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On Nearly α–Boundedness in L–Topological Spaces

Najah A. Alsaedi

Department of Mathematics, Faculty of Applied Science, Umm Al-Qura University, Makkah Al Mukarramah, Saudi Arabia

Abstract. In this paper, we introduce and study the concept of nearly α–boundedness on arbitrary L–subsets in L–topological spaces, which depends on the notion of α–regular closed remoted neighborhood system. Several characterizations of nearly α–boundedness in terms of convergence theory of α–filters, α–molecular nets and α–ideals are obtained. We prove that the concept is a good extension, productive, and topologically invariant.

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1. Introduction

Boundedness, as a natural generalization of relative compactness, was considered by several authors (see [1] and [2]). In 1949, Hu [3] introduced the notion of boundedness in general topological spaces and studied the closure, interior, base, and relativization of boundedness. In-depth analysis of boundedness and its various weaker forms was done by Lamprinos in [1] and [4]. A subset A of a space X is said to be bounded if every open cover of X has a finite subfamily that covers A. The concept of a bounded set is useful in investigating non-regular topological spaces, since bounded sets in regular spaces are compact.

In 1968, Chang [5] presented the concept of fuzzy compact. Since then, it has been a very important topic to define proper fuzzy compactness. Many authors have written on this problem and various kinds of fuzzy compactness have been presented [6, 7]. In 1984, Li [8] introduced the fuzzy Q_α–compactness based upon the concept of Q–neighborhoods. In 1992, Wang [9] generalized the Q_α–compactness to the L–fuzzy topological spaces.

In 1997, Georgiou and Papadopoulos [10] gave a characterization of fuzzy nearly compactness by using the notion of fuzzy weakly θ–upper limit of fuzzy nets. Also, he studied new fuzzy compactness and fuzzy boundedness in fuzzy topological spaces. Recently, Georgiou and Papadopoulos in [11, 12] extended the concept of a bounded set to fuzzy topology; and introduced the notion of fuzzy boundedness using the fuzzy compactness given by Chang [5], which is not a good extension of ordinary compactness; the Tychonoff product theorem does not hold, and it contradicts some kinds of separation

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Email addresses: dr-Najah2008@hotmail.com, nasadi@uqu.edu.sa (N. A. Alsaedi)

axioms. Hence, the notion of fuzzy boundedness in [10] is not a good extension of ordinary bounded, and so it is unsatisfactory.

In 2003, Nouh [13] introduced the concept of N–boundedness on an arbitrary L–subset in L–topological spaces, and he gave new characterizations and properties of N–boundedness in terms of the convergence theory of α–nets, α–filters, and α–ideals. He proved that the concept of N–boundedness is a good extension, productive, and topologically invariant.

Since there are not enough studies on the concept of boundedness in L–topological spaces. So in 2023, Alsaedi [14] introduced the concept of nearly Ω–boundedness on an arbitrary L–subset in L–topological spaces by using the notion of Ω–upper limit of Ω–nets.

In this paper, we generalize the nearly Ω–boundedness to nearly α–boundedness, where we will study this concept on arbitrary L–subsets in L–topological spaces along the line of nearly Q_α–compactness defined by Wang [9] and α–regular closed remoted neighborhood due to Zhao [15]. Then we give new characterizations and properties of nearly α–boundedness in terms of the convergence theory of constant α–filter, α–molecular nets, and α–ideals. We prove that the notion is a good extension, productive, and topologically invariant.

2. Preliminaries

Throughout this paper $L = \langle L, \leq, \wedge, \vee, ^* \rangle$ denotes a completely distributive complete lattice with a smallest element 0 and a largest element 1 $(0 \neq 1)$ and with an order–reversing involution on it. An $\alpha \in L$ is called a molecule of L if $\alpha \neq 0$ and $0 \leq \nu \vee \gamma \leq \alpha$ implies $\nu \leq \gamma$ or $\gamma \leq \nu$, for all $\nu, \gamma \in L$. The set of all molecules of L is denoted by $M(L)$.

Let X be a nonempty set. L^X denotes the family of all mappings from X to L. The elements of L^X are called L–subsets on X. L^X can be made into a lattice by inducing the order and involution from L. We denote the smallest element and the largest element of L^X by 0_X and 1_X, respectively. If $\alpha \in L$, then the constant mapping $\alpha_X : X \longrightarrow \{\alpha\}$ is L–subset [16].

An L–point (or molecule on L^X), denoted by x_α, $\alpha \in M(L)$ is a L–subset which is defined by

$$x_\alpha(y) = \begin{cases} \alpha & : x = y \\ 0 & : x \neq y \end{cases}$$

The family of all molecules of L^X is denoted by $M(L^X)$ [17]. For $\mu \in L^X$ and $\alpha \in L$ we defined the set $\mu_{w\alpha} = \{x \in X : \mu(x) \geq \alpha\}$, which it is called weak α–cut of μ. The set $\mu_{s\alpha} = \{x \in X : \mu(x) \nleq \alpha\}$, it is called strong α–cut of μ and $\text{Supp}(\mu) = \{x \in X : \mu(x) > 0\}$ is called support of μ [18]. For any $\lambda \in L^X$ and $\alpha \in M(L)$ with $\alpha' \geq \alpha$, we have $(\lambda_{w\alpha})' \subseteq (\lambda')_{w\alpha}$. For $\Psi \subseteq L^X$, we define $2^{(\Psi)}$ by the set $\{\varphi \subseteq \Psi : \varphi \text{ is finite subfamily of } \Psi\}$.

An L–topology on X is a subfamily τ of L^X closed under arbitrary unions and finite intersections. The pair (X, τ) is called an L–topological space (or L–ts, for short) [19].

If (L^X, τ) is an L–ts, then for $\eta \in L^X$, $cl(\eta), int(\eta)$ and η' will denote the closure, interior, and complement of η. A mapping $f : L^X \to L^Y$ is an L–valued Zadeh function induced by a mapping $f : X \to Y$, iff $f(\mu)(y) = \bigvee\{\mu(x) : f(x) = y\}$ for every $\mu \in L^X$ and every $y \in Y$ [17].

An L–ts (L^X, τ) is called fully stratified if for each $\alpha \in L$, $\underline{\alpha} \in \tau$[18]. If (L^X, τ) is an L–ts, then the family of all crisp open sets in τ is denoted by $[\tau]$ i.e., $(X, [\tau])$ is a crisp topological space [20].

Definition 2.1 [21]. If (L^X, τ) is L–ts, then $\mu \in L^X$ is called a regular open set iff $\mu = int(cl(\mu))$. The family of all regular open sets is denoted by $RO(L^X, \tau)$. The complement of a regular open set is called a regular closed set and satisfies $\mu^c = cl(int(\mu))$. The family of all regular closed sets is denoted by $RC(\mathrm{L}^X, \tau)$.

Definition 2.2 [21]. The L–valued Zadeh mapping $f_L : (L^X, \tau) \to (L^Y, \Delta)$ is called Almost L–continuous iff $f_L^{-1}(\eta) \in \tau'$ for each $\eta \in RC(L^Y, \Delta)$.

Definition 2.3 [9]. Let (L^X, τ) be an L–ts and $x_\alpha \in M(L^X)$. Then $\lambda \in \tau'$ is called an remoted neighborhood (R–nbd, for short) of x_α if $x_\alpha \notin \lambda$. The set of all R–nbds of x_α is called remoted neighborhood system and is denoted by R_{x_α}.

Definition 2.4 [15]: Let (L^X, τ) be an L-ts, $\mu \in L^X$ and $\alpha \in M(L)$. $\Psi \subset \tau'$ is called an:

(i) α-remoted neighborhood family of μ, briefly α-RF of μ, if for each L–point $x_\alpha \in \mu$ there is $\lambda \in \Psi$ such that $\lambda \in R_{x_\alpha}$.

(ii) $\bar{\alpha}$-remoted neighborhood family of μ, briefly $\bar{\alpha}$-RF of μ, if there exists $\gamma \in \beta^*(\alpha)$ such that Ψ is a γ-RF of μ, where $\beta^*(\alpha) = \beta(\alpha) \cap M(L)$, and $\beta(\alpha)$ denotes the union of all the minimal sets relative to α.

Definition 2.5 [22]: Let (L^X, τ) be an L-ts, $\mu \in L^X$ and $\alpha \in M(L)$. Then $\Psi \subset RC(L^X, \tau)$ is called an α-regular closed remoted neighborhood family of μ, briefly α–RCRF of μ, if for each L-point $x_\alpha \in \mu$ there is $\lambda \in \Psi$ such that $\lambda \in R_{x_\alpha}$.

Definition 2.6 [21]. Let (L^X, τ) be an L-ts, $\mu \in L^X$ and $\alpha' \in M(L)$. Then the family $\Psi \subseteq \tau$ is called an:

(i) α–cover of μ, if for each $x \in \mu_{w_{\alpha'}}$ there is $\lambda \in \Psi$ such that $\lambda(x) \nleq \alpha$.

(ii) Nearly α–cover of μ, if for each $x \in \mu_{w_{\alpha'}}$ there is $\lambda \in \Psi$ such that $\mathrm{int}(\mathrm{cl}(\lambda))(x) \nleq \alpha$.

Definition 2.7 [17]. Let (L^X, τ) be an L-ts and $\mu \in L^X$. Then $x_\alpha \in M(L^X)$ is called the δ–adherent point of μ and write $x_\alpha \in \delta cl(\mu)$ iff $\mu \nleq cl(\mathrm{int}(\lambda))$ for each $\lambda \in R_{x_\alpha}$. If $\mu = \delta cl(\mu)$, then μ is called a δ–closed L–subset. The family of all δ–closed L–subsets of X is denoted by $\delta C(L^X, \tau)$ and its complement is called the family of all δ–open L–subsets and denoted by $\delta O(L^X, \tau)$.

Definition 2.8 [23]. Let (L^X, τ) be an L-ts, $\mu \in L^X$ and $\alpha \in M(L)$. An α-RF $\Psi = \{\eta_j : j \in J\}$ of μ is called a directed if $\eta_1, \eta_2 \in \Psi$ there is $\eta_3 \in \Psi$ such that $\eta_3 \leq \eta_1 \wedge \eta_2$.

Definition 2.9 [9]: Let $(D, \leq)$ be a directed set. Then the mapping $S : D \to L^X$ and denoted by $S = \{\mu_n : n \in D\}$ is called a net of L-subsets in X. Specifically, the mapping

$S : D \to M(L^X)$ is said to be a molecular net in L^X. If $\mu \in L^X$ and for each $n \in D, S \in \mu$ then S is called a net in μ.

Definition 2.10 [9]: Let (L^X, τ) be an L-ts and $S = \{S(n) : n \in D\}$ be a molecular net in L^X. S is called a molecular α-net ($\alpha \in M(L)$), if for each $\gamma \in \beta^*(\alpha)$ there exists $n \in D$ such that $\vee(S(m)) \geq \gamma$ whenever $m \geq n$, where $\vee(S(m))$ is the height of the molecular $S(m)$. If $\vee(S(m)) = \alpha$ for each $m \in D$, then $\{S(m) : m \in D\}$ is called a constant molecular α-net.

Definition 2.11 [9]: Let $S = \{S(n) : n \in D\}$ and $T = \{T(m) : m \in E\}$ be molecular nets in (L^X, τ). Then T is said to be a molecular subnet of S if there is a mapping $f : E \to D$ that satisfies the following conditions:
(i) $T = S \circ f$
(ii) For each $n \in D$ there is $m \in E$ such that $f(l) \geq n$ for each $l \in E$, $l \geq m$.

Definition 2.12 [21]: Let (L^X, τ) be an L–ts and $\Delta = \{\mu_n : n \in D\}$ be a net of L–subsets in (L^X, τ) and $x_\alpha \in M(L^X)$.

Then:
(i) x_α is called the δ–limit point of Δ (or δ–converges) to the point x_α, in symbols $\Delta \xrightarrow{\delta} x_\alpha$ if for every $\eta \in R_{x_\alpha}$ there is an $n \in D$ such that for every $m \in D$ and $m \geq n$ then $\mu_n \notin cl(\text{int}(\eta))$. The union of all δ–limit points of Δ are denoted by $\delta.\underline{\lim}(\Delta)$.
(ii) x_α is called a δ–cluster (δ–adherent) point of Δ, in symbols $\Delta \overset{\delta}{\propto} x_\alpha$ if for every $\eta \in R_{x_\alpha}$ and every $n \in D$ there exists $m \in D$ such that $m \geq n$ and $\mu_n \notin cl(\text{int}(\eta))$. The union of all δ–cluster points of Δ are denoted by $\delta\overline{\lim}(\Delta)$.

If $\delta\underline{\lim}(\Delta) = \delta\overline{\lim}(\Delta) = \mu$, then we say that μ is the δ–limit of Δ, or we say that Δ δ–converges to μ, in symbol $\delta.\lim(\Delta) = \mu$.

The δ–limit and δ–cluster points of a molecular net are defined similarly in [16].

Definition 2.13 [17]: Let (L^X, τ) be an L-ts and S be a molecular α-net in L^X. Then $x_\alpha \in M(L^X)$ is called the δ–limit point of S, (or S δ–converges to x_α) in symbol $S \xrightarrow{\delta} x_\alpha$ if for every $\mu \in R_{x_\alpha}$ there is an $n \in D$ such that for each $m \in D$ and $m \geq n$ we have $S(m) \not\leq cl(\text{int}(\mu))$. The union of all limit points of S is denoted by $\delta.\lim(S)$.

Definition 2.14 [17]: Let (L^X, τ) be an L–ts and S be a molecular α–net in L^X. Then $x_\alpha \in M(L^X)$ is called a δ–cluster point of S, in symbol $S \overset{\delta}{\propto} x_\alpha$ if for every $\mu \in R_{x_\alpha}$ and every $n \in D$ there is $m \in D$ such that $m \geq n$ and $S(m) \notin cl(\text{int}(\mu))$. The union of all δ–cluster points of S is denoted by $\delta adh(S)$.

Theorem 2.15 [22]: Assume that $S = \{S(n) : n \in D\}$ is a molecular net in an L–ts (L^X, τ) and $x_\alpha \in M(L^X)$. Then the following results are true:
(i) $S \overset{\delta}{\propto} x_\alpha$ iff there exists a subnet T of S such that $T \xrightarrow{\delta} x_\alpha$.
(ii) If $S \xrightarrow{\delta} x_\alpha$, then $T \xrightarrow{\delta} x_\alpha$ for each subnet T of S.

Definition 2.16 [8]: Let (L^X, τ) be an L-ts, $\mu \in L^X$. Then μ is called nearly Q_α–compact (or NQ_α–compact) in (L^X, τ) if for each $\alpha \in M(L)$ and every α–RF Ψ of μ there is $\Psi_o \in 2^{(\Psi)}$ such that Ψ_o is an α–RCRF of μ.

If 1_X is nearly Q_α–compact, then (L^X, τ) is called a nearly Q_α–compact space.

Definition 2.17 [23]: An L–ts (L^X, τ) is said to be:
(i) LT_1–space iff for any $x_\alpha, y_\gamma \in M(L^X)$, $x \neq y$ there is $\lambda \in R_{x_\alpha}$ such that $y_\gamma \in \lambda$.
(ii) LT_2–space iff for any $x_\alpha, y_\gamma \in M(L^X)$, $x \neq y$ there is $\lambda \in R_{x_\alpha}$, $\eta \in R_{y_\gamma}$ such that $\lambda \vee \eta = 1_X$.
(iii) $LT_{2\frac{1}{2}}$–space iff for any $x_\alpha, y_\gamma \in M(L^X)$, $x \neq y$ there is $\lambda \in R_{x_\alpha}$, $\eta \in R_{y_\gamma}$ such that $\text{int}(\lambda) \vee \text{int}(\eta) = 1_X$.
(iv) LR_2–space (regular space) iff for all $\alpha \in M(L)$, $x \in X$ and for each $\lambda \in R_{x_\alpha}$ there is $\eta \in R_{x_\alpha}$, $\rho \in \tau'$ such that $\eta \vee \rho = 1_X$ and $\lambda \wedge \rho = 0_X$.
(v) LSR_2–space (Semi-regular space) iff for all $x_\alpha \in M(L^X)$ and for each $\lambda \in R_{x_\alpha}$ there is $\eta \in R_{x_\alpha}$ such that $\lambda \leq cl(\text{int}(\eta))$.
(vi) LT_3–space iff it is LR_2–space and LT_1–space.

Theorem 2.18 [8]: Let (L^X, τ) be an L–ts and $\mu \in L^X$. Then the following properties are true:

(i) Every set with finite support is nearly Q_α–compact.

(ii) Every nearly Q_α–compact set in a fully stratified and LT_2–space , then it is δ–closed.

Theorem 2.19 [8]: Let (L^X, τ) be an L–ts, $\alpha \in M(L)$ and $\mu \in L^X$. Then μ is NQ_α–compact iff for each constant molecular α–net S contained in μ has a δ–cluster point with height α in μ.

Theorem 2.20 [21]. If L–ts (L^X, τ) is LR_2–space, then it is LSR_2–space.

Theorem 2.21 [21]: An L–ts (L^X, τ) is LSR_2–space iff for any $\mu \in L^X$, $cl(\mu) = \delta cl(\mu)$.

Corollary 2.22 [19]. If L–ts (L^X, τ) is LSR_2–space, then a closed L–subset is a δ–closed L–subset and hence $\delta cl(\mu)$ is a δ–closed L–subset.

Definition 2.23 [15]: The nonempty family $\mathcal{F} \subset L^X$ is called an L–filter if the following conditions are satisfied, for each $\mu_1, \mu_2 \in L^X$
(i) $0_X \notin \mathcal{F}$
(ii) If $\mu_1 \leq \mu_2$ and $\mu_1 \in \mathcal{F}$, then $\mu_2 \in \mathcal{F}$.
(iii) If $\mu_1, \mu_2 \in \mathcal{F}$, then $\mu_1 \wedge \mu_2 \in \mathcal{F}$.

Definition 2.24 [15]: A filter $\mathcal{F}$ in L^X is called an α–filter ($\alpha \in M(L)$), if for every $\lambda \in \mathcal{F}$, $\bigvee_{x \in X} \lambda(x) \geq \alpha$.

Definition 2.25 [15]: Let (L^X, τ) be an L–ts and $\mathcal{F}$ be an L–filter in L^X.

Then $x_\alpha \in M(L^X)$ is called the δ–cluster point of $\mathcal{F}$, in symbol $\mathcal{F} \overset{\delta}{\propto} x_\alpha$ if for each $\lambda \in \mathcal{F}$ and each $\mu \in R_{x_\alpha}$, $\lambda \nsubseteq cl(\text{int}(\mu))$. The union of all δ–cluster points of $\mathcal{F}$ is denoted by $\delta adh(\mathcal{F})$.

Definition 2.26 [27]: The nonempty family $I \subset L^X$ is called an L–ideal if the following conditions are satisfied, for each $\mu_1, \mu_2 \in L^X$

(i) $1_X \notin I$

(ii) If $\mu_1 \leq \mu_2$ and $\mu_2 \in I$, then $\mu_1 \in I$.

(iii) If $\mu_1, \mu_2 \in I$, then $\mu_1 \vee \mu_2 \in I$.

Definition 2.27 [27]: Let I be an L–ideal in an L–ts (L^X, τ) and $\alpha \in M(L)$. Then I is said to be an α–ideal, if $\vee_{n \in X}\eta(x) < \alpha$ for each $\eta \in I$.

Theorem 2.28 [22]: Let $\mathcal{F}$ be a L–filter in an L–ts (L^X, τ) and $S(\mathcal{F})$ be the L–molecular net induced by $\mathcal{F}$. Then $\delta adh(\mathcal{F}) = \delta adh(S(\mathcal{F}))$.

Theorem 2.29 [22]: Suppose that S is a L–net in an L–ts (L^X, τ) and $\mathcal{F}(S)$ is the L–filter induced by S. Then $\delta adh(S) = \delta adh(\mathcal{F}(S))$.

Theorem 2.30 [22]: Suppose that I is an L–ideal in an L–ts (L^X, τ), and $S(I)$ is the L–molecular net induced by I. Then $\delta adh(I) = \delta adh(S(I))$.

3. Nearly α–Boundedness in L–topological spaces

In this section, we introduce the concept of nearly α–bounded sets in L–topological spaces. Then we obtain several characterizations of nearly α–bounded sets.

Definition 3.1. Let (L^X, τ) be an L−ts, $\mu \in L^X$ and $\alpha \in M(L)$, then $\mu \in L^X$ is called a nearly α−bounded ($N\alpha$−bounded, for short) set in (L^X, τ) iff for each $\alpha - RF$ Ψ of 1_X, there exists $\Psi_o \in 2^{(\Psi)}$ such that Ψ_o is an $\alpha - RCRF$ of μ.

Theorem 3.2. Suppose that $f_L : (L^X, \tau) \to (L^Y, \Delta)$ is a L−continuous and $\mu \in L^X$ is a $N\alpha$−bounded L−subset in (L^X, τ), then $f_L(\mu)$ is a $N\alpha$−bounded L−subset in (L^Y, Δ).

Proof. Let μ be a $N.\alpha$–bounded in L^X and let $\Psi \subset \Delta'$ be an α–RF of 1_Y ($\alpha \in M(L)$). To begin with, let us show that $f_L^{-1}(\Psi) = \{f_L^{-1}(\lambda) : \lambda \in \Psi\}$ is an α–RF of 1_X. Since f_L is a L–continuous, then $f_L^{-1}(\Psi) \subset \tau'$. Let $x \in X$, then $f_L(x_\alpha) = (f(x))_\alpha \in f_L(1_X)$ and by $\Psi \subset \Delta'$ is an α–RF of 1_Y there exists $\lambda \in \Psi$ with $\lambda \in R_{(f(x))_\alpha}$, i.e, $(f(x))_\alpha \notin \lambda$ or, equivalently, $\lambda(f(x)) \ngeq \alpha$. By the definition of inverse mapping, $f_L^{-1}(\lambda)(x) = \lambda(f(x)) \ngeq \alpha$, hence $x_\alpha \notin f_L^{-1}(\lambda)$. It follows that $f_L^{-1}(\lambda) \in R_{x_\alpha}$. Therefore $f_L^{-1}(\Psi)$ is an α–RF of 1_X. From the $N.\alpha$–boundedness of μ there exists $\Psi_\circ \in 2^{(\Psi)}$ such that $f_L^{-1}(\Psi_\circ)$ is an α–RCRF of μ, that is, for each $x_\alpha \in \mu$ there exists $\lambda \in \Psi$ such that $f_L^{-1}(\lambda) \in R_{x_\alpha}$, i.e., $f_L^{-1}(\lambda)(x) \ngeq \alpha$. Hence $\lambda(y) = \lambda(f(x)) \ngeq \alpha$ and so for each $y_\alpha \in f_L(\mu)$, there exists $x_\alpha \in \mu$ and $\lambda \in \Psi_\circ$ satisfying $y_\alpha = f_L(x_\alpha) \notin \lambda$. Hence $\lambda(y) \ngeq \alpha$, i.e., $\lambda \in R_{y_\alpha}$. This implies that $\Psi_\circ \in 2^{(\Psi)}$ is an α–RCRF of $f_L(\mu)$. By Definition 3.1, we have $f_L(\mu)$ is a $N.\alpha$–bounded L–subset in (L^Y, Δ).

Theorem 3.3. Let (L^X, τ) be an L–ts and $\alpha' \in M(L)$, then the set $\mu \in L^X$ is $N\alpha$–bounded iff for every α–cover $\Psi \subseteq \tau$ of 1_X there exists $\Psi_o \in 2^{(\Psi)}$ such that Ψ_o is a nearly α–cover of μ.

Proof. Let $\mu \in L^X$ be a $N.\alpha$–bounded set and let $\Psi \subseteq \tau$ is any α–cover of 1_X. Let $\varphi = \Psi' = \{\lambda' : \lambda \in \Psi\}$ and let $\gamma = \alpha'$. One can see that φ is an γ–RF of 1_X. Since $1_X(x) \geq \gamma$ for each $x_\gamma \in 1_X$, i.e., $x \in X$ for each $x_\gamma \in 1_X$, there exists $\lambda \in \Psi$ satisfying $\lambda(x) \ngeq \alpha = \gamma'$, this equivalently, there exists $\lambda' \in \varphi$ with $\gamma \nleq \lambda'(x)$, and so $\lambda' \in R_{x_\gamma}$. This implies that φ is an γ–RF of 1_X. Being μ is $N.\alpha$–bounded, then there exists $\varphi_\circ \in 2^{(\varphi)}$ such that $\varphi_\circ$ is an γ–RCRF of μ. We assert $\varphi'_\circ \in 2^{(\Psi)}$ is a nearly α–cover of μ. In fact, for each $x_\gamma \in \mu$ there is $\lambda' \in \varphi_\circ$ satisfying $cl(\text{int}(\lambda')) \in R_{x_\gamma}$, that is $\gamma \nleq cl(\text{int}(\lambda'(x)))$, equivalently, for each $x \in \mu_{w_\gamma}$ we have $\lambda \in \varphi'_\circ$ with $\text{int}(cl(\lambda(x))) = (cl(\text{int}(\lambda'))(x))' \nleq \alpha$. Therefore, $\varphi'_\circ$ is a nearly α–cover of μ.

Conversely, suppose that the condition is satisfied and let that Ψ is an γ–RF of 1_X. Put $\Psi' = \varphi$ and $\gamma' = \alpha$, then φ is an α–cover of 1_X, and then there exists $\varphi_\circ \in 2^{(\varphi)}$ such that $\varphi_\circ$ is a nearly α–cover of μ. Evidently, $\varphi'_\circ \in 2^{(\Psi)}$ is an γ–RCRF of μ. Hence μ is a $N.\alpha$–bounded.

Theorem 3.4. Let (L^X, τ) be an L–ts, $\mu \in L^X$ is a α–bounded [23], then μ is a $N\alpha$–bounded.

proof It follows directly from the fact that $RC(L^X, \tau) \subseteq \tau'$.

The following example shows that the converse is not true in general.

Example 3.5. Let $L = [0,1]$, $X = N$ and let $\tau = \{0_X, x_{.5}, 1_X\}$. Then (L^X, τ) is L–ts. Firstly, we show that 1_X is not α–bounded set. In fact, we suppose that constant 0.5–net $S = \{x_{.5} : x \in X\}$ in 1_X, Let $y_{0.3} \in M(L^X)$, $x \neq y$, then $R_{y_{0.3}} = \{0_X, x_{0.5}\}$. Since $S(n) = x_{0.5} \leq x_{0.5} \in R_{y_{0.3}}$. So $y_{0.3}$ is not cluster point of S in 1_X. On account of the arbitrariness of y it follows that the S has no cluster point in 1_X with height 0.3. Thus μ is not α–bounded set.

Now, we show that 1_X is $N\alpha$–bounded set. Let $S = \{x_\alpha : x \in X, \alpha \in L\}$ is any constant α–net in 1_X.

(i) If $\alpha > 0.5$ then $R_{x_\alpha} = \{0_X, x_{0.5}\}$, where $cl(int(x_{0.5})) = x_{0.5}$. Since $S(n) = x_\alpha \notin 0_X \quad \forall\, n \in N, \quad \forall \alpha > 0.5$. Hence, x_α is δ–cluster point of S with height α in 1_X.

(ii) If $\alpha \leq 0.5$, then $R_{x_\alpha} = \{0_X\}$ where $cl(int(0_X)) = 0_X$. Since $S(n) = x_\alpha \notin 0_X \quad \forall\, n \in N, \quad \forall \alpha \leq 0.5$. So x_α is δ–cluster point of S with height α in 1_X. Thus 1_X is $N\alpha$–bounded set.

Theorem 3.6. (The goodness of α−boundedness) Let $(L^{X_i}\omega_L(T))$ be the induced L-ts by the ordinary space (X,T), $\alpha \in M(L)$ and $\mu \in L^X$. Then μ is $N\alpha$−bounded in $(L^{X_i}, \omega_L(T))$ iff $\mu_{\omega\alpha} = \{x \in X : \mu(x) \geq \alpha\}$ is nearly bounded in (X,T).

proof Let $\mu \in L^X$ be a $N\alpha$−bounded and $\{U_j : j \in J\}$ be an open cover of X in (X,T). Then the family $\{1_{U_j} : j \in J\}$ is a α−covr of 1_X in $(L^{X_i}, \omega_L(T))$. Since μ is $N\alpha$−bounded, there is a finite subset J_o of J such that $\{1_{U_j} : j \in J_o\}$ is an nearly α−cover of μ in $(L^{X_i}\omega_L(T))$ in line with Theorem 3.3, i.e, for each $x \in \mu_{\omega\alpha}$ there exists $j \leq n$ and $1_{U_j} \in \varphi_o$ satisfying $\mathrm{int}(cl(1_{U_j}))(x) \nleq \alpha$. However $\mathrm{int}(cl(1_{U_j}))(x) = 1_{\mathrm{int}(cl(U_j))}$, and so $x \in \mathrm{int}(cl(U_j))$. This implies that $\bigcup_{j=1}^n \mathrm{int}(cl(U_j)) \supset \mu_{\omega\alpha}$. Hence $\mu_{\omega\alpha}$ is a nearly bounded in (X,T) for any $\alpha \in M(L)$.

Conversely, suppose that $\mu_{\omega\alpha}$ is a nearly bounded set for any $\alpha \in M(L)$ and φ is a α−cover of 1_X in $(L^{X_i}\omega_L(T))$. Then for any $x \in X$, $\alpha \in M(L)$ there exists $\eta_x \in \varphi$ such that $\eta_x(x) \nleq \alpha$. Put $(\eta_x)_{s\alpha} = \{y \in X : \eta_x(y) \nleq \alpha\}$, then $x \in (\eta_x)_{s\alpha}$. Since $\eta_x \in \omega_L(T)$ then $(\eta_x)_{s\alpha} \in T$. One can see that $U = \{(\eta_x)_{s\alpha} : x \in X\}$ is an open cover of X in (X,T). Since $\mu_{\omega\alpha}$ is a nearly bounded in (X,T), then there exists $x_1, x_2, ..., x_n \in \mu_{s\alpha}$ such that $U_o = \{(\eta_{x_i})_{s\alpha} : i = 1,2,...,n\}$ is an nearly open cover of $\mu_{\omega\alpha}$. Thus there exists $i \leq n$ with $x \in \mathrm{int}(cl((\eta_{x_i})_{s\alpha}))$ for each $x \in \mu_{\omega\alpha}$. However $\mathrm{int}(cl((\eta_{x_i})_{s\alpha})) \subset \mathrm{int}(cl((\eta_{x_i})))_{s\alpha}$ and so $x \in \mathrm{int}(cl((\eta_{x_i})))_{s\alpha}$, thus $\mathrm{int}(cl(\eta_{x_i}))(x) \nleq \alpha$. Hence $\varphi_o = \{\eta_{x_i} : i = 1,2,...,n\} \in 2^{(\varphi)}$ is a nearly α−cover of μ. According to Theorem 3.3, μ is $N\alpha$−bounded in $(L^{X_i}, \omega_L(T))$.

Theorem 3.7. Let (L^X, τ) be a $L-$ts. and let $\mu \in L^X$. If η is $N\alpha-$bounded and $\mu \leq \eta$, then μ is a $N\alpha-$bounded.

Proof. Let η be a $N\alpha-$bounded set and $\mu \leq \eta$. Let $\Psi \subset \tau'$ be an $\alpha-$RF of 1_X. Since η is $N\alpha-$bounded set, then there exists a finite subfamily $\Psi_o \in 2^{(\Psi)}$ such that Ψ_o is an $\alpha-$RCRF of η, since $\mu \leq \eta$, then Ψ_o is an $\alpha-$RCRF of μ and so μ is a $N\alpha-$bounded set.

Definition 3.8. Let (L^X, τ) be an $L-$ts and $x_\alpha \in M(L^X)$. If $\mu \in L^X$ is closed and $N\alpha-$bounded set, then μ is called the $N\alpha B-$remoted neighborhood of x_α ($N\alpha BR-$nbd, for short) of x_α if $x_\alpha \notin \mu$. The set of all $N\alpha BR-$nbds of x_α is denoted by $N\alpha BR_{x_\alpha}$.
We note that [24] $\alpha BR_{x_\alpha} \subseteq N\alpha BR_{x_\alpha} \subseteq R_{x_\alpha}$, $\forall x_\alpha \in M(L^X)$.
The following example shows that the converse is not true in general.

Example 3.9 Let $L = [0, 1]$, $X = \square$ and let $\tau = \{0_X, x_{.5}, 1_X\}$. Then (L^X, τ) is $L-$ts. Firstly, we show that $\mu = x_{.4} \in L^X$ is not a $\alpha-$bounded set. In fact, we suppose that constant 0.3−net $S = \{x_{.3} : x \in X\}$ in μ. Let $y_{.2} \in M(L^X)$, $x \neq y$, then $R_{y_{.2}} = \{0_X, x_{.5}\}$. Since $S(n) = x_{.3} \leq x_{.5} \in R_{y_{.2}}$. So $y_{.2}$ is not cluster point of S. On account of the arbitrariness of y it follows that the S has no cluster point in 1_X with height 0.3. Thus μ is not $\alpha-$bounded set.

Now, we show that μ is $N\alpha-$bounded set. Let $S = \{x_\alpha : x \in X, \alpha \leq .4\}$ is any constant $\alpha-$net in μ. If $\alpha < .4$ then $R_{x_\alpha} = \{0_X\}$, where $cl(int(0_X)) = 0_X$. Since $S(n) = x_\alpha \notin cl(int(0_X)) = 0_X \forall n \in N, \forall \alpha < .4$. If $\alpha = .4$, then $R_{x_{.4}} = \{0_X\}$ and $S(n) = x_\alpha \notin cl(int(0_X)) = 0_X$. So x_α is the $\delta-$cluster point of S in $1_X\, 1_X$. Thus μ is a $N\alpha-$bounded set.

Example 3.10 Let $X = \{2, 3, 4, \ldots\}$, $L = [0, 1]$,

$$\rho_n(x) = \begin{cases} 0 & : x = n \\ \frac{1}{2} + \frac{1}{n} & : x \neq n \quad n \in \mathbb{N} \end{cases}$$

$$\eta_2' = \begin{cases} 1 & : x = 2 \\ \frac{1}{2} & : x > 2 \end{cases}$$

$$\sigma_n(x) = \eta_n'(x) = \begin{cases} 1 & : x = n \\ \frac{1}{2} - \frac{1}{n} & : x \neq n \quad \text{for } n \in \{3, 4, \ldots\} \end{cases}$$

Then we have :

(i) $int(\rho_n) = \eta_n$, $\forall n \in X$.

(ii) $cl(int(\rho_2)) = \rho_2 \vee \sigma_2$.

(iii) $cl(int(\rho_3)) = \rho_3 \vee \sigma_2$, $\quad cl(int(\rho_n)) = \rho_n$, $\forall n \geq 4$.

We show that $\rho_5 \in L^X$ is not $N.\alpha$–bounded set. Put $\Psi = \{\rho_n : n \in X\}$, then then Ψ is 0.8–RF of 1_X where $\alpha = 0.8 \in M(L) = (0, 1]$ (because $\forall x \in X\ \exists \lambda = \rho_6 \in \Psi \ni cl(\text{int}(\rho_6)) \in R_{x_{0.8}}$), where $\rho_6(x) = 0$ at $x = 6$ and $\rho_6(x) = 0.6$ at $x \neq 6$.

But the family $\{cl(\text{int}(\rho_n)) : n \in X\} = \{\rho_2 \vee \sigma_2, \rho_3 \vee \sigma_3, \rho_n : n \geq 4\}$. Then any finite subfamily $\Psi_\circ = \{cl(\text{int}(\rho_n)) : i < n\} \in 2^{(\Psi)}$ is not 0.8–RF of ρ_5 (because $\exists x_{0.8} \in \rho_5$ and $\forall \lambda = \rho_2$ we have $\rho_2 \notin R_{x_{0.8}}$).

Where $cl(\text{int}(\rho_2))(x) = \rho_2 \vee \sigma_2(x) = 1$ at $x = 2$ and $cl(\text{int}(\rho_2))(x) = \rho_2 \vee \sigma_2(x) = 1$ at $x \neq 2$. Thus ρ_5 is not a $N.\alpha$–bounded set, however $\rho_5 \in R_{x_{0.8}}$.

Thus $N\alpha BR_{x_{0.8}} \subseteq R_{x_{0.8}}$.

Definition 3.11. Let (L^X, τ) be an $L-ts$ and $\mu \in L^X$. Then $x_\alpha \in M(L^X)$ is called a $N.\alpha - bounded$ adherent point of μ and write $x_\alpha \in N\alpha Bcl(\mu)$ iff $\mu \nleq \lambda$ for each $\lambda \in \alpha BR_{x_\alpha}$. If $\mu = N\alpha Bcl(\mu)$, then μ is called a $N\alpha B - closed\ L - subset$. The family of all $N\alpha B - closed\ L - subsets$ is denoted by $N\alpha BC(L^X, \tau)$ and its complement is called the family of all $N\alpha B - open\ L - subsets$ and denoted by $N\alpha BO(L^X, \tau)$.

Theorem 3.12. Let (L^X, τ) be an $L-ts$ and let $\mu \in L^X$. Then the following statements are true: (i) $\mu \leq cl(\mu) \leq N\alpha Bcl(\mu)$. Moreover, $N\alpha Bcl(\mu) \leq \alpha B.cl(\mu)$ [26]

(ii) If $\eta \in L^X$ and $\mu \leq \eta$ then $N\alpha Bcl(\mu) \leq N\alpha Bcl(\eta)$.

(iii) $N\alpha Bcl(N\alpha Bcl(\mu)) = N\alpha Bcl(\mu)$.

(iv) $N\alpha Bcl(\mu) = \wedge\{\eta \in L^X : \eta \in N\alpha BC(L^X, \tau), \mu \leq \eta\}$.

Proof. (i) Let $x_\alpha \in M(L^X)$ such that $x_\alpha \notin N\alpha Bcl(\mu)$, then there exists $\lambda \in N\alpha BR_{x_\alpha}$ such that $\mu \leq \lambda$. Since $N\alpha BR_{x_\alpha} \subseteq R_{x_\alpha}$ and so $\lambda \in R_{x_\alpha}$ and hence $x_\alpha \notin cl(\mu)$. Thus $cl(\mu) \leq N\alpha Bcl(\mu)$.

(ii) Let $x_\alpha \in M(L^X)$ such that $x_\alpha \notin N\alpha Bcl(\eta)$, then there exists $\lambda \in N\alpha BR_{x_\alpha}$ such that $\eta \leq \lambda$. Since $\mu \leq \eta$, then $\mu \leq \lambda$ and so $x_\alpha \notin N\alpha Bcl(\mu)$. Thus $N\alpha Bcl(\mu) \leq N\alpha Bcl(\eta)$.

(iii) Suppose $x_\alpha \in M(L^X)$ such that $x_\alpha \in N\alpha Bcl(N\alpha Bcl(\mu))$. According to Definition 3.11, we have $N\alpha Bcl(\mu) \nleq \lambda$ for each $\lambda \in N\alpha BR_{x_\alpha}$. Hence, there exists $y_\gamma \in M(L^X)$ such that $y_\gamma \in N\alpha Bcl(\mu)$ with $y_\gamma \notin \lambda$ and so $\mu \nleq \lambda$, that is, $x_\alpha \in N\alpha Bcl(\mu)$. This shows that $N\alpha Bcl(N\alpha Bcl(\mu)) \leq N\alpha Bcl(\mu)$. On the other hand, $\mu \leq N\alpha Bcl(\mu)$ follows from (i) and so $N\alpha Bcl(\mu) \leq N\alpha Bcl(N\alpha Bcl(\mu))$. Therefore, $N\alpha Bcl(N\alpha Bcl(\mu)) = N\alpha Bcl(\mu)$.

(iv) On account of (i) and (iii), $N\alpha Bcl(\mu)$ is a $N\alpha B - closed$ set containing μ, and so $N\alpha Bcl(\mu) \geq \wedge\{\eta \in L^X : \eta \in N\alpha BC(L^X, \tau), \mu \leq \eta\}$. Conversely, in case $x_\alpha \in M(L^X)$ and $x_\alpha \in N\alpha Bcl(\mu)$, then $\mu \nleq \lambda$ for each $\lambda \in N\alpha BR_{x_\alpha}$. Hence, if η is an $N\alpha B - closed$ set containing μ, then $\eta \nleq \lambda$, and then $x_\alpha \in N\alpha Bcl(\eta) = \eta$. This implies that $N\alpha Bcl(\mu) \leq \wedge\{\eta \in L^X : \eta \in N\alpha BC(L^X, \tau), \mu \leq \eta\}$. Hence

$N\alpha Bcl(\mu) = \wedge\{\eta \in L^X : \eta \in N\alpha BC(L^X, \tau), \mu \leq \eta\}$

From Theorem 3.12, one can see that every $N\alpha B - closed\ L - subset$ is a closed $L - subset$, but the inverse is not true since every closed $L - subset$ is not a $N.\alpha - bounded$ set in general, as the following example shows.

Example 3.13. By Example 3.10, let $\rho \in L^X$ be a L–subset, define as follows:

$$\rho(x) = \begin{cases} 1 & : x = 3, 4, 5, ... \\ \frac{1}{6} & : x = 2 \end{cases}$$

We note that ρ is a closed L–subset because $\rho \in \tau'$ where τ' is a L–topology with a subbase $\{\mu'_n, \eta'_n : n \in X\}$, and we have:

$$\eta_3(x) = \begin{cases} 0 & : x \geq 3 \\ \frac{5}{6} & : x < 3 \end{cases}$$

And so

$$\rho = \eta_3'(x) = \begin{cases} 1 & : x \geq 3 \\ \frac{1}{6} & : x < 3 \end{cases}$$

Therefore $\rho \in \tau'$. Now, the family $\Psi = \{\mu_n : n \in X\}$ is $0-$cover of 1_X. Since $\alpha = 0 \in Pr(L) = [0,1)$ $(\forall x \in X\ \exists \lambda \in \Psi\ \ni\ \lambda(x) > 0)$ which has no finite subfamily Ψ_o of Ψ such that $\{int(cl(\mu_n)) : i < n\}$ is a $0-$cover of ρ (since $\Psi_o = \{\mu_2 \vee \eta_2, \mu_n : n \geq 4\}$ is not a $0-$cover of ρ). Hence ρ is not a $N.\alpha-$bounded set.

Theorem 3.14. Let (L^X, τ) be an $L-$ts. The following statements hold:
(i) $0_X, 1_X \in N\alpha BC(L^X, \tau)$.
(ii) If $\mu_1, \mu_2, ..., \mu_n \in N\alpha BC(L^X, \tau)$, then $\bigvee_{i=1}^n \mu_i \in N\alpha BC(L^X, \tau)$.
(iii) If $\{\mu_i : i \in I\} \subseteq N\alpha BC(L^X, \tau)$, then $\bigwedge_{i\in I} \mu_i \in N\alpha BC(L^X, \tau)$.
(iv) Every $N\alpha-$bounded and closed set is $N\alpha B-$closed.
(v) $\mu \in L^X$ is $N\alpha B-$closed iff there exists $\lambda \in N\alpha BR_{x_\alpha}$ such that $\mu \leq \lambda$ for each $x_\alpha \in M(L^X)$ with $x_\alpha \notin \mu$.

Proof. (i) Obvious.

(ii) Let $\mu_1, \mu_2, ..., \mu_n \in N\alpha BC(L^X, \tau)$ and $x_\alpha \in M(L^X)$ such that $x_\alpha \in N\alpha B.cl\big(\bigvee_{i=1}^n \mu_i\big)$, then for each $\lambda \in N\alpha BR_{x_\alpha}$ we have $\bigvee_{i=1}^n \mu_i \nleq \lambda$ and so $\mu_i \nleq \lambda$ for some $i = 1, 2, ..., n$. Hence $x_\alpha \in N\alpha B.cl(\mu_i)$ for some $i = 1, 2, ..., n$. Since μ_i is a $N\alpha B$–closed set, then $N\alpha B.cl(\mu_i) \leq \mu_i$ for some $i = 1, 2, ..., n$ and so $x_\alpha \in \mu_i$ for some $i = 1, 2, ..., n$ and hence $x_\alpha \in \bigvee_{i=1}^n \mu_i$. Thus $N\alpha B.cl\big(\bigvee_{i=1}^n \mu_i\big) \leq \bigvee_{i=1}^n \mu_i \ \ \ldots\ (*)$
Conversely, since $\mu_i \leq N\alpha B.cl(\mu_i)$ then $\bigvee_{i=1}^n \mu_i \leq N\alpha B.cl\big(\bigvee_{i=1}^n \mu_i\big) \ \ \ldots\ (**)$.
Hence from $(*)$ and $(**)$ we have $N\alpha B.cl\big(\bigvee_{i=1}^n \mu_i\big) = \bigvee_{i=1}^n \mu_i$. Thus $\bigvee_{i=1}^n \mu_i \in N\alpha BC(L^X, \tau)$.

(iii) Let $\mu_1, \mu_2, ..., \mu_n \in N\alpha BC(L^X, \tau)$ and $x_\alpha \in M(L^X)$ such that $x_\alpha \in N\alpha B.cl(\bigwedge_{i\in I} \mu_i)$, then for each $\lambda \in N\alpha BR_{x_\alpha}$ we have $\bigwedge_{i\in I} \mu_i \nleq \lambda$ and so $\mu_i \nleq \lambda$ for each $i \in I$. Hence $x_\alpha \in N\alpha B.cl(\mu_i)$ for each $i \in I$. Since μ_i is a $N\alpha B$–closed set, then $N\alpha B.cl(\mu_i) \leq \mu_i$ for each $i \in I$ and so $x_\alpha \in \mu_i$ for each $i \in I$ and hence $x_\alpha \in \bigwedge_{i\in I} \mu_i$. Thus $N\alpha B.cl(\bigwedge_{i\in I} \mu_i) \leq \bigwedge_{i\in I} \mu_i \ \ \ldots\ (*)$.
Conversely, since $\mu_i \leq N\alpha B.cl(\mu_i)$ then $\bigwedge_{i\in I} \mu_i \leq N\alpha B.cl(\bigwedge_{i\in I} \mu_i) \ \ \ldots\ (**)$.
Hence from $(*)$ and $(**)$ we have $N\alpha B.cl(\bigwedge_{i\in I} \mu_i) = \bigwedge_{i\in I} \mu_i$. Thus $\bigwedge_{i\in I} \mu_i \in N\alpha BC(L^X, \tau)$.

(iv) Let $\mu \in L^X$ be a $N\alpha-$bounded and closed set and let $x_\alpha \in M(L^X)$ such that $x_\alpha \notin \mu$, since μ is a $N\alpha-$bounded and closed set, then $\mu \in N\alpha BR_{x_\alpha}$. Since $\mu \leq \mu$, then $x_\alpha \notin N\alpha Bcl(\mu)$ and so $N\alpha Bcl(\mu) \leq \mu$. Therefore μ is $N\alpha B-$closed set.

(v) Suppose that μ is a $N\alpha B-$closed set, $x_\alpha \in M(L^X)$ and $x_\alpha \notin \mu$. By Definition 3.11, there exists $\lambda \in N\alpha BR_{x_\alpha}$ with $\mu \leq \lambda$. Conversely, provided that the condition is satisfied. If μ is not a $N\alpha B-$closed set, then there exists $x_\alpha \in M(L^X)$ such that $x_\alpha \in N\alpha Bcl(\mu)$ and $x_\alpha \notin \mu$. Hence $\mu \nleq \lambda$ for each $\lambda \in N\alpha BR_{x_\alpha}$. It conflicts with the hypothesis, and so μ is a $N\alpha B-$closed set.

Theorem 3.15. Let (L^X, τ) be an $L-ts$ and $\mu \in L^X$. Then the mapping $N\alpha Bcl : L^X \to L^X$ is called a closure operator of $N\alpha-$boundedness iff it satisfies:
(i) $N\alpha Bcl(0_X) = 0_X$.
(ii) $\mu \leq N\alpha Bcl(\mu)$.
(iii) $N\alpha Bcl(\mu \vee \eta) = N\alpha Bcl(\mu) \vee N\alpha Bcl(\eta)$.
(iv) $N\alpha Bcl(N\alpha Bcl(\mu)) = N\alpha Bcl(\mu)$.

A closure operator of $N\alpha-$boundedness $N\alpha Bcl$ generates $L-$topology $\tau_{N\alpha Bcl}$ on L^X as: $\tau_{N\alpha Bcl} = \{\mu \in L^X : N\alpha Bcl(\mu') = \mu'\}$.

Proof. It follows directly from Theorems 3.12 and 3.14.

Theorem 3.16. Let (L^X, τ) be an $L-ts$. Then:
(i) $\tau_{\alpha B} \leq \tau_{N\alpha B} \leq \tau$.
(ii) If (L^X, τ) is $\alpha-$bounded (resp. $N\alpha-$bounded space), then $\tau = \tau_{\alpha B}$ (resp. $\tau = \tau_{N\alpha B}$).
(iii) If (L^X, τ) is LR_2-space, then $\tau_{\alpha B} = \tau_{N\alpha B}$.

Proof. (i) Let $\mu \in \tau_{\alpha B}$, then $\alpha Bcl(\mu') \leq \mu'$. Since $N\alpha Bcl(\mu') \leq \alpha Bcl(\mu')$ hence $N\alpha Bcl(\mu') \leq \mu'$ and so $\mu \in \tau_{N\alpha B}$. If $\mu \in \tau_{N\alpha B}$, then $N\alpha Bcl(\mu') \leq \mu'$ and so $\mu \in \tau$. also if $\mu \in \tau_{N\alpha B}$, then $N\alpha Bcl(\mu') \leq \mu'$. Since $cl(\mu') \leq NBcl(\mu')$, hence $cl(\mu') \leq \mu'$ and so $\mu \in \tau$.
(ii) We note that $\tau_{\alpha B} \leq \tau$ from (i). Now, let $\mu \in \tau$ then $\mu' \in \tau'$. Since 1_X is a $N\alpha-$bounded and $\mu' \leq 1_X$, μ' is $N\alpha-$bounded. By Theorem 3.7 and by Theorem 3.14 (iv), we have μ' that is a $N\alpha B-$closed set and so $\mu' \in \tau_{N\alpha B}$. Thus $\tau = \tau_{N\alpha B}$.

Definition 3.17. Let (L^X, τ) be an $L-ts$, $\mu \in L^X$ and $N\alpha B.int(\mu) = \vee\{\rho \in L^X : \rho \in N\alpha BO(L^X, \tau), \rho \leq \mu\}$. We say that $N\alpha B.int(\mu)$ is the $N\alpha B-$interior of μ.

The following Theorem shows the relationships between $N\alpha B-$closure operator and $N\alpha B-$interior operator.

Theorem 3.18. Let (L^X, τ) be an $L-ts$ and $\mu \in L^X$. Then the following are true:
(i) μ is $N\alpha B-$open iff $\mu = N\alpha B.int(\mu)$.
(ii) $(N\alpha Bcl(\mu))' = N\alpha B.int(\mu')$ and $(N\alpha B.int(\mu))' = N\alpha Bcl(\mu')$.
(iii) $N\alpha Bcl(\mu) = (N\alpha B.int(\mu'))'$ and $N\alpha int(\mu) = (N\alpha cl(\mu'))'$.
(iv) $N\alpha B.int(\mu) \leq \alpha B.int(\mu) \leq int(\mu) \leq \mu$.
(v) If $\eta \in L^X$ and $\mu \leq \eta$ then $N\alpha B.int(\mu) \leq N\alpha B.int(\eta)$.
(vi) $N\alpha B.int(N\alpha B.int(\mu)) = N\alpha B.int(\mu)$.

Proof. (i) Let $\mu \in L^X$ be an $N\alpha B-$ open set, then $N\alpha B.int(\mu) = \vee\{\rho \in L^X : \rho \in N\alpha BO(L^X, \tau), \rho \leq \mu\} = \mu$ and so $\mu = N\alpha B.int(\mu)$.

Conversely, let $\mu = N\alpha B.int(\mu)$, since $N\alpha B.int(\mu) = \vee\{\rho \in L^X : \rho \in N\alpha BO(L^X, \tau), \rho \leq \mu\} = \mu$. Therefore μ is $N\alpha B-$open set.

(ii) It follows directly from Theorem 3.12 (iv) and Definition 3.17.
(iii) It follows directly from (ii).
(iv) It follows directly from (ii) and Theorems 3.12 (i).
(v) It follows directly from (ii) and Theorem 3.12 (ii).
(vi) It follows directly from (ii) and Theorem 3.12 (iii).

Theorem 3.19. Let (L^X, τ) be an $L-ts$. The following statements hold:
(i) $0_X, 1_X \in N\alpha BO(L^X, \tau)$.
(ii) If $\mu_1, \mu_2, ..., \mu_n \in N\alpha BO(L^X, \tau)$, then $\bigwedge_{i=1}^{n} \mu_i \in N\alpha BO(L^X, \tau)$.

(iii) If $\{\mu_i : i \in I\} \subseteq N\alpha BO(L^X,\tau)$, then $\bigvee_{i\in I}\mu_i \in N\alpha BO(L^X,\tau)$.

Proof. It is similar to the proof of Theorem 3.14.

Definition 3.20. Let (L^X,τ) be an L–ts and S be a molecular net in L^X. Then $x_\alpha \in M(L^X)$ is called a $N\alpha$–bounded limit point of S, (or S $N\alpha B$–converges to x_α) in symbol $S \xrightarrow{N\alpha B} x_\alpha$ if for every $\mu \in N\alpha BR_{x_\alpha}$ and there is $n \in D$ such that $m \in D$ and $m \geq n$ we have $S(m) \notin \mu$.

The union of all $N\alpha$–bounded limit points of S is denoted by $N\alpha B.\lim(S)$.

Definition 3.21. Let (L^X,τ) be an L–ts and S be a molecular net in L^X. Then $x_\alpha \in M(L^X)$ is called a $N\alpha$–bounded cluster point of S, in symbol $S \overset{N\alpha B}{\propto} x_\alpha$, if for every $\mu \in N\alpha BR_{x_\alpha}$ and every $n \in D$ there is $m \in D$ such that $m \geq n$ and $S(m) \notin \mu$.

The union of all $N\alpha$–bounded cluster points of S is denoted by $N\alpha B.adh(S)$.

Theorem 3.22. Suppose that S is a molecular net in (L^X,τ), $\mu \in L^X$ and $x_\alpha \in M(L^X)$. Then the following statements hold:

(i) $x_\alpha \in N\alpha B.\lim(S)$ iff $S \xrightarrow{N\alpha B} x_\alpha$.

(ii) $x_\alpha \in N\alpha B.adh(S)$ iff $S \overset{N\alpha B}{\propto} x_\alpha$.

(iii) $\lim(S) \leq N\alpha B.\lim(S)$.

(iv) $adh(S) \leq N\alpha B.adh(S)$.

(v) $N\alpha B.\lim(S)$ and $N\alpha B.adh(S)$ are $N\alpha B$–closed in L^X.

(vi) $x_\alpha \in N\alpha B.cl(\mu)$ (resp. $x_\alpha \in \delta cl(\mu)$ [9]), iff there exists a molecular net S in μ such that S is $N\alpha B$–converges (resp. δ–converges) to x_α.

Proof. (i) Let $x_\alpha \in N\alpha B.\lim(S)$ and let $\lambda \in N\alpha BR_{x_\alpha}$. Since $x_\alpha \notin \lambda$, then $N\alpha B.\lim(S) \notin \lambda$. Therefore there exists $y_\gamma \in M(L^X)$ such that $y_\gamma \in N\alpha B.\lim(S)$ and $y_\gamma \notin \lambda$. Then $\lambda \in N\alpha BR_{y_\gamma}$, and so there is $n \in D$ such that for each $m \in D$ and $m \geq n$ we have $S(m) \notin \lambda$, but since $\lambda \in N\alpha BR_{x_\alpha}$ so $S \xrightarrow{N\alpha B} x_\alpha$. Conversely, let $S \xrightarrow{N\alpha B} x_\alpha$, then by Definition 3.20, we have $x_\alpha \in N\alpha B.\lim(S)$.

(ii) Let $x_\alpha \in N\alpha B.(S)$ and let $\lambda \in N\alpha BR_{x_\alpha}$. Since $x_\alpha \in N\alpha B.(S)$, then every $n \in D$ there is $m \in D$ such that $m \geq n$ and $S(m) \notin \lambda$, hence $S \overset{N\alpha B}{\propto} x_\alpha$. Conversely, let $S \overset{N\alpha B}{\propto} x_\alpha$, then by Definition 3.21 we have $x_\alpha \in N\alpha B.(S)$.

(iii) Let $x_\alpha \in \lim(S)$ and let $\eta \in N\alpha BR_{x_\alpha}$. Since $N\alpha BR_{x_\alpha} \subseteq R_{x_\alpha}$, then $\eta \in R_{x_\alpha}$. And since $x_\alpha \in \lim(S)$, then, for each $\lambda \in R_{x_\alpha}$ there is $n \in D$ such that for each $m \in D$ and $m \geq n$, we have $S(m) \notin \lambda$ and so $S(m) \notin \eta$. Hence, $x_\alpha \in N\alpha B.\lim(S)$. So $\lim(S) \leq N\alpha B.\lim(S)$.

(iv) Let $x_\alpha \in adh(S)$ and let $\eta \in N\alpha BR_{x_\alpha}$. Since $N\alpha BR_{x_\alpha} \subseteq R_{x_\alpha}$, then $\eta \in R_{x_\alpha}$. And since $x_\alpha \in adh(S)$, then, for each $\lambda \in R_{x_\alpha}$ and for each $n \in D$ there exists $m \in D$ such that $m \geq n$ we have $S(m) \notin \lambda$. And so $S(m) \notin \eta$. Hence $x_\alpha \in N\alpha B.(S)$. So $adh(S) \leq N\alpha B.(S)$.

(v) Let $x_\alpha \in N\alpha B.cl(N\alpha B.\lim(S))$, then $N\alpha B.\lim(S) \nleq \lambda$ for each $\lambda \in N\alpha BR_{x_\alpha}$ and then there exists $y_\gamma \in M(L^X)$ such that $y_\gamma \in N\alpha B.\lim(S)$ and $y_\gamma \notin \lambda$. Then for each

$\mu \in N\alpha BR_{y_\gamma}$, there is $n \in D$ such that for each $m \in D$ and $m \geq n$ we have $S(m) \notin \mu$, and so $S(m) \notin \lambda$. Hence $x_\alpha \in N\alpha B.\lim(S)$. Thus $N\alpha B.cl(N\alpha B.\lim(S)) \leq N\alpha B.\lim(S)$ and so $N\alpha B.\lim(S)$ is a $N\alpha B$–closed set.

Similarly, one can easily verify that $N\alpha B.cl(N\alpha B.(S)) \leq N\alpha B.(S)$.

(vi) Let $x_\alpha \in M(L^X)$ such that $x_\alpha \in N\alpha B.cl(\mu)$, then $\mu \nleq \lambda$ for each $\lambda \in N\alpha BR_{x_\alpha}$. Since $\mu \nleq \lambda$ then, there exists $\alpha(\mu,\lambda) \in M(L)$ such that $x_{\alpha(\mu,\lambda)} \in \mu$ with $x_{\alpha(\mu,\lambda)} \notin \lambda$. Since the pair $(N\alpha BR_{x_\alpha}, \geq)$ is a directed set so we can define a molecular net $S : N\alpha BR_{x_\alpha} \longrightarrow M(L^X)$ as follows $S(\lambda) = x_{\alpha(\mu,\lambda)}$ for each $\lambda \in N\alpha BR_{x_\alpha}$. Hence S is a molecular net in μ.

Now let $\eta \in N\alpha BR_{x_\alpha}$ such that $\lambda \leq \eta$, so we have there exists $S(\eta) = x_{\alpha(\mu,\eta)} \notin \eta$ and so $S(\eta) = x_{\alpha(\mu,\eta)} \notin \lambda$. Hence S is $N\alpha B$–converges to x_α.

Conversely, let S be a molecular net in μ such that S is $N\alpha B$–converges to x_α, then for each $\lambda \in N\alpha BR_{x_\alpha}$ there is $n \in D$ such for each $m \in D$ and $m \geq n$, we have $S(m) \notin \lambda$. Since $S(n) \in \mu$ for each $n \in D, m \in D$. So $S(m) \in \mu$ and $\mu \geq S(m) > \lambda$ hence $\mu \nleq \lambda$ for each $\lambda \in N\alpha BR_{x_\alpha}$. This means that $x_\alpha \in N\alpha B.cl(\mu)$.

Definition 3.23. Let (L^X, τ) be an L–ts and I be an ideal in L^X. Then $x_\alpha \in M(L^X)$ is called:

(i) limit point of I [25], (or I converges to x_α) in symbol $I \rightarrow x_\alpha$ if $R_{x_\alpha} \subseteq I$. The union of all limit points of I is denoted by $\lim(I)$.

(ii) $N\alpha B$–bounded limit point of I, (or I $N\alpha B$–converges to x_α) in symbol $I \xrightarrow{N\alpha B} x_\alpha$ if $N\alpha BR_{x_\alpha} \subseteq I$. The union of all $N\alpha B$–bounded limit points of I is denoted by $N\alpha B.\lim(I)$.

Definition 3.24. Let (L^X, τ) be an L–ts and I be an ideal in L^X. Then $x_\alpha \in M(L^X)$ is called:

(i) Cluster point of I [25], in symbol $I \propto x_\alpha$ if for every $\mu \in R_{x_\alpha}$ and every $\lambda \in I$, $\lambda \vee \mu \neq 1_X$. The union of all cluster points of I is denoted by $\mathrm{adh}(I)$.

(ii) $N\alpha B$–bounded cluster point of I, in symbol $I \overset{N\alpha B}{\propto} x_\alpha$ if for every $\mu \in N\alpha BR_{x_\alpha}$ and every $\lambda \in I$, $\lambda \vee \mu \neq 1_X$. The union of all $N\alpha B$–bounded cluster points of I is denoted by $N\alpha B.\mathrm{adh}(I)$.

Theorem 3.25. Suppose that I is an ideal in (L^X, τ), $\mu \in L^X$ and $x_\alpha \in M(L^X)$. Then the following statements hold:

(i) $x_\alpha \in N\alpha B.\lim(I)$ iff $I \xrightarrow{N\alpha B} x_\alpha$.

(ii) $x_\alpha \in N\alpha B.\mathrm{adh}(I)$ iff $I \overset{N\alpha B}{\propto} x_\alpha$.

(iii) $\lim(I) \leq N\alpha B.\lim(I) \leq \alpha B.\lim(I)$.

(iv) $\mathrm{adh}(I) \leq N\alpha B.\mathrm{adh}(I) \leq \alpha B.\mathrm{adh}(I)$.

(v) $x_\alpha \in N\alpha B.cl(\mu)$ iff there exists an ideal I in L^X such that $I \xrightarrow{N\alpha B} x_\alpha$ and $\mu \notin I$.

(vi) $N\alpha B.\lim(I)$ and $N\alpha B.\mathrm{adh}(I)$ are $N\alpha B$–closed set in L^X.

Proof. (i) Let $x_\alpha \in N\alpha B.\lim(I)$ and let $\lambda \in N\alpha BR_{x_\alpha}$. Since $x_\alpha \notin \lambda$ and $x_\alpha \in N\alpha B.\lim(I)$, then $N\alpha B.\lim(I) \nleq \lambda$. Therefore there exists $y_\gamma \in M(L^X)$ such that $y_\gamma \in N\alpha B.\lim(I)$ and $y_\gamma \notin \lambda$. Then $\lambda \in N\alpha BR_{y_\gamma}$ and so $N\alpha BR_{x_\alpha} \subseteq N\alpha BR_{y_\gamma} \subseteq I$ hence $N\alpha BR_{x_\alpha} \subseteq I$. Thus $I \xrightarrow{N\alpha B} x_\alpha$. Conversely, let $I \xrightarrow{N\alpha B} x_\alpha$, then by Definition 3.23 (ii) we have $x_\alpha \in N\alpha B.\lim(I)$.

(ii) Let $x_\alpha \in N\alpha B.adh(I)$ and let $\lambda \in N\alpha BR_{x_\alpha}$. Since $x_\alpha \notin \lambda$ and $x_\alpha \in N\alpha B.adh(I)$, therefore $N\alpha B.adh(I) \nleq \lambda$ and so there exists $y_\gamma \in M(L^X)$ such that $y_\gamma \in N\alpha B.adh(I)$ and $y_\gamma \notin \lambda$ hence $\lambda \in N\alpha BR_{y_\gamma}$ and so $N\alpha BR_{x_\alpha} \subseteq N\alpha BR_{y_\gamma}$ and $\mu \vee \lambda \neq 1_X$ for each $\mu \in I$ hence $I \overset{N\alpha B}{\propto} x_\alpha$. Conversely, let $I \overset{N\alpha B}{\propto} x_\alpha$, then by Definition 3.24 (ii) we have $x_\alpha \in N\alpha B.adh(I)$.

(iii) Let $x_\alpha \in \lim(I)$ and let $\eta \in N\alpha BR_{x_\alpha}$. Since $N\alpha BR_{x_\alpha} \subseteq R_{x_\alpha}$, then $\eta \in R_{x_\alpha}$. And since $x_\alpha \in \lim(I)$, then $R_{x_\alpha} \subseteq I$ so for each $\eta \in R_{x_\alpha}$, $\eta \in I$ and since $\eta \in N\alpha BR_{x_\alpha}$, so $N\alpha BR_{x_\alpha} \subseteq I$. Hence $x_\alpha \in N\alpha B.\lim(I)$. So $\lim(I) \leq N\alpha B.\lim(I)$.

Let $x_\alpha \in N\alpha B.\lim(I)$ and let $\eta \in \alpha BR_{x_\alpha}$. Since $\alpha BR_{x_\alpha} \subseteq N\alpha BR_{x_\alpha}$, then $\eta \in N\alpha BR_{x_\alpha}$. And since $x_\alpha \in N\alpha B.\lim(I)$, then $N\alpha BR_{x_\alpha} \subseteq I$ so for each $\eta \in N\alpha BR_{x_\alpha}$, $\eta \in I$ and since $\eta \in \alpha BR_{x_\alpha}$, $\eta \in I$ and since $\eta \in \alpha BR_{x_\alpha}$, so $\alpha BR_{x_\alpha} \subseteq I$. Hence $x_\alpha \in \alpha B.\lim(I)$. So $N\alpha B.\lim(I) \leq \alpha B.\lim(I)$.

(iv) Let $x_\alpha \in \text{adh}(I)$ and let $\eta \in N\alpha BR_{x_\alpha}$. Since $x_\alpha \in \text{adh}(I)$, so for each $\lambda \in R_{x_\alpha}$, $\lambda \in I$ and since $\eta \in N\alpha BR_{x_\alpha}$ so $\eta \in R_{x_\alpha}$. Hence $x_\alpha \in N\alpha B.\text{adh}(I)$. So $\text{adh}(I) \leq \alpha B.\text{adh}(I)$. Similarly, one can easily verify that $N\alpha B.\text{adh}(I) \leq \alpha B.\text{adh}(I)$.

(v) Let $x_\alpha \in M(L^X)$ such that $x_\alpha \in N\alpha B.cl(\mu)$. The family $I = \{\rho \in L^X : \exists \lambda \in N\alpha BR_{x_\alpha} \ni \rho \leq \lambda\}$ is an ideal in L^X. Now we show that $\mu \notin I$. Since $x_\alpha \in N\alpha B.cl(\mu)$, then for each $\lambda \in N\alpha BR_{x_\alpha}$, $\mu \nleq \lambda$. So by definition of I we have $\mu \notin I$. Finally, we show that $I \xrightarrow{N\alpha B} x_\alpha$. Let $\lambda \in N\alpha BR_{x_\alpha}$, since $\lambda \leq \lambda$, then $\lambda \in I$. So $N\alpha BR_{x_\alpha} \subseteq I$. Thus $I \xrightarrow{N\alpha B} x_\alpha$.

Conversely, let I be an ideal in L^X such that $I \xrightarrow{N\alpha B} x_\alpha$ and $\mu \notin I$. Then for each $\lambda \in N\alpha BR_{x_\alpha}$, $\lambda \in I$. Since $\lambda \in I$, $\mu \notin I$, $\mu \nleq \lambda$ and so $x_\alpha \in N\alpha B.cl(\mu)$.

(vi) Let $x_\alpha \in N\alpha B.cl(N\alpha B.\lim(I))$, then $N\alpha B.\lim(I) \nleq \lambda$ for each $\lambda \in N\alpha BR_{x_\alpha}$ and then there exists $y_\gamma \in M(L^X)$ such that $y_\gamma \in N\alpha B.\lim(I)$ and $y_\gamma \notin \lambda$. Since $\lambda \in N\alpha BR_{y_\gamma}$ and $I \xrightarrow{N\alpha B} x_\alpha$ then $\eta \in I$ for each $\eta \in N\alpha BR_{x_\alpha}$. Since $y_\gamma \notin \lambda$ then $\lambda \in I$. But $\lambda \in N\alpha BR_{x_\alpha}$ and so $x_\alpha \in N\alpha B.\lim(I)$. Thus $N\alpha B.cl(N\alpha B.\lim(I)) \leq N\alpha B.\lim(I)$ and so $N\alpha B.\lim(I)$ is a $N\alpha B$–closed set. Similarly, one can easily verify that $N\alpha B.cl(N\alpha B.\text{adh}(I)) \leq N\alpha B.\text{adh}(I)$.

Theorem 3.26. Suppose that S is a molecular net in L–ts (L^X, τ), $\mu \in L^X$. Then:

(i) $N\alpha B.\lim(S) = N\alpha B.\lim(I(S))$.

(ii) $N\alpha B.\text{adh}(S) \leq N\alpha B.\text{adh}(I(S))$.

Proof. (i) Let $x_\alpha \in M(L^X)$ such that $x_\alpha \in N\alpha B.\lim(S)$, by Theorem 3.25 (i), we have $S \xrightarrow{N\alpha B} x_\alpha$ then for each $\lambda \in N\alpha BR_{x_\alpha}$ there is $n \in D$ such that for each $m \in D$ and $m \geq n$ we have $S(m) \notin \lambda$, and hence by the definition of $I(S)$ we have

$\lambda \in I(S)$ for each $\lambda \in N\alpha BR_{x_\alpha}$ and so $N\alpha BR_{x_\alpha} \subseteq I(S)$. Thus $x_\alpha \in N\alpha B.\lim(I(S))$, i.e., $N\alpha B.\lim(S) \leq N\alpha B.\lim(I(S))$.

Conversely, let $x_\alpha \in N\alpha B.\lim(I(S))$, then $\lambda \in I(S)$ for each $\lambda \in N\alpha BR_{x_\alpha}$, and then there is $n \in D$ such that for each $m \in D$ and $m \geq n$ we have $S(m) \notin \lambda$. Therefore $x_\alpha \in N\alpha B.\lim(S)$. Thus $N\alpha B.\lim(I(S)) \leq N\alpha B.\lim(S)$. Hence $N\alpha B.\lim(S) = N\alpha B.\lim(I(S))$.

(ii) Let $x_\alpha \in N\alpha B.\text{adh}(S)$, then for each $\lambda \in N\alpha BR_{x_\alpha}$ and for each $n \in D$ such there is $m \in D$ and $m \geq n$ we have $S(m) \notin \lambda$. If $\mu \in I(S)$, then there is $n \in D$ such that for each $m \in D$ and $m \geq n$ then $S(m) \notin \mu$. Thus $\mu \vee \lambda \neq 1_X$ for each $\mu \in I(S)$ and for each $\lambda \in N\alpha BR_{x_\alpha}$, that is, $x_\alpha \in N\alpha B.\text{adh}(I(S))$. This implies that $N\alpha B.\text{adh}(S) \leq N\alpha B.\text{adh}(I(S))$.

Theorem 3.27. Let (L^X, τ) be a $L-$ts. and let $\mu \in L^X$. Then:

(i) If 1_X is $NQ_\alpha-$compact iff 1_X is $N\alpha-$bounded.

(ii) If μ is $NQ_\alpha-$compact, then μ is $N\alpha-$bounded.

(iii) If η is $NQ_\alpha-$compact and $\mu \leq \eta$, then μ is $N\alpha-$bounded.

(iv) If $\mu_1, \mu_2, ..., \mu_m$ are $N\alpha-$bounded sets, then $\bigvee_{i=1}^{m} \mu_i$ is $N\alpha-$bounded.

Proof (i) Let (L^X, τ) be a $NQ_\alpha-$compact space and let $\Psi = \{\lambda_i : i \in I\} \subseteq \tau'$ be an $\alpha-$RF of 1_X. Since (L^X, τ) is a $NQ_\alpha-$compact space, there exists a finite subfamily $\Psi_\circ = \{\lambda_i : i = 1, 2, ..., m\} \in 2^{(\Psi)}$ such that $\Psi_\circ$ is an $\alpha-$RCRF of 1_X and so 1_X is $N\alpha-$bounded set. Conversely, let 1_X be a $N\alpha-$bounded set and let $\Psi = \{\lambda_i : i \in I\} \subseteq \tau'$ be an $\alpha-$RF of 1_X. Since 1_X is a $N\alpha-$bounded set, then there exists a finite subfamily $\Psi_\circ = \{\lambda_i : i = 1, 2, ..., m\} \in 2^{(\Psi)}$ such that $\Psi_\circ$ is an $\alpha-$RCRF of 1_X and so 1_X is a $NQ_\alpha-$compact set.

(ii) Let μ be a $NQ_\alpha-$compact and let $\Psi = \{\lambda_i : i \in I\} \subseteq \tau'$ be an $\alpha-$RF of 1_X and so Ψ is an $\alpha-$RF of μ. Since μ is a $NQ_\alpha-$compact set, then there exists a finite subfamily $\Psi_\circ = \{\lambda_i : i = 1, 2, ..., m\} \in 2^{(\Psi)}$ such that $\Psi_\circ$ is an $\alpha-$RCRF of μ and so μ is a $N\alpha-$bounded set.

(iii) Let η be a $NQ_\alpha-$compact set and $\mu \leq \eta$. Let $\Psi = \{\lambda_i : i \in I\} \subseteq \tau'$ be an $\alpha-$RF of 1_X and so Ψ is $\alpha-$RF of η. Since η is a $NQ_\alpha-$compact set, then there exists a finite subfamily $\Psi_\circ = \{\lambda_i : i = 1, 2, ..., m\} \in 2^{(\Psi)}$ such that $\Psi_\circ$ is an $\alpha-$RCRF of η, since $\mu \leq \eta$, then $\Psi_\circ$ is an $\alpha-$RCRF of μ and so μ is a $N\alpha-$bounded set.

(iv) Let $\mu_1, \mu_2, ..., \mu_n$ be a $N\alpha-$bounded set and let $\Psi \subseteq \tau'$ be an $\alpha-$RF of 1_X. Since $\mu_1, \mu_2, ..., \mu_m$ are $N\alpha-$bounded sets, then there exist $\Psi_\circ^1, \Psi_\circ^2, ..., \Psi_\circ^n \in 2^{(\Psi)}$ such that $\Psi_\circ^1, \Psi_\circ^2, ..., \Psi_\circ^n$ are $\alpha-$RCRF of $\mu_1, \mu_2, ..., \mu_n$, respectively, and so for each $x_\alpha \in \mu_1$ there is $\lambda_1 \in \Psi_\circ^1$ such that $\lambda_1 \in R_{x_\alpha}$, for each $x_\alpha \in \mu_2$, there is $\lambda_2 \in \Psi_\circ^2$ such that $\lambda_2 \in R_{x_\alpha}$, ..., for each $x_\alpha \in \mu_n$ there is $\lambda_n \in \Psi_\circ^n$ such that $\lambda_n \in R_{x_\alpha}$. Hence, for each $x_\alpha \in \mu_1 \vee \mu_2 \vee ... \vee \mu_n$, we have $x_\alpha \in \mu_i$ for some $i \in \{1, 2, ..., n\}$ and so there is $\lambda_i \in \bigvee_{i=1}^{n} \Psi_\circ^i$ such that $\lambda_i \in R_{x_\alpha}$ and hence $\bigvee_{i=1}^{n} \Psi_\circ^i$ is an $\alpha-$RCRF of $\mu_1 \vee \mu_2 \vee ... \vee \mu_n$. Thus $\Psi_o = \vee_{i=1}^{n} \Psi_i^o$ is $\alpha-$RCRF of $\mu_1 \vee \mu_2 \vee \ldots \vee \mu_n$ and so $\vee\{\mu_i : i = 1, 2, \ldots, n\}$ is a $N\alpha-$bounded set.

The following Example shows that the converse of Theorem 3.27 (ii) is not true in general.

Example 3.28. Let $X = \{x\}$, $L = [0,1]$, and let $\tau = \{0_X, x_{\frac{1}{4}}, x_{\frac{8}{9}}, 1_X\}$. Then (L^X, τ) is $L-$ts and $\tau' = \{0_X, x_{\frac{3}{4}}, x_{\frac{1}{9}}, 1_X\}$. Firstly, we show that $\mu = x_{\frac{1}{2}} \in M(L^X)$ is $N\alpha$–bounded set. We suppose that $S = \{x_\alpha : x \in X, \alpha \leq \frac{1}{2}\}$ is any constant α–molecular net in μ. If $\alpha \leq \frac{1}{2}$, we take $x \in X$ so that $S(n) = x_\alpha \notin cl(\text{int}(\lambda))$ for each $\lambda \in R_{x_{0.5}} = \{0_X, x_{\frac{1}{9}}\}$, where $cl(\text{int}(0_X)) = 0_X$ and $cl(\text{int}(x_{\frac{1}{9}})) = 0_X$. Then x_α is a δ–cluster point of S in 1_X. Thus $\mu = x_{0.5}$ is a $N\alpha$–bounded set.

Now, we show that $\mu = x_{0.5} \in M(L^X)$ is not a NQ_α–compact set. Indeed, (L^X, τ) is not LR_2–space. Since there is $x_{0.8} \in M(L^X)$ and there is $\lambda = x_{\frac{3}{4}} \in R_{x_{0.8}}$ such that for each $\eta \in R_{x_{0.8}} = \{0_X, x_{\frac{1}{9}}, x_{\frac{3}{4}}\}$ we have $\lambda \not\leq \text{int}(\eta)$, where $\text{int}(0_X) = 0_X$, $\text{int}(x_{\frac{1}{9}}) = 0_X$ and $\text{int}(x_{\frac{3}{4}}) = x_{\frac{1}{4}}$. Hence (L^X, τ) is not a LR_2–space, and so $\mu = x_{0.5}$ is not a NQ_α–compact set.

Theorem 3.29: Every $L-$subset with finite support is a $N\alpha-$bounded set.

Proof. Let (L^X, τ) be a $L-$ts. and $\mu \in L^X$ with finite support. Then by Theorem 2.17 (i), we have μ is $NQ_\alpha-$compact and by Theorem 3.27 (ii) we have μ is a $N\alpha-$bounded set.

Theorem 3.30. Let (L^X, τ) be an L-ts and μ be a $N\alpha-$bounded set in (L^X, τ), then μ is a $N\alpha-$bounded set in (L^X, τ_Y).

Proof. Let μ be a $N\alpha$–bounded set in (L^X, τ) and $\phi \neq Y \subseteq X$. Let $\Psi = \{\rho_i = \eta_i \wedge 1_Y : \eta_i \in \tau', i \in I\} \subseteq \tau'_Y$ be an α–RF of 1_Y. Hence $\Psi^* = \{\eta_i : i \in I\} \subseteq \tau'$ is an α–RF of 1_X. Since μ is a $N\alpha$–bounded set in (L^X, τ), then there exists $\Psi^*_\circ = \{\eta_{i_m} : m = 1, 2, \ldots, n\} \in 2^{(\Psi^*)}$ such that $\Psi^*_\circ$ is an α–RCRF of μ and so $\Psi_\circ = \{\rho_{i_m} = \eta_{i_m} \wedge 1_Y : m = 1, 2, \ldots, n\} \in 2^{(\Psi)}$ is an α–RCRF of μ. Hence μ is a $N\alpha$–bounded set in (L^Y, τ_Y).

4. α–Nets' characterizations of $N.\alpha$–Boundedness

In this section, we give several characterizations of N.α–Boundedness in terms of both the upper δ–limit of nets of L–subsets and δ–cluster points of constant molecular α–nets.

Theorem 4.1. Let (L^X, τ) be an L − ts, $\alpha \in M(L)$ and $\mu \in L^X$. Then μ is N.α–bounded iff for each constant molecular α–net S contained in μ has δ–cluster point in X with height α.

Proof. Suppose that μ is N.α–bounded and let $S = \{S(n) : n \in D\}$ be a constant molecular α–net in μ with height α. If S does not have any δ–cluster point in X with height α. Then for all $x_\alpha \in M(L^X)$, x_α is not a δ–cluster point of S and so there exists $\lambda_x \in R_{x_\alpha}$ and $n_x \in D$ such that for every $m \in D$ and $m \geq n_x$, then $S(m) \in cl(int(\lambda_x))$. Put $\Psi = \{\lambda_x : x \in X \text{ and } \alpha \in M(L)\}$ is an α–RF of 1_X. Since μ is N.α–bounded, then there exist $\Psi_o = \{cl(int(\lambda_x)) : i = 1, 2, \ldots, k\} \in 2^{(\Psi)}$ such that Ψ_o is an α–RCRF of μ. Hence, for each $i \leq k$ we have $n_{x_i} \in D$ when $m \geq n_{x_i}$, $S(m) \in cl(int(\lambda_{x_i}))$. Since D is a directed set, then there is $n_o \in D$ such that $n_o \geq n_{x_i}$ $(i = 1, 2, \ldots, k)$. Hence $S(m) \in \bigwedge_{i=1}^{k} cl(\text{int}(\lambda_{x_i}))$ whenever $m \geq n_o$. This means that $S(m)$ not have α–RCRF in Ψ_o, and so Ψ_o is not α–RCRF of μ. This contradicts the hypothesis that μ is N.α–bounded. Thus S has a δ–cluster point in X with height α.

Conversely, suppose that μ is not N.α–bounded. Then there exist $\alpha \in M(L)$ and a family Ψ which is an α–RF of 1_X, but for any family $\Psi_o \in 2^{(\Psi)}$, we have Ψ_o is not an α–RCRF of μ. Then there is a point $x_\alpha \in \mu$ with height α such that $x_\alpha \le \wedge\Psi_o$ and x_α is denoted by $(x(\Psi_o))_\alpha$. Since $2^{(\Psi)}$ is a directed set with relation $\le$, then $S = \{x((\Psi_o))_\alpha : \Psi_o \in 2^{(\Psi)}\}$ is a constant molecular α–net in μ. Take an arbitrary point y_α in X with height α, since Ψ is an α–RF of 1_X, then there is $\lambda \in \Psi$ such that $\lambda \in R_{y_\alpha}$. Hence, for each $\Psi_o \in 2^{(\Psi)}$ such that $cl(int(\lambda)) \in \Psi_o$, there is $(x(\Psi_o))_\alpha \le \wedge\Psi_o \le cl(int(\lambda))$, i.e., $S \le cl(int(\lambda))$. This shows that y_α is not a δ–cluster point of S in X with height α, which contradicts the hypothesis. Thus μ is a N.α–bounded.

Theorem 4.2. (Alexander's subbase lemma) Suppose that (L^X, τ) is a L − ts, $\alpha \in M(L)$, $\mu \in L^X$ and ξ is a subbase of τ'. If for each α–RF Ψ of 1_X consisting of elements of ξ, there is $\Psi_o \in 2^{(\Psi)}$ which is an α–RCRF of μ, then μ is a N.α–bounded set.

Proof. It is similar to Theorem 5.1 in [15].

Theorem 4.3. Let $\{(L^{X_i}, \tau_i) : i \in I\}$ be a L–ts's and μ_i be a $N\alpha$–bounded set in (L^{X_i}, τ_i) for each $i \in I$, then the product set $\mu = \prod_{i\in I} \mu_i$ is a $N\alpha$–bounded in the product space.

Proof. Let $\alpha \in M(L)$ and let $\mu_i \in L^{X_i}$ be a $N\alpha$–bounded set in (L^{X_i}, τ_i) for each $i \in I$. Let the set $\xi = \{P_i^{-1}(\lambda_i) : i \in I, \lambda_i \in \tau_i'\}$ be a subbase of the family of all closed sets of the product space. To show that $\prod_{i\in I} \mu_i$ is a $N\alpha$–bounded set, we only need to verify that for each α–RF, $\Psi \subseteq \xi$ of the set $\prod_{i\in I} X_i$ there exists $\Psi_o \in 2^{(\Psi)}$ such that Ψ_o is an α–RCRF of the set $\prod_{i\in I} \mu_i$. Let $\Psi = \vee_{n\in\mathbb{N}}\{P_{i_n}^{-1}(\lambda) : \lambda \in \mathfrak{R}_{i_n}, \mathfrak{R}_{i_n} \subseteq \tau_{i_n}'\}$.

Now, we consider the following two cases:

(i) There exists $i_o \in I$ such that no molecular with height α is contained in μ_{i_o}. Then by the definition of a product set $\prod_{i\in I} \mu_i$ it follows immediately that no point no molecular with height α is contained in $\prod_{i\in I} \mu_i$ and hence for each $\Psi_o \in 2^{(\Psi)}$, we have Ψ_o is an α–RCRF of $\prod_{i\in I} \mu_i$ and so $\prod_{i\in I} \mu_i$ is a $N\alpha$–bounded set.

(ii) For every $i \in I$, X_i contains a molecular with height α, x_α^i say. Then there must be some $n \in \mathbb{N}$ such that $\mathfrak{R}_{i_n}$ is an α–RF of X_{i_n}. In fact, for each $n \in \mathbb{N}$, $\mathfrak{R}_{i_n}$ is not an α–RF of X_{i_n}, then there exists $y^{i_n} \in X_{i_n}$ with $y_\alpha^{i_n} \in 1_{X_{i_n}} \wedge (\wedge\mathfrak{R}_{i_n})$. For each $i \notin \{i_n : n \in \mathbb{N}\}$ take $y^i = x^i$. Let $y \in \prod_{i\in I} X_i$ with projections y^i and $i \in I$. On the other hand, for each $\eta \in \Psi$, $y_\alpha \in \eta$. To see this, let $\eta = P_{i_n}^{-1}(\lambda)$, where $\lambda \in \mathfrak{R}_{i_n}$, then $\eta(y) = P_{i_n}^{-1}(\lambda)(y) = \lambda(P_{i_n}(y)) = \lambda(y^{i_n}) \ge \alpha$. Since $x_\alpha^{i_n} \in \lambda$ and $y = (y^{i_1}, y^{i_2}, ..., y^{i_n})$, hence $y_\alpha \in \eta$. However, this is impossible since Ψ is an α–RF of $\prod_{i\in I} X_i$. Suppose that $\mathfrak{R}_{i_n}$ is an α–RF of $\prod_{i\in I} X_i$. By the $N\alpha$–boundedness of μ_{i_n}, there exists $\mathfrak{R}_{i_n}^* \in 2^{(\mathfrak{R}_{i_n})}$ such that $\mathfrak{R}_{i_n}^*$ is an α–RCRF of μ_{i_n}. Consider the $\Psi_o = \{P_{i_n}^{-1}(cl(int(\lambda))) : \lambda \in \mathfrak{R}_{i_n}^*\}$ which is a finite subset of Ψ, i.e., $\Psi_o \in 2^{(\Psi)}$, if $x_\alpha \in \prod_{i\in I} \mu_i$, then $\mu_{i_n}(P_{i_n}(x)) = \mu_{i_n}(x^{i_n}) \ge \alpha$ so there is $\lambda \in \mathfrak{R}_{i_n}^*$ with $\lambda \in R_{x_\alpha^{i_n}}$, i.e., $cl(int(\lambda(x^{i_n}))) = cl(int(\lambda(P_{i_n}(x)))) \not\ge \alpha$, then $P_{i_n}^{-1}(cl(int(\lambda)))(x) \not\ge \alpha$ and therefore $P_{i_n}^{-1}(cl(int(\lambda))) \in R_{x_\alpha}$. This shows that $\Psi_o \in 2^{(\Psi)}$ is an α–RCRF of $\prod_{i\in I} \mu_i$ and so $\prod_{i\in I} \mu_i$ is a $N\alpha$–bounded set.

Theorem 4.4. Let $S = \{S(n) : n \in D\}$ and $T = \{T(n) : n \in D\}$ be a molecular nets in a L–ts (L^X, τ) such that $T(n) \ge S(n)$ for each $n \in D$ and $x_\alpha \in M(L^X)$. Then the following results are true:

(i) If $S \xrightarrow{N\alpha B} x_\alpha$, then $T \xrightarrow{N\alpha B} x_\alpha$.
(ii) If $S \overset{N\alpha B}{\propto} x_\alpha$, then $T \overset{N\alpha B}{\propto} x_\alpha$.

Proof. (i) Let $x_\alpha \in M(L^X)$ such that $S \xrightarrow{N\alpha B} x_\alpha$, then for each $\lambda \in N\alpha BR_{x_\alpha}$, there exists $n \in D$ such that for each $m \in D$ and $m \geq n$ then $S(m) \notin \lambda$. Since $T(n) \geq S(n) > \lambda$, and so for each $\lambda \in N\alpha BR_{x_\alpha}$ there exists $n \in D$ such that for each $m \in D$ and $m \geq n$ then $T(m) \notin \lambda$. This shows that T is $N\alpha B$–converges to x_α.

(ii) Let $x_\alpha \in M(L^X)$ such that $S \overset{N\alpha B}{\propto} x_\alpha$, then for each $\lambda \in N\alpha BR_{x_\alpha}$ and each $n \in D$ there exists $m \in D$ such that $m \geq n$ then $S(m) \notin \lambda$. Since $T(n) \geq S(n)$ for each $n \in D$, then $T(n) \geq S(n) > \lambda$. Thus for each $\lambda \in N\alpha BR_{x_\alpha}$ and for each $n \in D$ there exists $m \in D$ such that $m \geq n$ then $T(m) \notin \lambda$. This shows that $T \overset{N\alpha B}{\propto} x_\alpha$.

Theorem 4.5. Let $\{\mu_n : n \in D\}$ be a net of closed L-subsets in L^X such that $\mu_{n_1} \leq \mu_{n_2}$ then $\delta.\overline{\lim}(\mu_n) \leq \wedge\{\mu_n : n \in D\}$ iff $n_2 \leq n_1$.

Proof. Let $x_\alpha \in \delta.\overline{\lim}(\mu_n)$ and let $x_\alpha \notin \wedge\{\mu_n : n \in D\}$ hence there is $n_\circ \in D$ such that $x_\alpha \notin \mu_{n_\circ}$. Let $\rho = \mu_{n_\circ}$, then $\rho \in R_{x_\alpha}$. Since $x_\alpha \in \delta.\overline{\lim}(\mu_n)$ and $\rho \in R_{x_\alpha}$, there is $n \in D$, $n \geq n_\circ$ such that $\mu_n \nleq cl(\text{int}(\rho))$, which contradicts the hypothesis that $x_\alpha \notin \wedge\{\mu_n : n \in D\}$. Thus $x_\alpha \in \wedge\{\mu_n : n \in D\}$.

Theorem 4.6. Let (L^X, τ) be an L–ts and $\mu \in L^X$. Then μ is a N.α–bounded set iff for every net $\{\eta_n : n \in D\}$ of closed L–subsets in L^X such that $\delta \lim(\eta_n)(x) < \alpha$, for each $x \in X$, there is $n_o \in D$ for which $\eta_n \wedge \mu = 0_X$ for every $n \in D, n \geq n_o$.

Proof. Let $\mu \in L^X$ be a N.α–bounded set and let $\{\eta_n : n \in D\}$ be a net of closed L–subsets in L^X such that $\delta.\overline{\lim}(\eta_n)(x) < \alpha$, for each $x \in X$. Then for every molecular $x_\alpha \in M(L^X)$ there exists $\rho_x \in R_{x_\alpha}$ and $n_x \in D$ such that $\eta_n \leq cl(int(\rho_x))$ for every $n \in D$, $n \geq n_x$. Since $\rho_x \in R_{x_\alpha}$ for every $x \in X$, then the family $\Psi = \{\rho_x : x \in X \text{ } and \text{ } \alpha \in M(L)\}$ is an α–RF of 1_X. Since μ is a N.α–bounded, there exist $\Psi_o = \{cl(int(\rho_{x_i})) : i = 1, 2, ..., k\} \in 2^{(\Psi)}$ such that Ψ_o is an α–RCRF of μ. Put $\rho = \bigwedge_{i=1}^{k} \rho_{x_i}$, then $cl(int(\rho)) \in R_{x_\alpha}$. Since D is a directed set, there is $n_o \in D$ such that $n_o \geq n_{x_i}$ for every $i = 1, 2, ..., k$. Then for every $n \in D, n \geq n_o$, we have $\eta_n \leq cl(int(\bigwedge_{i=1}^{k} \rho_{x_i}))$ and so $\eta_n \leq cl(int(\rho))$, whenever $n \geq n_o$. Since $cl(int(\rho)) \wedge \mu = 0_X$, then $\eta_n \wedge \mu = 0_X$ for every $n \in D, n \geq n_o$.

Conversely, suppose that $\mu \in L^X$ satisfies the condition of the Theorem. We prove that μ is a N.α–bounded set. Let $\Psi \subseteq \tau'$ be an α–RF of 1_X. Let $D = 2^{(\Psi)}$ be the set of all finite subsets of Ψ directed by inclusion, and let $\{\eta_\Psi : \Psi \in D\}$ be a net of closed L–subsets in L^X such that $\eta_\Psi = \wedge\{cl(int(\rho)) : \rho \in \Psi\}$. Obviously, $\eta_{\Psi_1} \leq \eta_{\Psi_2}$ iff $\Psi_2 \subseteq \Psi_1$. Hence by Theorem 4.5 it follows that $\delta.\overline{\lim}(\eta_\Psi)(x) \leq \wedge\{\eta_\Psi : \Psi \in D\}$. Then $\wedge\{\eta_\Psi : \Psi \in D\}(x) = \wedge(\wedge\{cl(int(\rho)) : \rho \in \Psi\})(x) < \alpha$ for each $x \in X$. Thus $\delta.\overline{\lim}(\eta_\Psi)(x) < \alpha$ for each $x \in X$. By assumption, there exists an element $\Psi_o \in D$ for which $\eta_\Psi \wedge \mu = 0_X$ for every $\Psi \in D$, $\Psi \geq \Psi_o$. By the above we have $\eta_{\Psi_o} \wedge \mu = 0_X$ and so $(\forall x_\alpha \in \mu)(\exists cl(int(\rho)) \in \Psi_o)(cl(int(\rho)) \in R_{x_\alpha})$. Thus $\Psi_o \in 2^{(\Psi)}$ is an α–RCRF of μ. Hence μ is a N.α–bounded.

Theorem 4.7. An L–ts (L^X, τ) is a NQ_α–compact iff for every a net $\{\eta_n : n \in D\}$ of closed L–subsets in L^X such that $\delta\overline{\lim}(\eta_n)(x) < \alpha$ for each $x \in X$, there is $n_o \in D$ for which $\eta_n = 0_X$ for every $n \in D,\ n \geq n_o$.

Proof. Let (L^X, τ) be a NQ_α–compact. Then 1_X is a NQ_α–compact and by Theorem 3.27 (i), we have 1_X is a $N.\alpha$–bounded. Let $\{\eta_n : n \in D\}$ be a net of closed L–subsets in L^X such that $\delta.\overline{\lim}(\eta_n)(x) < \alpha$ for each $x \in X$. Hence by Theorem 4.6, there exists $n_\circ \in D$ such that $\eta_n \wedge 1_X = 0_X$ for every $n \in D,\ n \geq n_\circ$ and so $\eta_n = 0_X$ for every $n \in D,\ n \geq n_\circ$.

Conversely, suppose that 1_X satisfies the condition. We prove that 1_X is a NQ_α–compact. Let Ψ be an α–RF of 1_X and let $D = 2^{(\Psi)}$ be the set of all finite subsets of Ψ directed by inclusion, and let $\{\eta_\Psi : \Psi \in D\}$ be a net of closed L–subsets in L^X such that $\eta_\Psi = \wedge\{cl(\text{int}(\rho)) : \rho \in \Psi\}$.

Obviously, $\eta_{\Psi_1} \leq \eta_{\Psi_2}$ iff $\Psi_2 \subseteq \Psi_1$. Hence, by Theorem 4.6, it follows that

$$\delta.\overline{\lim}(\eta_\Psi) \leq \wedge\{\eta_\Psi : \Psi \in D\}.$$

Then $\wedge\{\eta_\Psi : \Psi \in D\}(x) = \wedge\{cl(\text{int}(\rho)) : \rho \in \Psi\}(x) < \alpha$ for every $x \in X$.and so $\delta.\overline{\lim}(\eta_n)(x) < \alpha$, for each $x \in X$. By assumption, there exists an element $\Psi_\circ \in D$ for which $\eta_\Psi = 0_X$ for every $\Psi \in D,\ \Psi \geq \Psi_\circ$. Thus $\eta_{\Psi_\circ} = 0_X$ and so $x_\alpha \notin \eta_{\Psi_\circ} = \wedge\{cl(\text{int}(\rho)) : \rho \in \Psi_\circ\}$ for every $x_\alpha \in M(L^X)$ and hence $\Psi_\circ \in 2^{(\Psi)}$ is an α–RCRF of 1_X.Hence 1_X is a NQ_α–compact.

Theorem 4.8. If (L^X, τ) is fully stratified and LT_2–space, then $\mu \in L^X$ is a NQ_α–compact set iff μ is δ–closed and N.α–bounded.

Proof. Let $\mu \in L^X$ be an NQ_α–compact, then by Theorem 2.18 (ii), we have μ is δ–closed and by Theorem 3.27 (ii), we have μ is N.α–bounded. Conversely, let μ be a δ–closed and N.α–bounded set and let S be a constant α–molecular net in μ. Since μ is N.α–bounded, then by Theorem 4.1, we have S has δ–cluster point, say x_α in X with height α. By Theorem 2.15 (j), then there is a subnet T of S such that T δ–converges to x_α and so $x_\alpha \in \delta cl(\mu)$ by Theorem 3.22 (vi). Since μ is δ–closed, then $\delta cl(\mu) = \mu$ and so $x_\alpha \in \mu$, then by Theorem 2.19, we have μ is a compact set.

Theorem 4.9. If (L^X, τ) is LR_2–space, then $\mu \in L^X$ is a N.α–bounded set iff $\delta cl(\mu)$ is a N.α–bounded set.

Proof. If $\delta.cl(\mu)$ is a $N.\alpha$–bounded set, then μ is a $N.\alpha$–bounded set (by Theorem 3.7). Conversely, suppose that μ is $N.\alpha$–bounded and $\Psi = \{\eta_{x_j} : j \in J\}$ is an α–RF of 1_X. Then for each $x \in X$ there is $\eta_{x_j} \in \Psi$ such that $\eta_{x_j} \in R_{x_\alpha}$. Since (L^X, τ) is LR_2–space, then there is $\lambda_{x_j} \in R_{x_\alpha}$ and there is $\rho_{x_j} \in \tau'$ such that $\lambda_{x_j} \vee \rho_{x_j} = 1_X$ and $\rho_{x_j} \wedge \eta_{x_j} = 0_X$. Then the family $\{\lambda_{x_j} : x_\alpha \in M(L^X)\}$ is an α–RF of 1_X. Since μ is $N.\alpha$–bounded, then exists finite subset $J_\circ$ of J such that $\{\lambda_{x_j} : j \in J_\circ\}$ is an α–RCRF of μ. Since $\lambda_{x_j} \vee \rho_{x_j} = 1_X$, $x_\alpha \notin \lambda_{x_j}$, then $x_\alpha \in \rho_{x_j}$. Since $\rho_{x_j} \wedge \eta_{x_j} = 0_X$, then $\{\eta_{x_j} : j \in J_\circ\}$ is an α–RCRF of ρ_{x_j}. Therefore $\mu \leq \rho_{x_j}$ for $J \in J_\circ$. Since $\rho_{x_j} \in \tau'$ and (L^X, τ) is LSR_2–space, then by Theorem 2.21, we have $\delta.cl(\rho_{x_j}) = \rho_{x_j}$ and so $\{\eta_{x_j} : j \in J_\circ\}$ is an α–RCRF of $\delta.cl(\rho_{x_j})$ and since $\delta.cl(\mu) \leq \delta.cl(\rho_{x_j})$, then $\{\eta_{x_j} : j \in J_\circ\}$ is an α–RCRF of $\delta.cl(\mu)$. Hence $\delta.cl(\mu)$ is a $N.\alpha$–bounded set.

Theorem 4.10. If (L^X, τ) is a LT_3–space, then $\mu \in L^X$ is a N.α–bounded set iff μ is a L–subset of a NQ_α–compact set.

Proof. If μ is N.α–bounded, then by Theorem 4.9, and Corollary 2.22, we have $\delta cl(\mu)$ is δ–closed and N.α–bounded set, hence by Theorem 4.8, we have $\delta cl(\mu)$ is a NQ_α–compact set.

Conversely, If μ is a L–subset of NQ_α–compact set, then by Theorem 3.27 (iii), we have μ is a N.α–bounded set.

Theorem 4.11. Assume that $S = \{S(n) : n \in D\}$ is a molecular net in a L–ts (L^X, τ) and $x_\alpha \in M(L^X)$. Then the following results are true:

(i) x_α is a $N\alpha B$–cluster point of S iff there exists a subnet T of S such that T is $N\alpha B$–converges to x_α.

(ii) If x_α is a $N\alpha B$–cluster point of S, then T is a $N\alpha B$–converges to x_α for each subnet T of S.

Proof. (i) Provided that $S = \{S(n) : n \in D\}$ and x_α is a $N\alpha B$–cluster point of S, then for each $\lambda \in N\alpha BR_{x_\alpha}$ and each $n \in D$ there is $k \in D$ such that $S(k) \notin \lambda$ and $k \geq n$. Taking $k = g(n, \lambda)$, we get a mapping $g : D \times N\alpha BR_{x_\alpha} \longrightarrow D$ with $S(g(n, \lambda)) \notin \lambda$. Put $E = D \times N\alpha BR_{x_\alpha}$ and we define the relation $\leq$ on E as follows: $(n, \lambda_1) \leq (n_2, \lambda_2)$ iff $n_1 \leq n_2$ and $\lambda_1 \leq \lambda_2$, then $(E, \leq)$ is a directed set. For each $(n, \lambda) \in E$, we choose $T(n, \lambda) = S(g(n, \lambda))$, then $T = \{T(n, \lambda) : (n, \lambda) \in E\}$ is a subnet of S. Because: (*) There exists mapping $f : E \longrightarrow D$ define as follows $f(n, \lambda) = n$ and $T = S \circ f$.

(**) Let $n_1 \in D$, then there exists $(n_1, \lambda_1) \in E$ and $(n_1, \lambda_1) \leq (n_2, \lambda_2) \in E$ iff $n_1 \leq n_2$ and $\lambda_1 \leq \lambda_2$, $f(n_2, \lambda_2) = n_2 \geq n_1$. Now we prove that T is $N\alpha B$–converges to x_α, let $\lambda \in N\alpha BR_{x_\alpha}$ and $n \in D$, so $(n, \lambda) \in E$. Therefore, for each $(n, \lambda) \in E$ and $(n, \lambda) \leq (m, \eta)$ then $T(m, \eta) = S(g(m, \eta)) \notin \eta$ and $\lambda \leq \eta$, so $T(m, \eta) \notin \lambda$. Thus T is $N\alpha B$–converges to x_α.

Conversely, it follows directly from Definition 2.12.

(ii) It follows directly from Definition 2.12.

Theorem 4.12. Let (L^X, τ) be an L–ts and $\mu \in L^X$. Then μ is a N.α–bounded set iff every α–filter $\mathcal{F}$ containing μ as an element has a δ–cluster point in X with height α.

Proof. Suppose that μ is a N.α–bounded and $\mathcal{F}$ is a α–filter containing μ as an element $(\alpha \in M(L))$, then $\lambda \wedge \mu \in \mathcal{F}$ for each $\lambda \in \mathcal{F}$, hence $\bigvee_{x \in X} (\lambda \wedge \mu)(x) \geq \alpha$ for each $\lambda \in \mathcal{F}$ and for each $x_\alpha \in M(L^X)$ there exists a molecule $x_{(\lambda,\alpha)} \in \lambda \wedge \mu$ with height α. Put $S(\mathcal{F}) = \{x_{(\lambda,\alpha)} : (\lambda, \alpha) \in \mathcal{F} \times M(L)\}$. In $\mathcal{F} \times M(L)$ we define the relation that $(\lambda_1, \alpha_1) \geq (\lambda_2, \alpha_2) iff \lambda_1 \leq \lambda_2$ and $\alpha_1 \geq \alpha_2$. Then, $\mathcal{F} \times M(L)$ is a directed set with this relation, and $S(\mathcal{F})$ is a constant molecular α–net in μ. Since μ is a N.α–bounded set, then by Theorem 4.1, $S(\mathcal{F})$ has a δ–cluster point in X with height α, say x_α. So, by Theorem 2.28, $\mathcal{F}$ δ–cluster to x_α as well.

Conversely, suppose that the condition is satisfied and $S = \{S(n) : n \in D\}$ is a constant molecular α–net in μ. Let $\lambda_m = \bigvee(S(m))$ for each $m \in D$, $n \geq m$. Since D is a directed set, then the family $\{\lambda_m : m \in D\}$ can generate a filter $\mathcal{F}(S)$. Since S is a constant molecular α–net, then for each $\alpha \in M(L)$ $(\exists n \in D)(\forall m \in D, m \geq n)(\bigvee(S(m)) = \alpha)$, hence $\bigvee(\lambda_m(x)) = \bigvee(\bigvee(S(n))) = \alpha, n \geq m$ and so $\bigvee(\lambda_m(x)) = \alpha$. Since $\mathcal{F}(S)$ is produced by $\{\lambda_m : m \in D\}$, then for each $\lambda \in \mathcal{F}(S)$ contains some λ_m and therefore $\bigvee(\lambda(x)) = \alpha$. Hence $\mathcal{F}(S)$ is an α–filter. By assumption, $\mathcal{F}(S)$ has a δ–cluster point in X with height α, say x_α. Thus, for each $\mu \in R_{x_\alpha}$ and for each $\lambda \in \mathcal{F}(S)$. In particular, λ_m we have

$\lambda_m \not\leq \mu$, and by Theorem 2.29, we have S has a δ–cluster point x_α and by Theorem 4.1, we have μ is a N.α–bounded.

Theorem 4.13. If a set μ in a L–ts (L^X, τ) is a N.α–bounded, then every α–ideal I in L^X and $\mu \notin I$ has a δ–cluster point in X with height α.

Proof. Let I be an α–ideal in L^X and $\mu \in L^X$ be a $N\alpha$–bounded with $\mu \notin I$. Then for each $\eta \in I$ we have $\bigvee_{x\in X} \eta(x) < \alpha$, and then for each $\alpha \in M(L)$ there exists a molecule $S(\eta, \alpha) = x_{(\eta,\alpha)} \notin \eta$. Put $D(I) = \{(\eta, \alpha) : x_{(\eta,\alpha)} \in \mu, \eta \in I$ and $x_{(\eta,\alpha)} \notin \eta\}$.In $D(I)$ We define the relation that $(\eta_1, \alpha_1) \geq (\eta_2, \alpha_2)$ iff $\eta_1 \geq \eta_2$. Then $(D(I), \geq)$ is a directed set with this relation and $S(I) = \{S(\eta, \alpha) = x_{(\eta,\alpha)} : (\eta, \alpha) \in D(I)\}$ is a constant molecular α–net in μ. Since μ is a $N\alpha$–bounded set, then by Theorem 4.1, $S(I)$ has a δ–cluster point in X with height α, say x_α, by Theorem 2.30, we have x_α is also a δ–cluster point of I.

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