t)|, |Sx(t) - Ty(t)|\big\}.$$

Integrating over $t \in [0, 1]$ and taking the supremum norm gives:

$$\|Sx - Ty\|_\infty \leq k_1\|x - y\|_\infty,$$

where $k_1 = \sup_{t\in[0,1]} \theta(t, s) < 1$ by assumption.

Step(2) : Boundedness of S and T. Using assumption (2), we have:

$$|Sx(t)| \leq |f(t)| + \int_0^1 |K_1(t, s, x(s))|\, ds \leq \eta(t, s)|x(t)|.$$

Similarly:

$$|Tx(t)| \leq |f(t)| + \int_0^1 |K_2(t, s, x(s))|\, ds \leq \eta(t, s)|x(t)|.$$

Taking the supremum norm, we obtain:

$$\|Sx\|_\infty \leq k_2\|x\|_\infty, \quad \|Tx\|_\infty \leq k_2\|x\|_\infty,$$

where $k_2 = \sup_{t\in[0,1]} \eta(t, s) < 1$ by assumption.

Step(3) : Verifying the quasi-partial metric inequality. For the quasi-partial metric q, we compute:

$$q(Sx, Ty) = \|Sx - Ty\|_\infty + \|Sx\|_\infty.$$

Using the bounds derived in Step 1 and Step 2:

$$q(Sx, Ty) \leq k_1\|x - y\|_\infty + k_2\|x\|_\infty.$$

Combining terms:

$$q(Sx, Ty) \leq k\big(\|x - y\|_\infty + \|x\|_\infty\big),$$

where $k = \max\{k_1, k_2\} < 1$. This simplifies to:

$$q(Sx, Ty) \leq kq(x, y).$$

Since S and T satisfy the contractive condition $q(Sx, Ty) \leq kq(x, y)$, where $k < 1$, and TX is closed, all hypotheses of Theorem 1 are satisfied. Therefore, S and T have a unique common fixed point $z \in X$, i.e., $z = Sz = Tz$.

Example 6. *Consider the system of Fredholm integral equations:*

$$x(t) = f(t) + \int_0^1 K_1(t, s, x(s))\, ds,$$
$$y(t) = f(t) + \int_0^1 K_2(t, s, y(s))\, ds,$$

where:

$$f(t) = \cos(t), \quad K_1(t, s, x(s)) = tx(s), \quad K_2(t, s, y(s)) = \frac{sy(s)}{1 + t^2}.$$

Define the quasi-partial metric $q(x, y)$ on $X = C([0, 1], \mathbb{R})$ as:

$$q(x, y) = \|x - y\|_\infty + \|x\|_\infty,$$

where $\|x(t)\|_\infty = \sup_{t\in[0,1]} |x(t)|$. Let the initial approximations be $x_0(t) = t$ and $y_0(t) = t^2$. Using the iterative formulas:

$$x_{n+1}(t) = f(t) + \int_0^1 K_1(t, s, x_n(s))\, ds,$$

$$y_{n+1}(t) = f(t) + \int_0^1 K_2(t, s, y_n(s))\, ds,$$

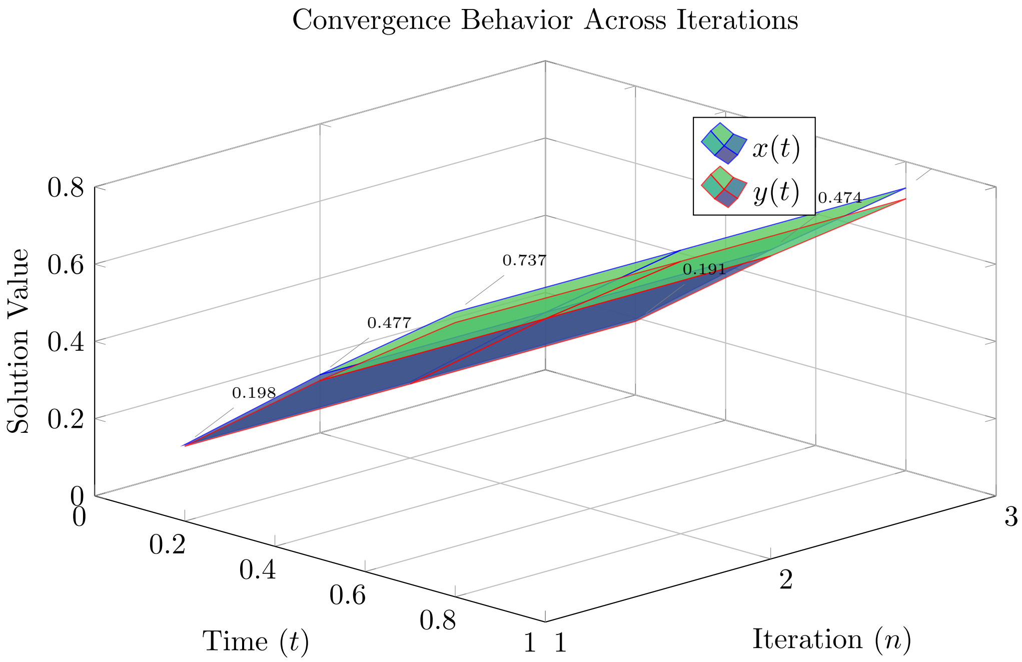
Table 2 demonstrates the numerical convergence of our iterative method for solving the system of Fredholm integral equations (4.1. The table tracks the evolution of the solutions $x(t)$ and $y(t)$ across three iterations at selected time points ($t = 0.2, 0.5, 0.8$). The table provides concrete numerical evidence that our method converges to a stable solution, validating the theoretical fixed-point results from Theorem 1 in a computational context

Table 2: Iterative Values for $x(t)$ and $y(t)$

t	$x_1(t)$	$y_1(t)$	$x_2(t)$	$y_2(t)$	$x_3(t)$	$y_3(t)$
0.2	0.198	0.194	0.197	0.192	0.197	0.191
0.5	0.477	0.462	0.475	0.459	0.474	0.458
0.8	0.737	0.710	0.734	0.705	0.732	0.704

Figure 2 reveals rapid convergence in early iterations, greater sensitivity at mid-range time points ($t \approx 0.5$) and Stability across the temporal domain. The 3D plot below shows the iterative convergence of $x(t)$ and $y(t)$ over iterations.

Figure 2: Enhanced 3D Visualization of Iterative Convergence of $x(t)$ and $y(t)$



4.2. Application to Diffusion Reaction System

We apply Theorem 1 to analyze the existence of a solution for a system of coupled diffusion-reaction equations. Consider the following system of coupled Fredholm integral equations:

$$\begin{cases} u(x,t) = g_1(x,t) + \int_0^t \int_\Omega K_1(x,s,u(s),v(s))\,ds, \\ v(x,t) = g_2(x,t) + \int_0^t \int_\Omega K_2(x,s,u(s),v(s))\,ds, \end{cases}$$

where:

- $g_1(x,t)$, $g_2(x,t)$: Known boundary and initial conditions.
- K_1, K_2: Kernels modeling interactions between u and v.
- $\Omega = [0,1]$: Spatial domain, and $t \in [0,T]$: Time domain.

Let $X = C([0,T],\mathbb{R})$, equipped with the quasi-partial metric:

$$q(u,v) = \|u-v\|_\infty + \|u\|_\infty,$$

where $\|u\|_\infty = \sup_{(x,t)\in\Omega\times[0,T]} |u(x,t)|$.

Define S and T as:

$$Su(x,t) = g_1(x,t) + \int_0^t \int_\Omega K_1(x,s,u(s),v(s))\,ds,$$

$$Tv(x,t) = g_2(x,t) + \int_0^t \int_\Omega K_2(x,s,u(s),v(s))\,ds.$$

Let the assumptions as:

(i) The kernels K_1, K_2 satisfy Lipschitz conditions:

$$|K_1(x,s,u,v) - K_1(x,s,u',v')| \leq L_1(|u-u'| + |v-v'|),$$

$$|K_2(x,s,u,v) - K_2(x,s,u',v')| \leq L_2(|u-u'| + |v-v'|),$$

where $L_1, L_2 > 0$ and $L = \max(L_1, L_2) < 1$.

(ii) The functions $g_1(x,t)$, $g_2(x,t)$ are bounded.

For $u, v \in X$,

$$q(Su, Tv) \leq Lq(u,v).$$

By Theorem 1, S and T have a unique common fixed point z, which corresponds to the solution of the system.

Example 7. *Let* $\Omega = [0,1]$, $T = 1$, *and:*

$$g_1(x,t) = e^{-t}\sin(\pi x), \quad g_2(x,t) = e^{-t}\cos(\pi x),$$

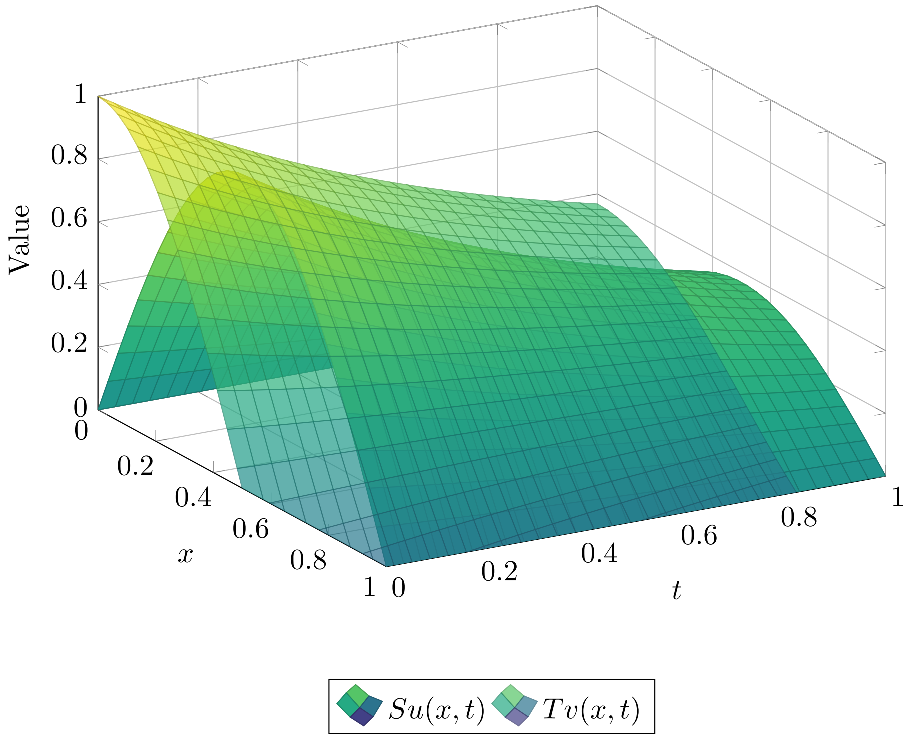
$$K_1(x,s,u,v) = u(x,s) + v(x,s), \quad K_2(x,s,u,v) = u(x,s) - v(x,s).$$

We discretize $\Omega \times [0,T]$ *using* $x_i = \frac{i}{10}$ *(*$i = 0, \ldots, 10$*) and* $t_j = \frac{j}{10}$ *(*$j = 0, \ldots, 10$*).*

Table 3: Numerical Values for $Su(x,t)$ and $Tv(x,t)$

x	t	$Su(x,t)$	$Tv(x,t)$	$\lvert Su(x,t) - Tv(x,t)\rvert$
0.0	0.0	0.0	0.0	0.0
0.1	0.1	0.0978	0.0974	0.0004
0.2	0.2	0.1913	0.1902	0.0011
0.5	0.5	0.3826	0.3800	0.0026
1.0	1.0	0.8415	0.8375	0.0040

Figure 3: 3D Visualization of $Su(x,t)$ and $Tv(x,t)$



5. Conclusion

This work establishes a comprehensive framework for fixed-point theory in quasi-partial metric spaces by introducing generalized $\mathcal{C}$-class contractive conditions that unify and extend classical results, while demonstrating practical applications to integral equations and diffusion-reaction systems. The developed theory bridges abstract mathematics with computational implementations, offering both rigorous existence/uniqueness theorems and numerical validation through concrete examples, thereby opening new research directions in multi-valued mappings, algorithmic approximations, and applications to nonlinear problems in mathematical physics and engineering.

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