EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 5921
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On the Generalization of the Bivariate extended Standard U-quadratic Distribution

Idzhar A. Lakibul[1,*], Daisy Lou L. Polestico[2,3], Arnulfo P. Supe[2,3]

[1] *Department of Mathematics, College of Science and Mathematics, Mindanao State University - Sulu, Jolo, Sulu, Philippines*
[2] *Department of Mathematics and Statistics, Mindanao State University - Iligan Institute of Technology, Iligan City, Philippines*
[3] *Premier Research Institute of Science and Mathematics - Center for Computational Analytics and Modeling, Mindanao State University - Iligan Institute of Technology, Iligan City, Philippines*

Abstract. This paper derives the generalized version of the Bivariate extended standard U-quadratic distribution using the compounding method. The joint probability and cumulative distribution functions of the derived distribution are obtained and it is observed that the said distribution can generate bivariate shape distributions with the following properties: (i) X and Y have bathtub shapes; (ii) X and Y have inverted bathtub shapes; (iii) X and Y have constant shapes; (iv) X has a constant distribution and Y has a bathtub shape; (v) X has a constant distribution and Y has inverted bathtub shape; and (vi) X has an inverted bathtub distribution and Y has a bathtub shape. Moreover, two special cases of the generalized BeSU distribution are constructed, and these are called the Special Bivariate extended Standard U-Quadratic Type I (SBeSU-Type I) and Special Bivariate extended Standard U-Quadratic Type II (SBeSU-Type II) distributions. Further, some properties of this proposed distribution are derived such as the marginal distribution, conditional distribution, conditional moments, conditional mean, conditional variance, product and ratio moments, Pearson correlation coefficient, joint moment generating function, Kendall's tau coefficient, Spearman's rho, and the stress - strength parameter. In addition, maximum likelihood estimation is performed to estimate the parameters of the derived distribution. A simulation study is carried out to evaluate the behavior of the parameter estimates. Finally, the proposed generalization of the Bivariate eSU distribution is applied to simulated data and compared with the Bivariate extended Standard U-quadratic distribution. The results show that the proposed generalization of the bivariate eSU distribution provides a better fit on the simulated data set than the bivariate eSU distribution.

2020 Mathematics Subject Classifications: 60E05, 62E10, 65C10
Key Words and Phrases: Standard U-quadratic distribution, Kumaraswamy distribution, Bivariate distribution, Bivariate extended Standard U-quadratic Distribution, bathtub shape distribution

---

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.5921

Email addresses: idzhar.lakibul@msusulu.edu.ph (I. A. Lakibul), daisylou.polestico@g.msuiit.edu.ph (D. L. Polestico), arnulfo.supe@g.msuiit.edu.ph (A. P. Supe)

1. Introduction

A prominent area of research in distribution theory is the generalization of univariate distributions to their corresponding bivariate cases. A bivariate distribution is useful for modeling the relationship between two random variables.

Filus and Filus [1] introduced a method for generating bivariate or multivariate distributions using linear combinations of random variables. Specifically, they derived the pseudo-Weibull and pseudo-gamma distributions as linear combinations of Weibull and gamma random variables, respectively. Shahbaz et al. [2] developed a bivariate exponential distribution as the compound distribution of two exponential random variables. In addition to the bivariate exponential distribution, many known bivariate distributions were derived using this, such as the bivariate pseudo-Weibull distribution [3], bivariate pseudo-Rayleigh distribution [4], bivariate pseudo-inverse Rayleigh distribution [5], bivariate pseudo-Gumbel distribution [6], bivariate inverse exponential distribution [7], among others.

Lakibul and Tubo [8] formed a probability distribution in the interval $[0, 1]$ called the extended standard U-quadratic distribution, and it is given in the following definition.

A random variable X is said to have an extended Standard U-quadratic distribution denoted by "eSU" if the probability density function (pdf) of X is given by

$$f(x) = 1 - \lambda + 3\lambda(2x - 1)^2, x \in [0, 1], \tag{1}$$

where $\lambda \in [-0.5, 1]$. It was investigated that this distribution can form three different types of shapes namely, the inverted bathtub for $\lambda \in [-0.5, 0)$, constant for $\lambda = 0$ and bathtub for $\lambda \in (0, 1]$. Moreover, several properties of this distribution are detailed in the paper of Lakibul and Tubo [9]. Furthermore, the generalized version of this distribution is given in the paper of Lakibul, Polestico and Supe [10]. This distribution can be used as an option to the Beta distribution and the Kumaraswamy [11] distribution to model data with support on $[0, 1]$, particularly those data that follow the bathtub, the inverted bathtub, and constant behavior.

Lakibul, Polestico and Supe [12] expanded the eSU distribution into the Bivariate extended Standard U-quadratic Distribution (BeSU), and it is define in the following statement.

A bivariate random vector (X, Y) is said to have a Bivariate extended Standard U-quadratic (BeSU) distribution if the joint pdf of X and Y is given by

$$f(x, y) = \left[1.5 - 1.5x + 3(1.5x - 0.5)(2y - 1)^2\right] \left[1 - \lambda + 3\lambda(2x - 1)^2\right], \tag{2}$$

where $0 \leq (x, y) \leq 1$ and $\lambda \in [-0.5, 1]$. It was observed that this BeSU distribution can describe three different types of bivariate shapes, namely, X and Y have bathtub shapes, X has a constant distribution and Y has a bathtub shape, and X has an inverted bathtub and Y has a bathtub shape. However, this distribution cannot model the bivariate shape distribution with the following properties: (i) X and Y have inverted bathtub shapes; (ii) X and Y have constant shapes; and (iii) X has a constant distribution and Y has inverted shape.

In this paper, we will use the idea of Shahbaz et al. [2] to generalize the extended standard bivariate U-quadratic distribution by adding an additional parameter to it to accommodate other combinations of the bivariate shape distribution. We will also derive some properties of the proposed generalized distribution such as the marginal distribution, conditional distribution, conditional moments, conditional mean, conditional variance, product and ratio moments, Pearson correlation coefficient, joint moment generating function, Kendall's tau coefficient, Spearman's rho, and the stress - strength parameter. Observation on the performance of the proposed generalized distribution is done by applying it on a simulated dataset.

The rest of the paper is structured as follows: Section 2 presents the construction of the proposed generalized Bivariate extended standard U-quadratic distribution. Section 3 provides derivations of some properties of the proposed generalized BeSU distribution. Section 4 discusses the maximum likelihood estimation for estimating the parameter of the proposed bivariate distribution. Section 5 deals with the random number generation of the proposed bivariate distribution. Section 6 presents the simulation results for assessing the behavior of the maximum likelihood estimate of the proposed bivariate distribution's parameter. Section 7 presents the application of the proposed bivariate distribution on a simulated dataset. Section 8 gives some concluding remarks about the paper and recommendations for future studies.

2. The ρ - Bivariate extended Standard U-quadratic distribution

This section presents the derivation of the generalized bivariate eSU distribution called as the ρ - Bivariate extended Standard U-quadratic (ρ-BeSU) Distribution.

Let X be a random variable that follows an extended Standard U-quadratic distribution with pdf given in Equation (1). Let Y be another random variable such that the conditional probability density function of Y given $X = x$ follows an extended Standard U-quadratic distribution, that is,

$$f(y|x) = 1 - v(x) + 3v(x)(2y-1)^2, y \in [0,1], \tag{3}$$

where $v(x) \in [-0.5, 1]$. Following the idea of Shahbaz [2] and by the definition of the conditional probability, the joint probability distribution function of X and Y is given by

$$\begin{aligned} f(x,y) =& f(y|x)f(x) \\ =& \left[1 - v(x) + 3v(x)(2y-1)^2\right]\left[1 - \lambda + 3\lambda(2x-1)^2\right]. \end{aligned} \tag{4}$$

Note that $v(x) \in [-0.5, 1]$ can be defined in many ways. In this paper, we define $v(x) = 1.5x^\rho - 0.5$, $\rho \geq 0$ for $x \in [0,1]$, which reduces to $v(x)$ defined in the construction of the simple BeSU distribution, when $\rho = 1$. Thus, the joint pdf of X and Y is given by

$$\begin{aligned} f(x,y) &= \left[1 - (1.5x^\rho - 0.5) + 3(1.5x^\rho - 0.5)(2y-1)^2\right]\left[1 - \lambda + 3\lambda(2x-1)^2\right] \\ &= \left[1.5 - 1.5x^\rho + 3(1.5x^\rho - 0.5)(2y-1)^2\right]\left[1 - \lambda + 3\lambda(2x-1)^2\right]. \end{aligned} \tag{5}$$

Definition 1. *A bivariate random vector (X,Y) is said to have a ρ - Bivariate extended Standard U-quadratic (ρ - BeSU) distribution if the joint pdf of X and Y is given by*

$$f(x,y)=\left[1.5-1.5x^{\rho}+3(1.5x^{\rho}-0.5)(2y-1)^2\right]\left[1-\lambda+3\lambda(2x-1)^2\right], \tag{6}$$

where $0\leq(x,y)\leq 1$, $\rho\geq 0$ and $\lambda\in[-0.5,1]$.

Remark 1. *If $\rho=1$, then ρ - BeSU distribution reduces to the BeSU distribution.*

Theorem 1. *Let (X,Y) be the bivariate random vector with joint pdf given in Equation (6), then the joint cdf of (X,Y) is given by*

$$F(x,y)=3y\left(2y^2-3y+1\right)M_{\rho}(x)-(2y-3)y^2F(x), \tag{7}$$

where

$$M_{\rho}(x)=\frac{(1+2\lambda)x^{\rho+1}}{\rho+1}-\frac{12\lambda x^{\rho+2}}{\rho+2}+\frac{12\lambda x^{\rho+3}}{\rho+3},$$

and

$$F(x)=(1+2\lambda)x-6\lambda x^2+4\lambda x^3.$$

Proof. The joint cumulative distribution function of X and Y is defined as

$$\begin{aligned}F(x,y)&=\int_0^x\int_0^y f(u,v)dvdu\\&=\int_0^x f(u)\left[\int_0^y f(v|u)dv\right]du.\end{aligned}$$

Observe that,

$$\begin{aligned}\int_0^y f(v|u)dv&=\int_0^y\left[1.5-1.5u^{\rho}+3(1.5u^{\rho}-0.5)(2v-1)^2\right]dv\\&=(1.5-1.5u^{\rho})\int_0^y dv+3\left(1.5u^{\rho}-0.5\right)\int_0^y\left(4v^2-4v+1\right)dv\\&=(1.5-1.5u^{\rho})\left(v\Big|_0^y\right)+3\left(1.5u^{\rho}-0.5\right)\left(4\frac{v^3}{3}\Big|_0^y-4\frac{v^2}{2}\Big|_0^y+v\Big|_0^y\right)\\&=(1.5-1.5u^{\rho})\,y+(1.5u^{\rho}-0.5)\left(4y^3-6y^2+3y\right)\\&=\left[1.5-1.5u^{\rho}+3\left(1.5u^{\rho}-0.5\right)\right]y+\left(1.5u^{\rho}-0.5\right)\left(4y^3-6y^2\right)\\&=3u^{\rho}y+\left(1.5u^{\rho}-0.5\right)\left(4y^3-6y^2\right)\\&=3u^{\rho}y+1.5u^{\rho}\left(4y^3-6y^2\right)-0.5\left(4y^3-6y^2\right)\\&=\left(6y^3-9y^2+3y\right)u^{\rho}-(2y^3-3y^2).\end{aligned}$$

Thus,

$$F(x,y)=\int_0^x\left(1-\lambda+3\lambda(2u-1)^2\right)\left[\left(6y^3-9y^2+3y\right)u^{\rho}-(2y^3-3y^2)\right]du$$

$$\begin{aligned} F(x,y) =& \left(6y^3 - 9y^2 + 3y\right) \int_0^x u^\rho f(u)du - (2y^3 - 3y^2) \int_0^x f(u)du \\ =& 3y\left(2y^2 - 3y + 1\right) M_\rho(x) - (2y-3)y^2 F(x), \end{aligned}$$

where

$$M_\rho(x) = \frac{(1+2\lambda)x^{\rho+1}}{\rho+1} - \frac{12\lambda x^{\rho+2}}{\rho+2} + \frac{12\lambda x^{\rho+3}}{\rho+3},$$

and

$$F(x) = (1+2\lambda)x - 6\lambda x^2 + 4\lambda x^3.$$

Corollary 1. *Let (X,Y) be a bivariate random variable with $\rho - BeSU$ distribution joint CDF given in Equation (7). If $\lambda = 0$, then the joint CDF of (X,Y) simplifies to*

$$F(x,y) = 3y\left(2y^2 - 3y + 1\right) \frac{x^{\rho+1}}{\rho+1} - (2y-3)y^2 x, \tag{8}$$

where $0 \leq (x,y) \leq 1$ and $\rho \geq 0$.

The proof follows easily from Theorem 1 by setting $\lambda = 0$ in Equation (7).

Corollary 2. *Let (X,Y) be a bivariate random variable with $\rho - BeSU$ distribution joint CDF given in Equation (7). If $\rho = 0$, then the joint CDF of (X,Y) is given by*

$$F(x,y) = \left(4y^2 - 6y + 3\right)\left(1 + 2\lambda - 6\lambda x + 4\lambda x^2\right) xy, \tag{9}$$

where $0 \leq (x,y) \leq 1$ and $\lambda \in [-0.5, 1]$.

The proof is straightforward by inserting $\rho = 0$ into Equation (7) of Theorem 1.

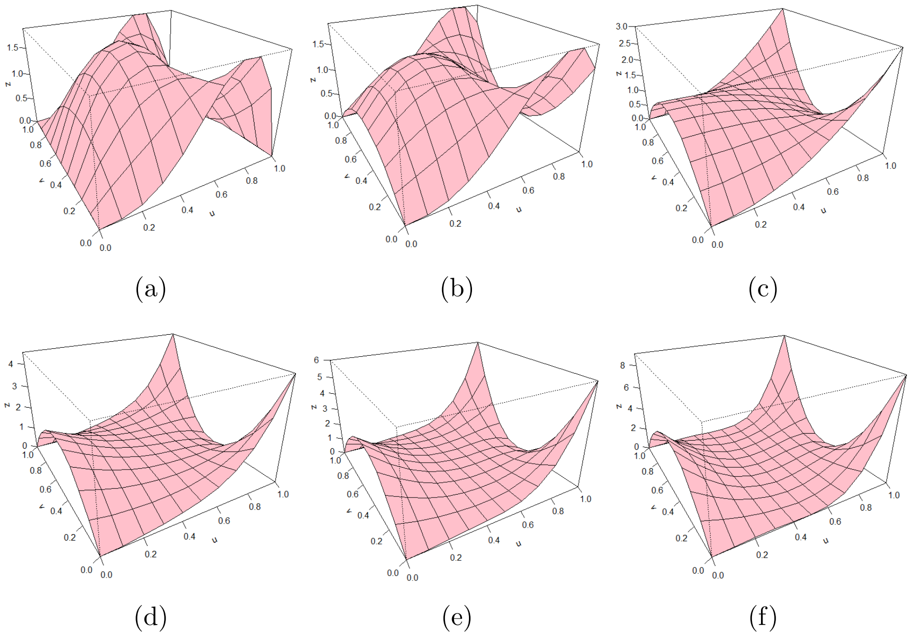


Figure 1: PDF plots of ρ-BeSU distribution for $\rho = 2$ and different values of λ: (a) $\lambda = -0.5$; (b) $\lambda = -0.25$; (c) $\lambda = 0$; (d) $\lambda = 0.25$; (e) $\lambda = 0.5$; and (f) $\lambda = 1$.

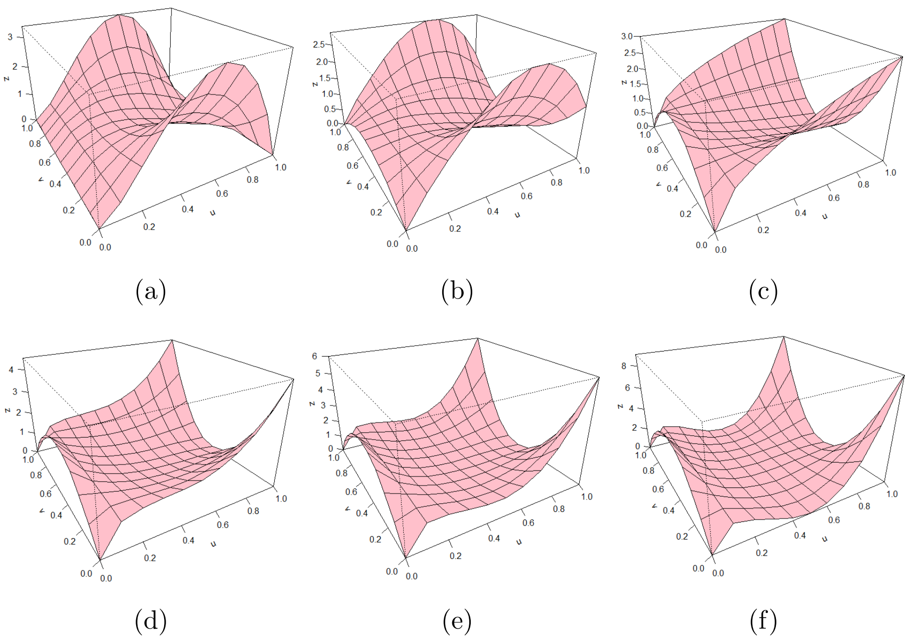


Figure 2: PDF plots of ρ - BeSU distribution for $\rho = 0.5$ and different values of λ: (a) $\lambda = -0.5$; (b) $\lambda = -0.25$; (c) $\lambda = 0$; (d) $\lambda = 0.25$; (e) $\lambda = 0.5$; and (f) $\lambda = 1$.

Figures 1 - 2 present the bivariate plots of the joint PDF of the ρ-BeSU distribution. It is observed from the said figures that the ρ-BeSU distribution can generate different bivariate behaviors such as combinations of bathtub and inverted bathtub shapes, bathtub and bathtub shapes, inverted bathtub and inverted bathtub shapes, among others.

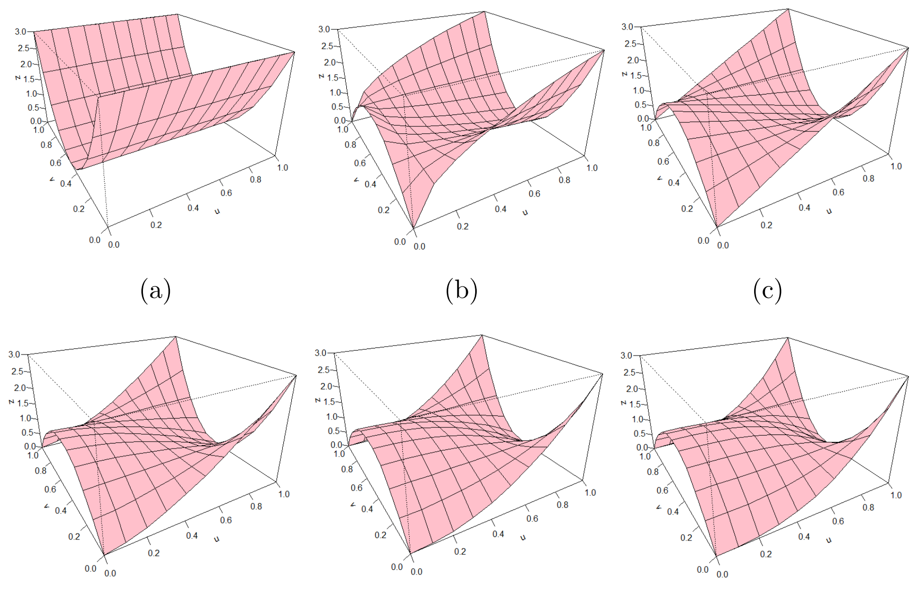
2.1. Special Cases of the ρ - BeSU distribution

This section presents two new special cases of the proposed generalized BeSU distribution.

1. If $\lambda = 0$, then the $\rho - BeSU$ distribution in Equation (6) reduces to

$$f(x, y) = \left[1.5 - 1.5x^{\rho} + 3(1.5x^{\rho} - 0.5)(2y - 1)^2\right]. \tag{10}$$

We refer to the PDF in Equation (10) the *special bivariate extended standard U-quadratic - Type I (SBeSU-Type I) distribution.*



(a) (b) (c)

(d) (e) (f)

Figure 3: PDF plots of SBeSU-Type I distribution for different values of ρ: (a) $\rho = 0$; (b) $\rho = 0.5$; (c) $\rho = 1$; (d) $\rho = 1.5$; (e) $\rho = 2$; and (f) $\rho = 2.5$.

Figure 3 presents the plots of the joint PDF of the SBeSU-Type I distribution for varying values of ρ. It is observed that the SBeSU-Type I distribution can represent bivariate behaviors with the following combination: (i) X has constant and Y has bathtub shapes; and (ii) X has constant and Y has inverted bathtub behaviors.

2. If $\rho = 0$, then the $\rho - BeSU$ distribution reduces to

$$f(x, y) = 3(2y-1)^2 \left[1 - \lambda + 3\lambda(2x-1)^2\right]. \tag{11}$$

Equation (11) is the PDF of the *Special Bivariate extended Standard U-quadratic Type - II (SBeSU - Type II) distribution.*

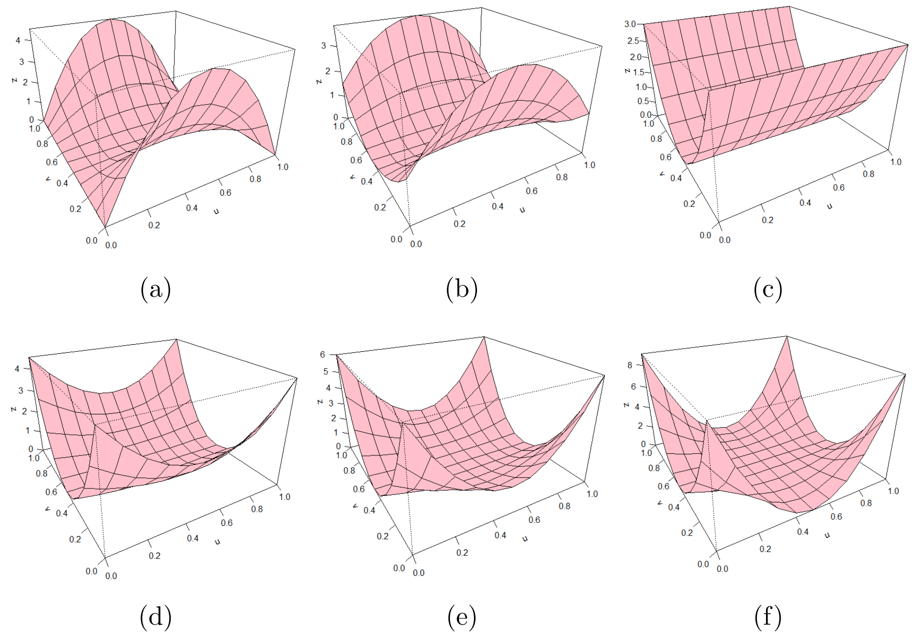


Figure 4: PDF plots of SBeSU-Type II distribution for different values of λ: (a) $\lambda = -0.5$; (b) $\lambda = -0.25$; (c) $\lambda = 0$; (d) $\lambda = 0.25$; (e) $\lambda = 0.5$; and (f) $\lambda = 1$.

Figure 4 presents the plots of the joint PDF of the SBeSU-Type II distribution for varying values of λ. It is observed that the SBeSU-Type II distribution can generate bivariate behaviors with the following combination: (i) X and Y have bathtub shapes; (ii) X has bathtub and Y has inverted bathtub behaviors; (iii) X has bathtub and Y has constant shapes.

3. Some properties of the ρ - Bivariate extended Standard U-quadratic distribution

This section presents some properties of the proposed generalized BeSU distribution such as the marginal distributions, conditional distribution, conditional moment, conditional mean, conditional variance, product and ratio moments, covariance, Pearson correlation, joint moment generating function, Kendall's tau coefficient, Spearman's rho coefficient, and the Stress - Strength parameter .

Theorem 2. *Let (X, Y) be a bivariate random vector with joint probability density function given in Equation (6). Then the marginal density function of Y follows an extended Standard U-quadratic (eSU) distribution with parameter $\lambda^* = 1.5\delta - 0.5$, where $\delta = \frac{(1+2\lambda)\rho^2+(5-2\lambda)\rho+6}{(\rho+1)(\rho+2)(\rho+3)}$, $\lambda \in [-0.5, 1]$ and $\rho \geq 0$.*

Proof. The marginal distribution of Y is defined as

$$\begin{aligned} f(y) =& \int_0^1 f(x,y)dx \\ =& \int_0^1 \left[1.5 - 1.5x^\rho + 3(1.5x^\rho - 0.5)(2y-1)^2\right] f(x)dx \\ =& 1.5\left[1-(2y-1)^2\right]\int_0^1 f(x)dx - 1.5\left[1-3(2y-1)^2\right]\int_0^1 x^\rho f(x)dx. \end{aligned} \tag{12}$$

Observe that the first part of Equation (12) simplifies to $1.5\left[1-(2y-1)^2\right]$ since $f(x)$ is a pdf of an eSU distribution for a random variable X. Furthermore,

$$\int_0^1 x^\rho f(x)dx = \int_0^1 x^\rho\left[(1-\lambda)+3\lambda(2x-1)^2\right]dx = \frac{(1+2\lambda)\rho^2+(5-2\lambda)\rho+6}{(\rho+1)(\rho+2)(\rho+3)} = \delta.$$

It follows that Equation (12) simplifies to

$$f(y) = 1 - \lambda^* + 3\lambda^*(2y-1)^2,$$

where $y \in [0, 1]$ and $\lambda^* = 1.5\delta - 0.5$. Thus, the marginal distribution of Y is eSU with parameter λ^*.

Corollary 3. *Let Y be a random variable that follows an eSU distribution with parameter λ^*, where λ^* and δ are given in Theorem 2. If $\lambda = 0$, then the PDF of Y follows an eSU distribution with parameter $\lambda^* = \frac{1-0.5\rho}{\rho+1}$, where $\rho \geq 0$.*

Corollary 4. *Let Y be a random variable that follows an eSU distribution with parameter λ^*, where λ^* and δ are given in Theorem 2. If $\rho = 0$, then the PDF of Y follows an eSU distribution with parameter $\lambda^* = 1$.*

Theorem 3. *Let X and Y be any two random variables with joint pdf given in Equation (6). If the marginal distribution of Y is given in Theorem 2, then the conditional distribution of X given $Y = y$ is*

$$f(X|Y=y) = \frac{\left[1.5 - 1.5x^\rho + 3(1.5x^\rho - 0.5)(2y-1)^2\right]\left[1-\lambda+3\lambda(2x-1)^2\right]}{1-(1.5\delta-0.5)+3(1.5\delta-0.5)(2y-1)^2}, \tag{13}$$

where $\delta = \frac{(1+2\lambda)\rho^2+(5-2\lambda)\rho+6}{(\rho+1)(\rho+2)(\rho+3)}$, $\lambda \in [-0.5, 1]$ and $\rho \geq 0$.

The proof follows directly from the definition of the conditional distribution of X given $Y = y$.

Theorem 4. *Let X and Y be any two random variables with joint pdf given in Equation (6). If the marginal distribution of Y is given in Theorem 2 then the rth conditional moment of X given $Y = y$ is*

$$\mathbb{E}\left[X^r|y\right] = \frac{1.5}{f(y)}\left\{\left[1-(2y-1)^2\right]\mathbb{E}\left[X^r\right] - \left[1-3(2y-1)^2\right]\mathbb{E}\left[X^{r+\rho}\right]\right\}, \tag{14}$$

where $f(y)$ is the marginal distribution of Y of the $\rho - BeSU$ distribution,

$$\mathbb{E}\left[X^r\right] = \frac{(1+2\lambda)r^2 + (5-2\lambda)r + 6}{(r+1)(r+2)(r+3)},$$

and

$$\mathbb{E}\left[X^{r+\rho}\right] = \frac{(1+2\lambda)(r+\rho)^2 + (5-2\lambda)(r+\rho) + 6}{(r+\rho+1)(r+\rho+2)(r+\rho+3)}.$$

Proof. The rth conditional moment of X given $Y = y$ is defined as

$$\mathbb{E}\left[X^r|y\right] = \int_0^1 x^r f(X|Y=y)dx$$
$$\mathbb{E}\left[X^r|y\right] = \int_0^1 x^r \frac{f(x,y)}{f(y)}dx.$$

It follows that the rth conditional moment of X given $Y = y$ becomes

$$\begin{aligned}
\mathbb{E}\left[X^r|y\right] =& \frac{1}{f(y)}\int_0^1 x^r\left[1.5\left(1-(2y-1)^2\right) - 1.5x\left(1-3(2y-1)^2\right)\right] f(x|\lambda)dx \\
=& \frac{1.5\left(1-3(2y-1)^2\right)}{f(y)}\left[\frac{(1+2\lambda)r^2 + (5-2\lambda)r + 6}{(r+1)(r+2)(r+3)}\right] \\
&- \frac{1.5\left(1-3(2y-1)^2\right)}{f(y)}\left[\frac{(1+2\lambda)(r+\rho)^2 + (5-2\lambda)(r+\rho) + 6}{(r+\rho+1)(r+\rho+2)(r+\rho+3)}\right] \\
=& \frac{1.5}{f(y)}\left\{\left[1-(2y-1)^2\right]\mathbb{E}\left[X^r\right] - \left[1-3(2y-1)^2\right]\mathbb{E}\left[X^{r+\rho}\right]\right\},
\end{aligned}$$

where $f(y)$ is the marginal distribution of Y of the $\rho - BeSU$ distribution,

$$\mathbb{E}\left[X^r\right] = \frac{(1+2\lambda)r^2 + (5-2\lambda)r + 6}{(r+1)(r+2)(r+3)},$$

and

$$\mathbb{E}\left[X^{r+\rho}\right] = \frac{(1+2\lambda)(r+\rho)^2 + (5-2\lambda)(r+\rho) + 6}{(r+\rho+1)(r+\rho+2)(r+\rho+3)}.$$

Remark 2. *The conditional mean of X given Y is given by*

$$\mathbb{E}\left[X|y\right] = \frac{1.5}{f(y)}\left\{0.5\left[1-(2y-1)^2\right] - \left[1-3(2y-1)^2\right]\mathbb{E}\left[X^{\rho+1}\right]\right\}, \tag{15}$$

where $\rho \geq 0$, $f(y)$ is the marginal distribution of Y of the ρ - BeSU distribution, and

$$\mathbb{E}\left[X^{\rho+1}\right] = \frac{(1+2\lambda)(\rho+1)^2 + (5-2\lambda)(\rho+1) + 6}{(\rho+2)(\rho+3)(\rho+4)}.$$

Remark 3. *The conditional variance of X given Y is given by*

$$Var(X|y) = \mathbb{E}[X^2|y] - \left(\mathbb{E}[X|y]\right)^2, \tag{16}$$

where $\mathbb{E}[X|y]$ is the conditional mean of X given Y,

$$\mathbb{E}\left[X^2|y\right] = \frac{1.5}{f(y)}\left\{\left[1-(2y-1)^2\right]\left(\frac{5+\lambda}{15}\right) - \left[1-3(2y-1)^2\right]\mathbb{E}\left[X^{\rho+2}\right]\right\}, \tag{17}$$

$\rho \geq 0$, $\lambda \in [-0.5, 1]$, $f(y)$ is the marginal distribution of Y of the ρ - BeSU distribution, and

$$\mathbb{E}\left[X^{\rho+2}\right] = \frac{(1+2\lambda)(\rho+2)^2 + (5-2\lambda)(\rho+2) + 6}{(\rho+3)(\rho+4)(\rho+5)}.$$

Theorem 5. *Let X and Y be any two random variables with joint pdf given in Equation (6), then the product and ratio moments are given by*

$$\mathbb{E}\left[X^rY^s\right] = \frac{1}{(s+1)(s+2)(s+3)}\left[6(s+1)\mathbb{E}[X^r] + 3s(s-1)\mathbb{E}[X^{r+\rho}]\right] \tag{18}$$

and

$$\mathbb{E}\left[X^rY^{-s}\right] = \frac{1}{(1-s)(2-s)(3-s)}\left[6(1-s)\mathbb{E}[X^r] + 3s(s+1)\mathbb{E}[X^{r+\rho}]\right], \tag{19}$$

where $\mathbb{E}[X^r] = \frac{(1+2\lambda)r^2+(5-2\lambda)r+6}{(r+1)(r+2)(r+3)}$ and $\mathbb{E}[X^{r+\rho}] = \frac{(1+2\lambda)(r+\rho)^2+(5-2\lambda)(r+\rho)+6}{(r+\rho+1)(r+\rho+2)(r+\rho+3)}$.

Proof. The product moment of X and Y is defined as

$$\begin{aligned}
\mathbb{E}\left[X^rY^s\right] = &\int_0^1\int_0^1 x^ry^sf(x,y)dxdy \\
=&\frac{1}{(s+1)(s+2)(s+3)}\left[6(s+1)\int_0^1 x^rf(x)dx\right. \\
&\left.+ 3s(s-1)\int_0^1 x^{r+\rho}f(x)dx\right] \\
=&\frac{1}{(s+1)(s+2)(s+3)}\left[6(s+1)\mathbb{E}[X^r] + 3s(s-1)\mathbb{E}[X^{r+\rho}]\right],
\end{aligned}$$

where

$$\mathbb{E}[X^r] = \frac{(1+2\lambda)r^2 + (5-2\lambda)r + 6}{(r+1)(r+2)(r+3)},$$

and

$$\mathbb{E}[X^{r+\rho}] = \frac{(1+2\lambda)(r+\rho)^2 + (5-2\lambda)(r+\rho) + 6}{(r+\rho+1)(r+\rho+2)(r+\rho+3)}.$$

Next, the ratio moment of X and Y is defined as

$$\begin{aligned}\mathbb{E}\left[\frac{X^r}{Y^s}\right] =& \mathbb{E}\left[X^r Y^{-s}\right] \\ =& \frac{1}{(1-s)(2-s)(3-s)}\Bigg[6(1-s)\mathbb{E}[X^r] + 3s(s+1)\mathbb{E}[X^{r+\rho}]\Bigg].\end{aligned}$$

Corollary 5. *Let X and Y be random variables with joint PDF of a $\rho-BeSU$ distribution with product moment given in Equation (18). If $r = s = 1$, then*

$$\mathbb{E}[XY] = \frac{1}{4}. \tag{20}$$

Proof. Substituting $r = s = 1$ to Equation (18) gives us the $\mathbb{E}[XY] = \frac{1}{4}$ since $\mathbb{E}[X] = \frac{1}{2}$.

Remark 4. *The covariance of X and Y is given by*

$$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{4} - \frac{1}{2}\left(\frac{1}{2}\right) = 0.$$

Remark 5. *The Pearson correlation or correlation coefficient of X and Y is zero.*

Note that the correlation coefficient of X and Y is zero since the covariance of X and Y is zero. This implies that there is no linear relationship between X and Y. A nonlinear form may best describe the relationship between X and Y.

Theorem 6. *Let (X,Y) be a bivariate random vector that follows a ρ - bivariate extended Standard U-quadratic distribution, then the joint moment generating function of X and Y is given by*

$$M_{X,Y}(t_1,t_2) = \frac{3}{(s+2)(s+3)}\sum_{r=0}^{\infty}\frac{t_1^r}{r!}\sum_{s=0}^{\infty}\frac{t_2^s}{s!}\left[2\mathbb{E}[X^r] + \frac{s(s-1)\mathbb{E}[X^{r+\rho}]}{(s+1)}\right],$$

where

$$\mathbb{E}[X^r] = \frac{(1+2\lambda)r^2 + (5-2\lambda)r + 6}{(r+1)(r+2)(r+3)},$$

and

$$\mathbb{E}[X^{r+\rho}] = \frac{(1+2\lambda)(r+\rho)^2 + (5-2\lambda)(r+\rho) + 6}{(r+\rho+1)(r+\rho+2)(r+\rho+3)}.$$

Proof. The joint moment generating function of X and Y is defined as

$$\begin{aligned} M_{X,Y}(t_1,t_2) &= \int_0^1 \int_0^1 e^{t_1x+t_2y} f(x,y)dydx \\ &= \int_0^1 \int_0^1 e^{t_1x} e^{t_2y} f(y|x)f(x)dydx. \end{aligned}$$

Recall that $e^{tx} = \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r$, then we have

$$\begin{aligned} M_{X,Y}(t_1,t_2) &= \int_0^1 \int_0^1 \sum_{r=0}^{\infty} \frac{t_1^r}{r!} x^r \sum_{s=0}^{\infty} \frac{t_2^s}{s!} y^s f(y|x)f(x)dydx \\ &= \sum_{r=0}^{\infty} \frac{t_1^r}{r!} \sum_{s=0}^{\infty} \frac{t_2^s}{s!} \int_0^1 \int_0^1 x^r y^s f(y|x)f(x)dydx \\ &= \sum_{r=0}^{\infty} \frac{t_1^r}{r!} \sum_{s=0}^{\infty} \frac{t_2^s}{s!} \int_0^1 x^r f(x) \left(\int_0^1 y^s f(y|x)dy \right) dx \\ &= \sum_{r=0}^{\infty} \frac{t_1^r}{r!} \sum_{s=0}^{\infty} \frac{t_2^s}{s!} \int_0^1 x^r f(x) \left(\frac{6(s+1) + 3s(s-1)x^\rho}{(s+1)(s+2)(s+3)} \right) dx \\ &= \frac{1}{(s+2)(s+3)} \sum_{r=0}^{\infty} \frac{t_1^r}{r!} \sum_{s=0}^{\infty} \frac{t_2^s}{s!} \left[6 \int_0^1 x^r f(x)dx + \frac{3s(s-1)}{s+1} \int_0^1 x^{r+\rho} f(x)dx \right] \\ &= \frac{3}{(s+2)(s+3)} \sum_{r=0}^{\infty} \frac{t_1^r}{r!} \sum_{s=0}^{\infty} \frac{t_2^s}{s!} \left[2\mathbb{E}[X^r] + \frac{s(s-1)}{s+1} \mathbb{E}[X^{r+\rho}] \right], \end{aligned}$$

where

$$\mathbb{E}[X^r] = \frac{(1+2\lambda)r^2 + (5-2\lambda)r + 6}{(r+1)(r+2)(r+3)},$$

and

$$\mathbb{E}[X^{r+\rho}] = \frac{(1+2\lambda)(r+\rho)^2 + (5-2\lambda)(r+\rho) + 6}{(r+\rho+1)(r+\rho+2)(r+\rho+3)}.$$

Lemma 1. *Let X be a random variable that follows an eSU distribution, then*

$$\int_0^1 F(x)f(x)dx = \frac{1}{2},$$

where $F(x)$ and $f(x)$ are CDF and PDF of the eSU distribution.

Proof. Let X be a random variable that follows an eSU distribution with CDF $F(x)$ and PDF $f(x)$, respectively. Then, using the rth moment of eSU distribution, we have

$$\begin{aligned}\int_0^1 F(x)f(x)dx =& \int_0^1 \left[(1+2\lambda)x - 6\lambda x^2 + 4\lambda x^3\right] f(x)dx \\ =&(1+2\lambda)\mathbb{E}[X] - 6\lambda\mathbb{E}[X^2] + 4\lambda\mathbb{E}[X^3] \\ =&\frac{1}{2}.\end{aligned}$$

Theorem 7. *Let (X, Y) be the random vector that follows a $\rho - BeSU$ distribution, then the Kendall's tau coefficient is zero.*

Proof. Let (X, Y) be the random vector that follows a $\rho - BeSU$ distribution with joint PDF and CDF given in Equations (6) and (7), respectively. Then, the Kendall's tau coefficient is defined as

$$\tau =4\int_0^1 \int_0^1 F_{X,Y}(x,y)f_{X,Y}(x,y)dxdy - 1, \tag{21}$$

where $F_{X,Y}(x,y)$ and $f_{X,Y}(x,y)$ are joint CDF and PDF of the random variables X and Y, respectively. Let us first consider

$$\begin{aligned}\int_0^1 F_{X,Y}(x,y)f_{X,Y}(x,y)dv =& \int_0^1 \left[3y\left(2y^2-3y+1\right)M_\rho(x) - (2y-3)y^2F(x)\right] \\ &\left\{\left[1.5 - 1.5x^\rho + 3(1.5x^\rho - 0.5)(2y-1)^2\right] f(x)\right\}dy \\ =&M_\rho(x)f(x)\int_0^1 3y\left(2y^2-3y+1\right) \\ &\left\{\left[1.5 - 1.5x^\rho + 3(1.5x^\rho - 0.5)(2y-1)^2\right]\right\}dy \\ &- F(x)f(x)\int_0^1 (2y-3)y^2 \times \\ &\left[1.5 - 1.5x^\rho + 3(1.5x^\rho - 0.5)(2y-1)^2\right] dy.\end{aligned}$$

Now set $A \equiv 1.5 - 1.5x^\rho + 3(1.5x^\rho - 0.5)(2y-1)^2$, which is the $f(y|x)$, where $Y|X$ follows an eSU distribution with parameter $1.5x^\rho - 0.5$.

$$\int_0^1 F_{X,Y}(x,y)f_{X,Y}(x,y)dy =M_\rho(x)f(x)\int_0^1 6y^3Ady - M_\rho(x)f(x)\int_0^1 9y^2Ady$$

$$
\begin{aligned}
&+ M_\rho(x)f(x)\int_0^1 3yAdy - F(x)f(x)\int_0^1 2y^3Ady \\
&+ F(x)f(x)\int_0^1 3y^2Ady \\
=&M_\rho(x)f(x)\Big(6\mathbb{E}[Y^3|X=x] - 9\mathbb{E}[Y^2|X=x] \\
&+ 3\mathbb{E}[Y|X=x]\Big) - F(x)f(x)\Big(2\mathbb{E}[Y^3|X=x] \\
&+ 3\mathbb{E}[Y^2|X=x]\Big).
\end{aligned}
$$

Observe from the construction of the $\rho - BeSU$ distribution that the random variable $Y|X$ follows an eSU distribution with parameter $1.5x^\rho - 0.5$. Now, using the rth moment of eSU distribution for $Y|X$, we have

$$
\begin{aligned}
\int_0^1 F_{X,Y}(x,y)f_{X,Y}(x,y)dv =&M_\rho(x)f(x)\left[6\left(\frac{4+3x^\rho}{20}\right) - 9\left(\frac{3+x^\rho}{10}\right) + 3\left(\frac{1}{2}\right)\right] \\
&- F(x)f(x)\left[2\left(\frac{4+3x^\rho}{20}\right) + 3\left(\frac{3+x^\rho}{10}\right)\right] \\
=&\frac{1}{2}F(x)f(x).
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
\tau =&4\int_0^1 \frac{1}{2}f(x)F(x)dx - 1 \\
=&2\int_0^1 f(x)F(x)dx - 1.
\end{aligned}
$$

From Lemma (1), we have

$$
\int_0^1 F(x)f(x)du = \frac{1}{2}.
$$

Thus, the Kendall's tau coefficient becomes $\tau = 0$.

Theorem 8. *Let (X,Y) be the random vector that follows a $\rho - BeSU$ distribution, then the Spearman's rho coefficient is zero.*

Proof. Let (X,Y) be the random vector that follows a $\rho - BeSU$ distribution with joint PDF and CDF given in Equations (6) and (7), respectively. Then, the Spearman's rho coefficient is defined as

$$
\rho^* =12\int_0^1\int_0^1 \left[F(x,y) - F(x)F(y)\right] f(x)f(y)dxdy
$$

$$\rho^* = 12\int_0^1\int_0^1 F(x,y)f(x)f(y)dxdy - 12\int_0^1\int_0^1 F(x)F(y)f(x)f(y)dxdy.$$

Set $S = \int_0^1 F(x,y)f(x)f(y)dy$. It follows that

$$\begin{aligned}
S =& f(x)\int_0^1 F(x,y)f(y)dy \\
=& f(x)\int_0^1 \left[3y\left(2y^2-3y+1\right)M_\rho(x) - (2y-3)y^2F(x)\right] f(y)dy \\
=& f(x)M_\rho(x)\int_0^1 \left(6y^3-9y^2+3y\right)f(y)dy - f(x)F(x)\int_0^1\left(2y^3-3y^2\right)f(y)dy \\
=& \frac{1}{2}f(x)F(x).
\end{aligned}$$

Also, from Lemma (1), we have

$$\int_0^1 F(x)f(x)dx = \frac{1}{2}.$$

Hence, the Spearman's rho is simplified to

$$\rho^* = 12\int_0^1\left(\int_0^1 F(x,y)f(x)f(y)dy\right)dx - 12\int_0^1 f(x)F(x)\left(\int_0^1 F(y)f(y)dy\right)dx = 0.$$

In the following theorem we derive the Stress-Strength parameter for the $\rho - BeSU$ distribution. The Stress-Strength parameter is a measure used in reliability engineering and Statistics to evaluate the performance and reliability systems under stress. The variable X represents the strength of a system or component, while variable X represents the applied stress. The parameter $P(Y < X)$ measures the probability that the system's strength exceeds the applied stress, which is critical in assessing the reliability of materials and components.

Theorem 9. *Let (X,Y) be a bivariate random vector that follows a ρ - bivariate extended Standard U-quadratic distribution, then the Stress - Strength parameter of X and Y is given by*

$$\begin{aligned}
P(Y<X) =& 3\left(\frac{(1+2\lambda)(\rho+1)^2+(5-2\lambda)(\rho+1)+6}{(\rho+2)(\rho+3)(\rho+4)}\right) \\
&- 9\left(\frac{(1+2\lambda)(\rho+2)^2+(5-2\lambda)(\rho+2)+6}{(\rho+3)(\rho+4)(\rho+5)}\right) \\
&+ 6\left(\frac{(1+2\lambda)(\rho+3)^2+(5-2\lambda)(\rho+3)+6}{(\rho+4)(\rho+5)(\rho+6)}\right) \\
&+ \frac{5+4\lambda}{10},
\end{aligned} \tag{22}$$

where $\rho \geq 0$ and $\lambda \in [-0.5, 1]$.

Proof. The stress - strength parameter of random variables X and Y is defined by

$$\begin{aligned}
\mathbb{P}(Y<X) &= \int_0^1\int_0^x f(x,y)dydx \\
&= \int_0^1\int_0^x \left[1.5-1.5x^\rho+3(1.5x^\rho-0.5)(2y-1)^2\right]\left[1-\lambda+3\lambda(2x-1)^2\right]dydx \\
&= 3\mathbb{E}[X^{\rho+1}]-9\mathbb{E}[X^{\rho+2}]+3\mathbb{E}[X^2]+6\mathbb{E}[X^{\rho+3}]-2\mathbb{E}[X^3] \\
&= 3\left(\frac{(1+2\lambda)(\rho+1)^2+(5-2\lambda)(\rho+1)+6}{(\rho+2)(\rho+3)(\rho+4)}\right) \\
&\quad -9\left(\frac{(1+2\lambda)(\rho+2)^2+(5-2\lambda)(\rho+2)+6}{(\rho+3)(\rho+4)(\rho+5)}\right) \\
&\quad +6\left(\frac{(1+2\lambda)(\rho+3)^2+(5-2\lambda)(\rho+3)+6}{(\rho+4)(\rho+5)(\rho+6)}\right) \\
&\quad +\frac{5+4\lambda}{10},
\end{aligned}$$

where $\rho \geq 0$ and $\lambda \in [-0.5, 1]$.

4. Maximum Likelihood Estimation

Let (X_1,Y_1),(X_2,Y_2),...,(X_n,Y_n) be a random sample of size n from a ρ - Bivariate extended Standard U-quadratic (ρ - BeSU) Distribution. Then the likelihood function is defined by

$$L=\prod_i^n\left[1.5-1.5x_i^\rho+3(1.5x_i^\rho-0.5)(2y_i-1)^2\right]\left[1-\lambda+3\lambda(2x_i-1)^2\right],$$

with its log-likelihood function is given by

$$\begin{aligned}
\log L =& \sum_i^n \log\left[1.5-1.5x_i^\rho+3(1.5x_i^\rho-0.5)(2y_i-1)^2\right] \\
&+\sum_i^n \log\left[1-\lambda+3\lambda(2x_i-1)^2\right].
\end{aligned}$$

The partial derivative of $\log L$ with respect to the parameters λ and ρ are respectively, given by

$$\frac{\partial \log L}{\partial\lambda}=\sum_i^n\frac{3(2x_i-1)^2-1}{1-\lambda+3\lambda(2x_i-1)^2},$$

and

$$\frac{\partial \log L}{\partial\rho}=\sum_i^n\frac{4.5x_i^\rho\log x_i-1.5x_i^\rho\log x_i(2y_i-1)^2}{1.5-1.5x_i^\rho+3(1.5x_i^\rho-0.5)(2y_i-1)^2}.$$

The maximum likelihood estimates of the parameters λ and ρ of the BeSU distribution are computed by solving the following system of non-linear equations:

$$\sum_{i}^{n} \frac{3(2x_i-1)^2-1}{1-\lambda+3\lambda(2x_i-1)^2}=0,$$

and

$$\sum_{i}^{n} \frac{4.5x_i^{\rho}\log x_i - 1.5x_i^{\rho}\log x_i(2y_i-1)^2}{1.5-1.5x_i^{\rho}+3(1.5x_i^{\rho}-0.5)(2y_i-1)^2}=0.$$

5. Random Number Generation

This section presents the algorithm for the generation of bivariate random numbers from the ρ - Bivariate extended Standard U-quadratic (ρ - BeSU) distribution.

Let us first consider the random number generation from the T-extended Standard U-quadratic (TeSU)-G family of distributions. To generate random samples from TeSU-G family of distributions, we follow the algorithm proposed by Lakibul and Tubo [8]. The cumulative distribution function of the TeSU-G family is given by

$$F(x)=(1+2\lambda)G(x)-6\lambda(G(x))^2+4\lambda(G(x))^3, \tag{23}$$

where $\lambda \in [-0.5, 1]$ and $G(x)$ is any baseline cumulative distribution function. Due to the nonlinear nature of the CDF, it is not possible to obtain a simple closed-form expression for its inverse. Therefore, a numerical algorithm is employed to approximate the inverse CDF, enabling efficient random number generation. The algorithm to generate random numbers from TeSU-G family is given as follows. Let v follow a uniform distribution $(0, 1)$.

Step 1. Compute

$$Q=\frac{1}{2}\left(1-\frac{1}{\lambda}\right), \lambda \neq 0;$$

$$R=\frac{1-2v}{16\lambda}.$$

Step 2. If $R^2 > Q^3$, then compute

$$A=-sign(R)\left(|R|+\sqrt{R^2-Q^3}\right)^{\frac{1}{3}};$$

$$B=\begin{cases} A, & if A=0 \\ \frac{Q}{A}, & otherwise \end{cases};$$

$$x=G^{-1}\left(A+B+\frac{1}{2}\right).$$

Otherwise,

$$\theta = arccos\left(\frac{R}{\sqrt{Q^3}}\right);$$

$$x = G^{-1}\left(\frac{1}{2} - 2\sqrt{Q}\ cos\left(\frac{\theta - 2\pi}{3}\right)\right),$$

where $G^{-1}(x)$ is the inverse function of any baseline distribution function $G(x)$. If $\lambda = 0$, then $x = G^{-1}(v)$. Note that, the extended Standard U-quadratic distribution is derived from the T-extended Standard U-quadratic - G family of distributions by taking $G(x) = x$. Thus, we have the following modified algorithm to generate random numbers from an extended Standard U-quadratic distribution. Let v follow a uniform distribution $(0, 1)$. If $\lambda = 0$, then $x = v$. Otherwise, it is given as follows:

Step 1.* Compute

$$Q = \frac{1}{2}\left(1 - \frac{1}{\lambda}\right), \lambda \neq 0;$$

$$R = \frac{1 - 2v}{16\lambda}.$$

Step 2.* If $R^2 > Q^3$, then

$$A = -sign(R)\left(|R| + \sqrt{R^2 - Q^3}\right)^{\frac{1}{3}};$$

$$B = \begin{cases} A, & if A = 0 \\ \frac{Q}{A}, & otherwise \end{cases};$$

$$x = A + B + \frac{1}{2}.$$

Otherwise,

$$\theta = arccos\left(\frac{R}{\sqrt{Q^3}}\right);$$

$$x = \frac{1}{2} - 2\sqrt{Q}\ cos\left(\frac{\theta - 2\pi}{3}\right).$$

The inverse CDF of the TeSU-G family is not readily available in a simple closed form due to its nonlinear and complex structure. As a result, the numerical algorithm presented above is used to approximate the inverse CDF, enabling random number generation in a computationally feasible manner. This approach is a standard method for generating random variables from distributions where the inverse CDF is difficult to compute directly.

In addition, to generate a random sample from the ρ - bivariate extended Standard U-quadratic distribution, we use the conditional approach given in the following algorithm:

Steps	Description
1	Draw a random sample X of size n from an extended Standard U-quadratic distribution with parameter λ.
2	For each observation X, draw a sample of size 1 from an extended Standard Standard U-quadratic distribution with parameter $1.5x^{\rho} - 0.5$. Repeat this process for all observations of X. Denote this sample as Y.
3	Finally, the desired random sample is (x, y).

6. Simulation Study

This section presents the simulation results to assess the behavior of the maximum likelihood estimate of the parameter of the proposed bivariate distribution. The simulation algorithm is given as follows:

Steps	Description
1	Draw sample of size n, $n = 50, 100, 200, 500, 1000$ from a ρ - bivariate extended Standard U-quadratic distribution with parameter λ and ρ using the algorithm given in the previous section.
2	Using the bivariate sample (x, y) obtained in Step 1 above, compute the maximum likelihood estimate of λ and ρ.
3	Repeat the preceding Steps 1-2 $N = 1000$ times to get 1000 estimates of λ and ρ.
4	Compute the mean, bias, and mean squared error (MSE) of the 1000 estimates obtained in Step 3 to get the desired results

The mean (AE), Bias, and MSE are, respectively, defined by

$$AE = \sum_{i=1}^{N} \frac{\lambda_i}{N}, \; Bias = AE - \lambda \text{ and } MSE = \sum_{i=1}^{N} \frac{(\lambda_i - \lambda)^2}{N}.$$

Considering two different sets of values of the parameters λ and ρ, the following table shows that as n becomes large, the average estimate (AE) of the parameters λ and ρ are going closer to the true values, respectively, while the bias and the MSE diminish to zero. Thus, the maximum likelihood estimates of the proposed distribution parameters are consistent.

Table 1: Results of the simulation study for the following set of values of the parameters of ρ - BeSU distribution: (a.) $\lambda = -0.3$ and $\rho = 0.1$ (Left); and (b.) $\lambda = 0.5$ and $\rho = 1.5$ (Right)

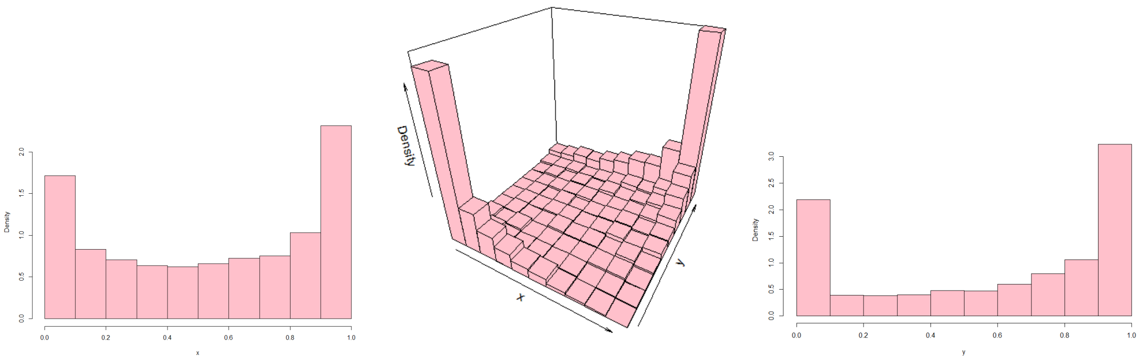
n	MLE	AE	$Bias$	MSE
50	$\hat{\lambda}$	−0.304	−0.004	0.017
100	$\hat{\lambda}$	−0.298	0.002	0.009
200	$\hat{\lambda}$	−0.300	0.000	0.004
500	$\hat{\lambda}$	−0.301	−0.001	0.002
1000	$\hat{\lambda}$	−0.301	−0.001	0.001
50	$\hat{\rho}$	0.106	0.006	0.008
100	$\hat{\rho}$	0.102	0.002	0.004
200	$\hat{\rho}$	0.102	0.002	0.002
500	$\hat{\rho}$	0.102	0.002	0.001
1000	$\hat{\rho}$	0.101	0.001	0.000

n	MLE	AE	$Bias$	MSE
50	$\hat{\lambda}$	0.495	−0.005	0.024
100	$\hat{\lambda}$	0.504	0.004	0.011
200	$\hat{\lambda}$	0.499	−0.001	0.006
500	$\hat{\lambda}$	0.499	−0.001	0.002
1000	$\hat{\lambda}$	0.498	−0.002	0.001
50	$\hat{\rho}$	1.808	0.308	1.720
100	$\hat{\rho}$	1.598	0.098	0.271
200	$\hat{\rho}$	1.541	0.041	0.122
500	$\hat{\rho}$	1.525	0.025	0.045
1000	$\hat{\rho}$	1.515	0.015	0.023

7. Application

In this section, we apply the proposed generalized bivariate distribution and compare with the BeSU distribution.

Here, we use the simulated bivariate data from the study of Lakibul, Polestico and Supe [12]. This bivariate data has the following property: X and Y have bathtub shapes. It was generated from the Bivariate Kumaraswamy distribution of Lakibul, Polestico and Supe [12], with $a = 0.5$ and $b = 0.5$. The following figures show the joint and marginal distributions of X and Y of the said bivariate simulated data.



(a) (b) (c)

Figure 5: histogram plot of the simulated data : (a) for variable X; (b) for variables X and Y; and (c) for variable Y.

In the analysis, we use the R-package "*bbmle*" to compute the maximum likelihood estimates of the parameters of the proposed ρ - BeSU and BeSU distributions. In addition, the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are used to assess and compare the performance of the proposed generalized bivariate distributions.

Table 2: Estimates and some diagnostic values of the fitted models for the simulated dataset.

$Distribution$	$Estimate$	$Std.Error$	$-2logLik$	AIC	BIC
$\rho - BeSU$	$\hat{\lambda} = 0.4891090$	**0.0142492**	**-2724.289**	**-2720.289**	**-2707.255**
	$\hat{\rho} = 0.0953629$	**0.0055758**			
$BeSU$	$\hat{\lambda} = 0.489109$	0.014249	2243.733	2245.733	2252.25

Table 2 shows some diagnostic statistics and maximum likelihood estimates of the fitted models for the simulated dataset. It is observed that the ρ - BeSU distribution has smaller values of the AIC and BIC than the BeSU distribution. Thus, the ρ - BeSU distribution provides a better fit for this simulated data than the BeSU distribution.

8. Conclusions and Recommendations

In this paper, we have derived the generalized version of the Bivariate extended Standard U-quadratic distribution, referred to as the ρ-Bivariate extended Standard U-quadratic (ρ-BeSU) distribution. Several important properties of the proposed ρ-BeSU distribution were computed, including the marginal and conditional distributions, conditional moments, conditional mean, conditional variance, product and ratio moments, Pearson correlation coefficient, joint moment generating function, Kendall's tau coefficient, Spearman's rho coefficient, and the stress-strength parameter. Maximum likelihood estimation was applied to estimate the parameters of the ρ-BeSU distribution, and a simulation study was conducted to assess the behavior of the parameter estimates. The results of the simulation study demonstrated that the maximum likelihood estimate of the parameter of the ρ-BeSU distribution is consistent. Furthermore, it was observed that the proposed ρ-BeSU distribution provides a better fit for the simulated bivariate dataset compared to the standard BeSU distribution, highlighting its potential as a more flexible model for bivariate data. While the ρ-BeSU distribution offers notable improvements, it is important to acknowledge some potential limitations. For instance, the current study focuses on a specific class of bivariate distributions, and there may be alternative distributions or estimation techniques that could provide additional benefits, particularly in cases with more complex dependence structures. Future research could explore extensions of the ρ-BeSU distribution to higher dimensions, which would enhance its applicability to multivariate data. Furthermore, alternative estimation methods, such as Bayesian inference or the use of copulas, could be considered to better account for varying dependence structures in different applications. In addition to these theoretical advancements, we recommend further investigations using the ρ-BeSU distribution to model the failure rates of two related components in a system or to model the lifetimes of two related electronic devices, particularly when the failure rates or lifetimes exhibit bathtub, inverted, or constant shapes on the interval $[0, 1]$. Additionally, future studies could consider extending the ρ-BeSU distribution by comparing it to other bivariate families of distributions, such as the bivariate Marshall-Olkin family or copulas, to evaluate its performance in diverse practical contexts.

Acknowledgements

The authors are grateful to the anonymous referees for their valuable comments and suggestions. Moreover, Idzhar A. Lakibul is also grateful to the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP) for giving him financial support to study at Mindanao State University - Iligan Institute of Technology (MSU-IIT). In addition, this paper is also supported by the DOST-ASTHRDP, MSU-Iligan Institute of Technology and the Mindanao State University - Sulu.

References

[1] J. K. Filus and L. Z. Filus. On some new classes of Multivariate Probability Distributions. *Pakistan Journal of Statistics*, 22:21–42, 2006.

[2] S. Shahbaz, M. Q. Shahbaz, and M. Mohsin. On Concomitants of Order Statistics for Bivariate Pseudo Exponential distribution. *World Applied Sciences Journal*, 6:1151–1156, 2009.

[3] S. Shahbaz and M. Ahmad. Concomitants of Order Statistics for Bivariate Pseudo-Weibull Distribution. *World Applied Sciences Journal*, 6:1409–1412, 2009.

[4] M. Q. Shahbaz and S. Shahbaz. Order Statistics and Concomitants of Bivariate Pseudo- Rayleigh Distribution. *World Applied Sciences Journal*, 7:826–828, 2009.

[5] M. Mohsin, M. Ahmad, S. Shahbaz, and M. Q. Shahbaz. Concomitants of Lower Records for Bivariate Pseudo – Inverse Rayleigh Distribution. *Sci. Int. (Lahore)*, 21:21–23, 2009.

[6] M. Q. Shahbaz, S. Shahbaz, and A. Rafiq. A New Bivariate Gumbel distribution. *Nonlinear Analysis Forum*, 16:133–136, 2011.

[7] S. H. Shahbaz and M. Q. Shahbaz. On Concomitants of Dual Generalized Order Statistics for a Bivariate Inverse Exponential Distribution. *European Journal of Pure and Applied Mathematics*, 11:929–936, 2018.

[8] I. A. Lakibul and B. F. Tubo. On the TeSU-G family of distributions applied to life data analysis. *Reliability:Theory and Applications*, 18:24–38, 2023.

[9] I. A. Lakibul and B. F. Tubo. On the Four-Parameter T-extended Standard U-quadratic Exponentiated Weibull distribution. *The Mindanawan Journal of Mathematics*, 5:17–33, 2023.

[10] I. A. Lakibul, D. L. Polestico, and A. P. Supe. On the Generalized Version of the extended Standard U-quadratic Distribution. *European Journal of Pure and Applied Mathematics*, 18(2), 2025.

[11] P. Kumaraswamy. A generalized probability density function for double-bounded random processes. *Journal of Hydrology*, 46:79–88, 1980.

[12] I. A. Lakibul, D. L. Polestico, and A. P. Supe. On the Bivariate Extension of the extended Standard U-quadratic Distribution. *European Journal of Pure and Applied Mathematics*, 17:790–809, 2024.