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# Hop $k$-Rainbow Domination in Graphs

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**Abstract.** Let $G = (V(G), E(G))$ be a graph. A function $f$ that assigns to each vertex of $G$ a subset of colors from the set $\{1, 2, \ldots, k\}$, i.e., $f : V(G) \rightarrow P(\{1, 2, 3, \ldots, k\})$, is called a *hop $k$-rainbow dominating function* (H$k$RDF) of $G$ if for every vertex $v \in V(G)$ with $f(v) = \varnothing$, we have $\bigcup_{u \in N_G^2(v)} f(u) = \{1, 2, \ldots, k\}$ where $N_G^2(v)$ is the set of vertices of $G$ at distance two from $v$. The *weight* of $f$, denoted $\omega(f)$, is defined as $\omega(f) = \sum_{x \in V(G)} |f(x)|$. The *hop $k$-rainbow domination number* of $G$, denoted $\gamma_{hrk}(G)$, is the minimum weight of a hop $k$-rainbow dominating function of $G$. A hop $k$-rainbow dominating function of $G$ with weight $\gamma_{hrk}(G)$ is a $\gamma_{hrk}$-function of $G$. In this paper, we initiate the study of hop $k$-rainbow domination in graphs. We begin by exploring fundamental properties of this parameter and then establish various bounds on $\gamma_{hrk}(G)$. Furthermore, we identify the graphs for which $\gamma_{hrk}(G) = n$ and determine exact values for certain graph classes, including complete graphs, complete bipartite graphs, paths, and cycles. Additionally, for any positive integer $a$, we construct connected graphs satisfying $\gamma_{hr2}(G) = \gamma_{r2}(G) = a$. Finally, we provide a characterization of all graphs where $\gamma_{hr2}(G) = n$.

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**Key Words and Phrases**: Hop domination number, $k$-rainbow domination number, hop $k$-rainbow domination number

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## 1. Introduction

Domination in graphs is frequently used as a model for real-world applications, where the vertices in a dominating set provide a service or product that must be accessible to every vertex in the network. In 2020, Haynes et al. ([1]) published a comprehensive survey on domination in graphs. Rainbow domination extends this concept by introducing multiple service types, represented by different colors, and ensuring that every vertex without direct service has access to all service types in its neighborhood. Rainbow domination is being studied because it has a lot to do with domination in Cartesian products of graphs and what that means for Vizing's conjecture, even though its practical uses are still unknown. The rainbow domination number was first proposed by Brešar et al. in 2008 [2]. They showed how it works in paired-domination within Cartesian products of graphs and explained how it is related to standard domination. Later, in 2014, Z. Shao determined bounds for the $k$-rainbow domination number of any arbitrary graph for any positive integer $k$ [3]. Since then, this concept has been widely explored (see, for example, [4–9]).

In this paper, we introduce the study of hop $k$-rainbow domination in graphs, integrating the ideas of hop domination and $k$-rainbow domination parameters. The incorporation of hop distance constraints influences the behavior of minimum rainbow dominating functions, leading to new theoretical bounds and extremal results. The concept of a hop dominating set was first proposed by Natarajan et al. in 2015 [10], and it has since been expanded by many researchers who have applied it to various domination variants. For more details on hop domination, see for instance, [11–14].

## 2. Terminology and Notation

Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$, and of order $n = |V(G)|$ and size $m = |E(G)|$. The set of neighbors of a vertex $u$ in $G$ is called the *open neighborhood* of $u$ in $G$, denoted by $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of $u$ in $G$ is the set $N_G[u] = N_G(u) \cup \{u\}$ and the *closed neighborhood* of a subset $S$ of $V(G)$ is the set $N_G[S] = N_G(S) \cup S$. The *degree* of a vertex $u$ in $G$ is the number of neighbors $u$ in $G$, denoted by $\deg(u)$. The maximum (minimum) degree among the vertices of $G$ is denoted by $\Delta(G)$ ($\delta(G)$, respectively). The *distance* $d_G(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u$-$v$ path in $G$ while the *diameter* of $G$, denoted by $\mathrm{diam}(G)$, is the maximum distance among all pairs of vertices in $G$. A *complete graph* on $n$ vertices is denoted by $K_n$, while a *complete bipartite graph* with partite sets of size $p$ and $q$ is denoted by $K_{p,q}$. We write $P_n$ for the *path* of order $n$, $C_n$ for the *cycle* of length $n$ and $\overline{K_n}$ for the graph with $n$ vertices and no edges, as defined by Harary in [15].

A vertex $v$ in $G$ is called an *$\ell$-neighbor* of a vertex $u$ in $G$ if $d_G(u, v) = \ell$. The set $N^{\ell}_G(u) = \{v \in V(G) : d_G(v, u) = \ell\}$ is called the *open $\ell$-neighborhood* of $u$. The *closed $\ell$-neighborhood* of $u$ in $G$ is given by $N^{\ell}_G[u] = N^{\ell}_G(u) \cup \{u\}$. The *open $\ell$-neighborhood* of $X \subseteq V(G)$ is the set $N^{\ell}_G(X) = \bigcup_{u\in X} N^{\ell}_G(u)$. The *closed $\ell$-neighborhood* of $X$ in $G$ is the set $N_G^{\ell}[X] = N_G^{\ell}(X) \cup X$.

A set $S \subseteq V(G)$ is said to be a *dominating set* if $N_G[S] = V(G)$. A dominating set $S$ is a minimal dominating set if no proper subset of $S$ is a dominating set. The *domination number* of a graph $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A dominating set $S$ with the cardinality equal to $\gamma(G)$ is said to be a $\gamma$-set of $G$ or $\gamma(G)$-set.

A set $S \subseteq V(G)$ is a *hop dominating set* of $G$ if $N_G^2[S] = V(G)$, that is, for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u,v) = 2$. The *hop domination number* of $G$, denoted by $\gamma_h(G)$, is The minimum cardinality of a hop dominating set of $G$. A hop dominating set with cardinality equal to $\gamma_h(G)$ is called a $\gamma_h$*-set* of $G$, as defined by Natarajan et al. in [10]. For a positive integer $\ell$, the $\ell$-degree of a vertex $v$ in a graph $G$, denoted by $\deg_\ell(v)$, is defined as the number of vertices at distance $\ell$ from $v$ in $G$. The maximum $\ell$-degree among the vertices of $G$ is denoted by $\Delta_\ell(G)$. In the special case $\ell = 2$, a *2-neighbor* is called a *hop-neighbor*, denoted by $N_G^2(u)$. The *2-degree* is called the *hop-degree*, denoted by $\deg_2(v)$. The maximum hop-degree among the vertices of $G$ is denoted by $\Delta_h(G)$, as defined by Shabani et al. in [16].

A function $f : V \to \{0, 1, 2\}$ is a *Roman dominating function* (RD-function, for short) on $G$ if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$, as defined by E. J. Cockayne in [17]. A hop Roman dominating function (HRD-function) of $G$ is a function $g$ defined on $V(G)$ into $\{0, 1, 2\}$ having the property that for every vertex $u \in V$ with $g(u) = 0$ there is a vertex $w$ with $g(w) = 2$ and $d(u,w) = 2$. The *hop Roman domination number* $\gamma_{hR}(G)$ is equal to the minimum weight of a HRD-function in $G$. For more details on hop Roman domination see for example [16].

Let $G$ be a graph and let $f$ be a function that assigns to each vertex a set of colors chosen from the set $\{1, 2, 3, \ldots, k\}$, that is, $f : V(G) \to P(\{1, 2, 3, \ldots, k\})$. If for each vertex $v \in V(G)$ with $f(v) = \varnothing$, we have $\bigcup_{u \in N_G(v)} f(u) = \{1, 2, 3, \ldots, k\}$, then $f$ is called the *k-rainbow dominating function* (kRDF) of $G$. The *weight* $\omega(f)$ of $f$ is defined as $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The *k-rainbow domination number* of $G$, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF. A $k$-rainbow dominating function of $G$ with weight $\gamma_{rk}(G)$ is a $\gamma_{rk}$-function of $G$, as defined by B. Brešar in [2].

A function $f : V(G) \to P(\{1, 2, \ldots, k\})$ is a hop $k$-rainbow dominating function (H$k$RD-function) if for every vertex $v \in V(G)$ with $f(v) = \varnothing$, we have $\bigcup_{u \in N_G^2(v)} f(u) = \{1, 2, \ldots, k\}$. The *weight* of a hop $k$-rainbow dominating function is $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The *hop k-rainbow domination number* of $G$, denoted $\gamma_{hrk}(G)$, is the minimum weight of a hop $k$-rainbow dominating function of $G$. A hop $k$-rainbow dominating function of $G$ with weight $\gamma_{hrk}(G)$ is a $\gamma_{hrk}$-function of $G$. For the sake of simplicity, we will write H$k$RD-number instead of hop $k$-rainbow domination number. Clearly, when $k = 1$, $\gamma_{h1r}(G)$ matches with the usual hop domination number $\gamma_h(G)$.

**Example 1.** Consider the graph $G$ in Figure 1. Let $S = \{v_1, v_2, v_5, v_6, v_7, v_8\}$ and let $f : V(G) \to P(\{1,2\})$ be a function defined by $f(v_1) = f(v_5) = f(v_7) = f(v_8) = \{2\}$, $f(v_2) = f(v_6) = \{1\}$, and $f(v_3) = f(v_4) = \varnothing$. Observe that $d_G(v_3, v_1) = 2$, and $d_G(v_4, v_2) = 2$. Thus, $S$ is a hop dominating set of $G$. Further, notice that $\bigcup_{u_1 \in N_G^2(v_3)} f(u_1) = \{1,2\}$ and $\bigcup_{u_2 \in N_G^2(v_4)} f(u_2) = \{1,2\}$. Then $f$ is a hop 2-rainbow dominating function of $G$. The weight of $f$ is given

$$\begin{aligned}\omega(f) &= |f(v_1)| + |f(v_2)| + |f(v_3)| + |f(v_4)| + |f(v_5)| + |f(v_6)| + |f(v_7)| + |f(v_8)| \\ &= 1 + 1 + 0 + 0 + 1 + 1 + 1 + 1 \\ &= 6.\end{aligned}$$

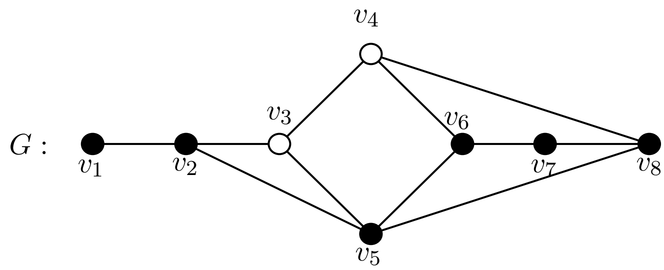


Figure 1: A graph $G$ of order 8 with a hop 2-rainbow dominating function of weight 6.

## 3. Preliminary Results

In this section, we study the basic properties of H$k$RD-function and we establish various bounds for H$k$RD-number of a graph $G$.

**Theorem 1.** *Let $k$ be a positive integer, and let $G$ be a graph of order $n = n_1 + n_2 + \cdots + n_p$ with $p$ disjoint components $G_1, G_2, \ldots, G_p$ with $n_1, n_2, \ldots, n_p$ vertices, respectively. Then*

$$\gamma_{hrk}(G) = \sum_{i=1}^{p} \gamma_{hrk}(G_i).$$

*Proof.* Let $G$ be a graph of order $n = n_1 + n_2 + \cdots + n_p$ consisting of $p$ disjoint components $G_1, G_2, \ldots, G_p$, where each $G_i$ has $n_i$ vertices.

First, for each $i \in \{1, 2, \ldots, p\}$, let $f_i$ be a $\gamma_{hrk}$-function of $G_i$, meaning that $f_i$ is a hop $k$-rainbow dominating function of $G_i$ of weight $\omega(f_i) = \gamma_{hrk}(G_i)$. Define a function $f : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ by $f(v) = f_i(v)$ if $v \in V(G_i)$, for each $i \in \{1, 2, \ldots, p\}$. Since $G_1, G_2, \ldots, G_p$ are disjoint, it follows that $f$ is a hop $k$-rainbow dominating function of $G$ of weight $\omega(f) = \sum_{i=1}^{p} \omega(f_i)$. By the minimality of $\gamma_{hrk}(G)$, we obtain

$$\gamma_{hrk}(G) \leq \omega(f) = \sum_{i=1}^{p} \gamma_{hrk}(G_i).$$

Conversely, suppose $f'$ is a $\gamma_{hrk}$-function of $G$. Since the components of $G$ are disjoint, we can define $f'_i$ as the restriction of $f'$ to $G_i$, i.e., $f'_i = f'|_{V(G_i)}$. Since each $f'_i$ is a hop $k$-rainbow dominating function of $G_i$, we have $\omega(f'_i) \geq \gamma_{hrk}(G_i)$. Thus, $\omega(f') = \sum_{i=1}^{p} \omega(f'_i) \geq \sum_{i=1}^{p} \gamma_{hrk}(G_i)$. Since $f'$ is an optimal function for $G$, it follows that

$$\gamma_{hrk}(G) = \omega(f') \geq \sum_{i=1}^{p} \gamma_{hrk}(G_i).$$

From the two inequalities, we conclude that

$$\gamma_{hrk}(G) = \sum_{i=1}^{p} \gamma_{hrk}(G_i).$$

This completes the proof. □

Thus, in the following discussion, we consider only connected graphs.

For any graph $G$ and a $\gamma_{hrk}$-function $f$ of $G$, define the set

$$V_i^f = \{x \in V(G) \mid |f(x)| = i\}, \quad \text{for } i \in \{0, 1, \dots, k\}.$$

**Example 2.** Consider the graph $G$ in Figure 1 and let $f : V(G) \to \mathcal{P}(\{1, 2\})$ be the hop 2-rainbow dominating function of $G$ defined in Example 1. Then, the sets of vertices corresponding to each function value are as follows:

$$V_0^f = \{v_3, v_4\}, \quad V_1^f = \{v_1, v_2, v_5, v_6, v_7, v_8\}, \quad V_2^f = \varnothing.$$

Let $G$ be a connected graph and consider the graph $G^2$ whose vertex set is $V(G)$, where two vertices $u$ and $v$ are adjacent in $G^2$ if $d_G(u, v) = 2$. Clearly, any $k$RD-function of $G^2$ is an H$k$RD-function of $G$ and vice versa. Thus, we have the following.

**Remark 1.** *For any graph $G$, $\gamma_{hrk}(G) = \gamma_{rk}(G^2)$.*

**Remark 2.** *If $G$ is a graph of order $n \geq 1$ with $\delta(G) \geq 1$. Let $f$ be a $\gamma_{hrk}$-function of $G$, then*

*(i)* $n = \sum_{j=0}^{k} |V_j^f|$,

*(ii)* $\gamma_{hrk}(G) = \sum_{j=1}^{k} j|V_j^f|$, *and*

*(iii)* $|V_0^f| \geq \sum_{j=2}^{k} (j-1)|V_j^f|$.

*Proof.* Suppose $G$ is a graph of order $n \geq 1$ with $\delta(G) \geq 1$, and let $f$ be a $\gamma_{hrk}$-function of $G$. By definition, $V_i^f = \{v \in V(G) : |f(v)| = i\}$ for each $i \in \{0, 1, \dots, k\}$. Since every vertex belongs to exactly one of these sets, we have $n = \sum_{j=0}^{k} |V_j^f|$. Hence, (i) holds. Since $\omega(f) = \sum_{v \in V(G)} |f(v)|$, we can rewrite this sum in terms of $V_j^f$ by $\gamma_{hrk}(G) = \sum_{j=1}^{k} j|V_j^f|$. Therefore, (ii) holds. Let $v$ be any vertex in $G$ with $f(v) = \varnothing$. Then $\bigcup_{u \in N_G^2(v)} f(u) =$

$\{1, 2, \ldots, k\}$. This implies that vertices in $V_0^f$ rely on vertices in $V_j^f$ for $j \geq 2$ to contribute enough colors. Since each vertex in $V_j^f$ contributes $j$ colors, but at least one color is required per vertex in $V_0^f$, it follows that $|V_0^f| \geq \sum_{j=2}^{k}(j-1)|V_j^f|$. This proves (iii). The proof is complete. $\square$

**Theorem 2.** *Let $k$ be a positive integer and $G$ be a graph of order $n = n_1 + n_2 + \cdots + n_p$. If $G$ is a disjoint union of cliques and isolates, then $\gamma_{hrk}(G) = n$.*

*Proof.* Assume that $G$ is a disjoint union of cliques and isolates. Let $G = \bigcup_{i=1}^{p} G_i$, where $G_i = K_r$ for some $r \geq 1$. By Theorem 1, $\gamma_{hrk}(G) = \sum_{i=1}^{p} \gamma_{hrk}(G_i)$. Therefore, we have

$$\gamma_{hrk}(G) = \sum_{i=1}^{p} \gamma_{hrk}(G_i) = \sum_{i=1}^{p} |V(G_i)| = n_1 + n_2 + \cdots + n_p = n.$$

$\square$

The next result is a direct consequence of Theorem 2.

**Corollary 1.** *Let $k$ and $n$ be positive integers. Then*

$$\gamma_{hrk}(K_n) = \gamma_{hrk}(\overline{K}_n) = n.$$

**Theorem 3.** *Let $k \geq 2$ be an integer and let $G$ be a connected graph of order $n \geq k+1$. Then*

$$\gamma_{hrk}(G) \geq k + 1.$$

*Moreover, the bound is sharp for stars $K_{1,m}$ $(m \geq k)$.*

*Proof.* Since $G$ is a connected graph of order $n \geq k+1$, we assume $\gamma_{hrk}(G) = k$. Let $S = V(G) \setminus V_0^g$, that is the set of vertices assigned at least one color. Given $n \geq k+1$, it follows that $V_0^g \neq \varnothing$. Now, consider any $x \in V_0^g$ and any $y \in S$. Since $x$ is not assigned a color, there exist a vertex $z$ such that $z \in N_G(x) \cap N_G(y)$, ensuring $x$ is at distance 2 from $y$. However, since $x$ is at distance 2 from every vertex in $S$, it follows that $z \notin S$, meaning $z \in V_0^g$. But then, $z$ would also need to be assigned the set of colors from some vertex in $S$, contradicting the assumption that $\gamma_{hrk}(G) = k$. Therefore, $\gamma_{hrk}(G) \geq k + 1$. $\square$

**Theorem 4.** *Let $k \geq 1$ be an integer and $G$ be a graph of order $n \geq 1$. Then*

$$\min\{n, k+1\} \leq \gamma_{hrk}(G) \leq n.$$

*In particular, if $1 \leq n \leq k+1$, then $\gamma_{hrk}(G) = n$.*

*Proof.* If $n \geq k+1$, then it follows from Theorem 3 that $\gamma_{hrk}(G) \geq k+1$. Let $n \leq k$ and let $g$ be $\gamma_{hrk}$-function of $G$. If $V_0^g \neq \varnothing$ and $v \in V_0^g$, then by the definition we have $\bigcup_{u \in N_G^2(v)} g(u) = \{1, 2, \ldots, k\}$, and so $\gamma_{hrk}(G) \geq \sum_{u \in N_G^2(v)} |g(u)| \geq k \geq n$. Assume that $V_0^g = \varnothing$. Then by Remark 2, we have $\gamma_{hrk}(G) = \sum_{j=1}^{k} j|V_j^f| \geq \sum_{j=1}^{k} |V_j^f| = n$. Consequently, we have $\gamma_{hrk}(G) \geq \min\{n, k+1\}$.

For the upper bound, consider the function $h : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ defined by $h(v) = \{1\}$ for all $v \in V(G)$. Clearly, $h$ is an H$k$RD-function of $G$ of weight $n$ leading to $\gamma_{hrk}(G) \leq n$.

Therefore, we have established that $\min\{n, k+1\} \leq \gamma_{hrk}(G) \leq n$. In particular, if $n \leq k+1$, then the lower bound is the $\min\{n, k+1\} = n$, implying that $\gamma_{hrk}(G) = n$. □

**Proposition 1.** *Let $k \geq 2$ be an integer and $G$ be a graph of order $n \geq k$ with $\delta(G) \geq n-2$. Then $\gamma_{hrk}(G) = n$.*

*Proof.* Let $f$ be a $\gamma_{hrk}$-function of $G$ such that $|V_0^f|$ is minimized. If $V_0^f = \varnothing$, then we are done. Assume, for contradiction, that $V_0^f \neq \varnothing$ and let $x \in V_0^f$. Hence, we must have $\bigcup_{u \in N_G^2(x)} f(u) = \{1, 2, \ldots, k\}$. It follows from $\delta(G) \geq n-2$ that $|N_G^2(x)| = 1$. Assume that $y \in N_G^2(x)$. Then $f(y) = \{1, 2, \ldots, k\}$. Since $y$ is adjacent to all vertices other than $x$, the function $g$ defined on $g(x) = g(y) = \{1\}$ and $g(z) = f(z)$ for other vertices is an H$k$RD-function of $G$ of weight at most $\omega(f)$ with $|V_0^g| < |V_0^f|$, contradicting the choice of $f$. Therefore, $V_0^f = \varnothing$, and it follows that $\gamma_{hrk}(G) = \omega(f) = n$. □

In the next theorem we provide a sufficient condition for a graph $G$ to have $\gamma_{hrk}(G) = n$.

**Theorem 5.** *For positive integers $n$ and $k \geq 2$, let $G$ be a graph of order $n \geq k$ with $k > \Delta_h(G^2)$. Then $\gamma_{hrk}(G) = n$.*

*Proof.* Suppose, for the sake of contradiction, that $\gamma_{hrk}(G) < n$. Let $f$ be a $\gamma_{hrk}$-function of $G$ such that $|V_0^f|$ is as large as possible, where $V_0^f = \{v \in V(G) : f(v) = \varnothing\}$. Since $\gamma_{hrk}(G) < n$, there is a vertex $v \in V_0^f$ and by the definition we have $\bigcup_{x \in N_G^2(v)} f(x) = \{1, 2, \ldots, k\}$. Since $k > \Delta_h(G^2)$, there exists a vertex $x \in N_G^2(v)$ such that $|f(x)| \geq \Delta_h(G) + 1$. Assume, without loss of generality, that $\{1, 2, \ldots, \Delta_h(G) + 1\} \subseteq f(x)$. Let $x_1, x_2, \ldots, x_{\deg_2(x)}$ be the vertices of $G$ at distance 2 from $x$ and define the function $g : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ by $g(x_i) = f(x_i) \cup \{i\}$ for each $i \in \{1, 2, \ldots, \deg_2(x)\}$, $g(x) = f(x) - \{1, \ldots, \deg_2(x)\}$ and $g(y) = f(y)$ for other vertices. Then $g$ is an H$k$RD-function of $G$ of weight $\omega(f)$ such that $|V_0^g| < |V_0^f|$ a contradiction with the choice of $f$. Consequently, $\gamma_{hrk}(G) = n$. □

The next result is immediate from Theorem 5.

**Corollary 2.** *For positive integers $n$ and $k$ with $n \geq k \geq 5$,*

$$\gamma_{hrk}(P_n) = \gamma_{hrk}(C_n) = n.$$

**Theorem 6.** *Let $k \geq 1$ be an integer and $G$ be a connected graph of order $n \geq k$ with hop maximum degree $\Delta_h$. Then $\gamma_{hrk}(G) \geq nk/(\Delta_h + k)$.*

*Proof.* Let $g$ be a $\gamma_{hrk}$-function of $G$, i.e $\omega(g) = \gamma_{hrk}(G)$, and let $V_0^g$ be the set of vertices assigned $\varnothing$ under $g$. Clearly, $\gamma_{hrk}(G) \geq n - |V_0^g|$. On the other hand, sine each vertex $x$ in $V_0^g$ must have each of $k$ colors in $N_G^2(x)$, we have $\Delta_h\gamma_{hrk}(G) \geq k|V_0^g|$. Since $\gamma_{hrk}(G) \geq n - |V_0^g|$, it would imply that $k\gamma_{hrk}(G) \geq kn - k|V_0^g|$. Moreover, since $k|V_0^g| \leq \Delta_h\gamma_{hrk}(G)$, it follows that $k\gamma_{hrk}(G)+\Delta_h\gamma_{hrk}(G) \geq kn$. Hence, $(k+\Delta_h)\gamma_{hrk}(G) \geq kn$. Therefore, $\gamma_{hrk}(G) \geq nk/(\Delta_h + k)$ as desired. □

The next result will be used in the subsequent discussion.

**Proposition 2.** [18] *For any positive integer $n$, we have:*

$$\gamma_{rk}(K_n) = \begin{cases} k, & \text{if } n \geq k, \\ n, & \text{if } n < k. \end{cases}$$

**Proposition 3.** *Let $m$, $n$, and $k$ be positive integers with $k \geq 1$ and $m \leq n$. Then*

$$\gamma_{hrk}(K_{m,n}) = \begin{cases} 2k & \text{if } m \geq k, \\ k+m & \text{if } m < k \text{ and } n \geq k, \\ m+n & \text{if } n < k. \end{cases}$$

*Proof.* It is easy to see that $K_{m,n}^2 = K_m \cup K_n$. By Proposition 2, for any positive integer $s$, we have $\gamma_{rk}(K_s) = \min\{k, s\}$. So, applying Remark 1 yields

$$\gamma_{hrk}(K_{m,n}) = \gamma_{rk}(K_{m,n}^2) = \gamma_{rk}(K_m) + \gamma_{rk}(K_n) = \min\{k, m\} + \min\{k, n\},$$

as desired. □

In the next theorem we provide an upper and a lower bound on the hop $k$-rainbow domination number in terms of hop domination number. We recall that $\gamma_h(K_n) = n$ and $\gamma_h(K_{m,n}) = 2$ (see [10]).

**Theorem 7.** *Let $k \geq 2$ be an integer and let $G$ be a graph of order $n \geq k$. Then*

$$\gamma_h(G) \leq \gamma_{hrk}(G) \leq k\gamma_h(G).$$

*Moreover, these bounds are sharp.*

*Proof.* We first show that the left inequality holds. Let $f$ be a $\gamma_{hrk}$-function of $G$. Then clearly the set $S = \{v \in V(G) : f(v) \neq \varnothing\}$ is a hop dominating set of $G$. Therefore, $|S| \geq \gamma_h(G)$. Since the weight $\omega(f) = \sum_{v\in V(G)} |f(v)|$ and $|f(v)| \geq 1$ for each $v \in S$, it follows that $\omega(f) \geq |S| \geq \gamma_h(G)$. This implies that $\gamma_h(G) \leq \gamma_{hrk}(G)$.

Next, we will show that $\gamma_{hrk}(G) \leq k\gamma_h(G)$. Consider a minimum hop dominating set $S$ of $G$ with $|S| = \gamma_h(G)$. Define a function $f : V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ by $f(v) = \{1, 2, \ldots, k\}$ for each vertex $v \in S$ and $f(v) = \varnothing$ for every $v \in V(G) \setminus S$. For each $v \in V(G) \setminus S$, there exists a vertex $u \in S$ such that $d(u, v) = 2$ since $S$ is a hop dominating set. Since $f(u) = \{1, 2, \ldots, k\}$ for each $u \in S$, it follows that $\bigcup_{u\in N_G^2(v)} f(u) = \{1, 2, \ldots, k\}$

for each $v \in V(G) \setminus S$. Thus, $f$ is a hop $k$-rainbow dominating function of $G$ of weight $\omega(f) = k\gamma_h(G)$. Hence, we have $\gamma_{hrk}(G) \leq k\gamma_h(G)$.

For the sharpness of the lower bound, let $G = K_n$. Then by Corollary 1, we have $\gamma_h(G) = \gamma_{hrk}(G)$. For the upper bound, let $G = K_{m,n}$, where $n \geq m \geq k$. Then by Proposition 3, we have $\gamma_{hrk}(K_{m,n}) = 2k = k\gamma_h(K_{m,n})$. □

## 4. Exact values

The exact values of the $k$-rainbow domination number for $k \in \{2, 3\}$ of paths and cycles are determined as follows.

**Theorem 8.** [3, 6] *Let $n$ be a positive integer. Then*
*(i)* $\gamma_{r2}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

*(ii) For $n \geq 3$,* $\gamma_{r2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor$.

*(iii) For $n \geq 5$,* $\gamma_{r3}(P_n) = \begin{cases} \left\lceil \frac{3n}{4} \right\rceil + 1 & \text{if} \quad n \equiv 0 \pmod 4, \\ \left\lceil \frac{3n}{4} \right\rceil & \text{if} \quad n \equiv 1, 2, 3 \pmod 4. \end{cases}$

*(iv) For $n \geq 5$,* $\gamma_{r3}(C_n) = \left\lceil \frac{3n}{4} \right\rceil$.

Using Theorem 8 and Proposition 1, we determine the exact value of hop $k$-rainbow domination number for $k \in \{2, 3\}$ of paths and cycles in the next Theorem.

**Theorem 9.** *Let $n$ be a positive integer. Then*

*(i)* $\gamma_{hr2}(P_n) = \begin{cases} \left\lfloor \frac{n+1}{4} \right\rfloor + \left\lfloor \frac{n-1}{4} \right\rfloor + 2 & \text{if} \quad n \equiv 1, 3 \pmod 4, \\ 2\left\lfloor \frac{n}{4} \right\rfloor + 2 & \text{if} \quad n \equiv 0, 2 \pmod 4. \end{cases}$

*(ii) For $n \geq 2$,* $\gamma_{hr3}(P_n) = \begin{cases} 2\left\lceil \frac{3n}{8} \right\rceil + 2 & \text{if} \quad n \equiv 0 \pmod 8, \\ \left\lceil \frac{3(n+1)}{8} \right\rceil + \left\lceil \frac{3(n-1)}{8} \right\rceil + 1 & \text{if} \quad n \equiv 1, 7 \pmod 8, \\ 2\left\lceil \frac{3n}{8} \right\rceil & \text{if} \quad n \equiv 2, 4, 6 \pmod 8, \\ \left\lceil \frac{3(n+1)}{8} \right\rceil + \left\lceil \frac{3(n-1)}{8} \right\rceil & \text{if} \quad n \equiv 3, 5 \pmod 8. \end{cases}$

*(iii) For $n \geq 4$,* $\gamma_{hr2}(C_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 1 \pmod 2, \\ 2\left(\left\lfloor \frac{n}{4} \right\rfloor + \left\lceil \frac{n}{8} \right\rceil - \left\lfloor \frac{n}{8} \right\rfloor\right) & \text{otherwise.} \end{cases}$

*(iv) For $n \geq 5$,* $\gamma_{hr3}(C_n) = \begin{cases} \left\lceil \frac{3n}{4} \right\rceil & \text{if } n \equiv 1 \pmod 2, \\ 2\left\lceil \frac{3n}{8} \right\rceil & \text{otherwise.} \end{cases}$

*Proof.* Let $P_n = [v_1, v_2, \dots v_n]$ be a path on $n$ vertices. By Theorem 4 and Proposition 1, we have $\gamma_{hr2}(P_n) = \gamma_{hr3}(P_n) = n$ for $n \in \{1, 2, 3, 4\}$. Assume that $n \geq 5$. If $n$ is even, then $P_n^2$ is the union of two paths:

$$P = [v_1, v_3, \dots, v_{n-1}] \quad \text{and} \quad Q = [v_2, v_4, \dots, v_n].$$

If $n$ is odd, then $P_n^2$ is the union of two paths:

$$P' = [v_1, v_3, \dots, v_n] \quad \text{and} \quad Q' = [v_2, v_4, \dots, v_{n-1}].$$

**(i)** Since both $P$ and $Q$ have $n/2$ vertices, we can use Theorem 8-(i) to find that $\gamma_{r2}(P) = \left\lfloor \frac{n/2}{2} \right\rfloor + 1 = \lfloor n/4 \rfloor + 1$, and similarly, $\gamma_{r2}(Q) = \lfloor n/4 \rfloor + 1$. If $n$ is even, then by Remark 1, we obtain

$$\gamma_{hr2}(P_n) = \gamma_{r2}(P_n^2) = \gamma_{r2}(P) + \gamma_{r2}(Q) = 2\left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right) = 2\left\lfloor \frac{n}{4} \right\rfloor + 2,$$

as desired. Assume that $n$ is odd. Then $P$ and $Q$ have lengths $\frac{n+1}{2}$ and $\frac{n-1}{2}$, respectively. Applying Theorem 8-(i) again, we obtain $\gamma_{r2}(P) = \left\lfloor \frac{(n+1)/2}{2} \right\rfloor + 1 = \left\lfloor \frac{n+1}{4} \right\rfloor + 1$ and $\gamma_{r2}(Q) = \left\lfloor \frac{(n-1)/2}{2} \right\rfloor + 1 = \left\lfloor \frac{n-1}{4} \right\rfloor + 1$. Therefore, by Remark 1, we have

$$\gamma_{hr2}(P_n) = \gamma_{r2}(P) + \gamma_{r2}(Q) = \left(\left\lfloor \frac{n+1}{4} \right\rfloor + 1\right) + \left(\left\lfloor \frac{n-1}{4} \right\rfloor + 1\right).$$

Simplifying, we get

$$\gamma_{hr2}(P_n) = \left\lfloor \frac{n+1}{4} \right\rfloor + \left\lfloor \frac{n-1}{4} \right\rfloor + 2 = \left\lfloor \frac{n}{4} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil + 2,$$

as desired.

**(ii)** If $n \equiv 0 \pmod 8$, then $|V(P)| = \frac{n}{2} \equiv 0 \pmod 4$, $|V(Q)| = \frac{n}{2} \equiv 0 \pmod 4$ and by Remark 1 and Theorem 8-(iii), we have

$$\gamma_{hr3}(P_n) = \gamma_{r3}(P_n^2) = \gamma_{r3}(P) + \gamma_{r3}(Q) = 2\left(\left\lceil \frac{3\frac{n}{2}}{4} \right\rceil + 1\right) = 2\left(\left\lceil \frac{3n}{8} \right\rceil + 1\right),$$

as desired. If $n \equiv 1 \pmod 8$ (the case $n \equiv 7 \pmod 8$ is similar), then $|V(P)| = \frac{n+1}{2} \equiv 1 \pmod 4$ and $|V(Q)| = \frac{n-1}{2} \equiv 0 \pmod 4$. Applying Remark 1 and Theorem 8-(iii), it follows that

$$\begin{aligned} \gamma_{hr3}(P_n) &= \gamma_{r3}(P) + \gamma_{r3}(Q) \\ &= \left\lceil \frac{3\frac{n+1}{2}}{4} \right\rceil + \left\lceil \frac{3\frac{n-1}{2}}{4} \right\rceil + 1 \end{aligned}$$

$$= \left\lceil \frac{3(n+1)}{8} \right\rceil + \left\lceil \frac{3(n-1)}{8} \right\rceil + 1.$$

If $n \equiv 3 \pmod 8$ (the case $n \equiv 5 \pmod 8$ is similar), then $|V(P)| = \frac{n+1}{2} \equiv 2 \pmod 4$ and $|V(Q)| = \frac{n-1}{2} \equiv 1 \pmod 4$. Applying Remark 1 and Theorem 8-(iii), it follows that

$$\begin{aligned}\gamma_{hr3}(P_n) &= \gamma_{r3}(P) + \gamma_{r3}(Q)\\ &= \left\lceil \frac{3\frac{n+1}{2}}{4} \right\rceil + \left\lceil \frac{3\frac{n-1}{2}}{4} \right\rceil\\ &= \left\lceil \frac{3(n+1)}{8} \right\rceil + \left\lceil \frac{3(n-1)}{8} \right\rceil.\end{aligned}$$

If $n \equiv 2, 4, 6 \pmod 8$, then $|V(P)| = |V(Q)| = \frac{n}{2} \not\equiv 0 \pmod 4$. Applying Theorem 8-(iii), it follows that

$$\gamma_{hr3}(P_n) = \gamma_{r3}(P_n^2) = \gamma_{r3}(P) + \gamma_{r3}(Q) = 2\left\lceil \frac{3\frac{n}{2}}{4} \right\rceil = 2\left\lceil \frac{3n}{8} \right\rceil.$$

**(iii)** Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$ be a cycle on $n$ vertices. By Theorem 4 and Remark 1, we have $\gamma_{hr2}(C_3) = 3$ and $\gamma_{hr2}(C_4) = 4$. Assume that $n \geq 5$. If $n$ is even, then $C_n^2$ is the union of two cycles:

$$C = [v_1, v_3, \ldots, v_{n-1}, v_1] \text{ and } C' = [v_2, v_4, \ldots, v_n, v_2],$$

of order $\frac{n}{2}$. By Remark 1 and Theorem 8-(ii), we get

$$\begin{aligned}\gamma_{hr2}(C_n) &= \gamma_{r2}(C_n^2)\\ &= \gamma_{r2}(C) + \gamma_{r2}(C')\\ &= 2\left(\left\lfloor \frac{\frac{n}{2}}{2} \right\rfloor + \left\lceil \frac{\frac{n}{2}}{4} \right\rceil - \left\lfloor \frac{\frac{n}{2}}{4} \right\rfloor\right)\\ &= 2\left(\left\lfloor \frac{n}{4} \right\rfloor + \left\lceil \frac{n}{8} \right\rceil - \left\lfloor \frac{n}{8} \right\rfloor\right),\end{aligned}$$

as desired. Assume that $n$ is odd. Then, clearly, $C_n^2 = C_n$, and by Theorem 8-(ii), we have

$$\gamma_{hr2}(C_n) = \gamma_{r2}(C_n^2) = \gamma_{r2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.$$

**(iv)** The proof is similar to that of item (iii). Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$ be a cycle of order $n$. We know from Theorem 4 and Remark 1 that $\gamma_{hr3}(C_n) = \gamma_{r3}(C_n^2)$. Assume that $n \geq 5$. If $n$ is odd. Then clearly $C_n^2 = C_n$. Hence, $\gamma_{hr3}(C_n) = \gamma_{r3}(C_n^2) = \gamma_{r3}(C_n)$. Applying Theorem 8(iv), we get

$$\gamma_{hr3}(C_n) = \left\lceil \frac{3n}{4} \right\rceil,$$

as desired. If $n$ is even. Then $C_n^2$ is the union of two cycles, each of length $\frac{n}{2}$:

$$C = [v_1, v_3, \ldots, v_{n-1}, v_1] \quad \text{and} \quad C' = [v_2, v_4, \ldots, v_n, v_2].$$

Thus,

$$\gamma_{hr3}(C_n) = \gamma_{r3}(C_n^2) = \gamma_{r3}(C) + \gamma_{r3}(C').$$

Since both $C$ and $C'$ are cycles of order $\frac{n}{2}$, we can apply Theorem 8-(iv) again, $\gamma_{r3}(C) = \left\lceil \frac{3\frac{n}{2}}{4} \right\rceil = \left\lceil \frac{3n}{8} \right\rceil$, and similarly, $\gamma_{r3}(C') = \left\lceil \frac{3n}{8} \right\rceil$. Hence,

$$\gamma_{hr3}(C_n) = \left\lceil \frac{3n}{8} \right\rceil + \left\lceil \frac{3n}{8} \right\rceil = 2\left\lceil \frac{3n}{8} \right\rceil,$$

This completes the proof. □

Applying Theorem 9 and a similar method used in [3, 9], we obtain the following result.

**Corollary 3.** *For any connected graph $G$ of order $n$,*

*(i)* $$\gamma_{hr2}(G) \leq \begin{cases} n + 2 - \frac{\operatorname{diam}(G)}{2}, & \text{if } \operatorname{diam}(G) \equiv 0 \pmod 4, \\ n + 1 - \frac{\operatorname{diam}(G)}{2}, & \text{if } \operatorname{diam}(G) \equiv 2 \pmod 4, \\ n + 1 - \frac{\operatorname{diam}(G)+1}{2}, & \text{if } \operatorname{diam}(G) \equiv 1 \pmod 4, \\ n + 1 + \frac{1-\operatorname{diam}(G)}{2}, & \text{if } \operatorname{diam}(G) \equiv 3 \pmod 4. \end{cases}$$

*(ii)* $$\gamma_{hr3}(G) \leq \begin{cases} n - \operatorname{diam}(G) + \left\lceil \frac{3\operatorname{diam}(G)+6}{8} \right\rceil + \left\lceil \frac{3\operatorname{diam}(G)}{8} \right\rceil, & \text{if } \operatorname{diam}(G) \equiv 0, 6 \pmod 8, \\ n - 1 - \operatorname{diam}(G) + 2\left\lceil \frac{3\operatorname{diam}(G)+3}{8} \right\rceil, & \text{if } \operatorname{diam}(G) \equiv 1, 3, 5 \pmod 8, \\ n - 1 - \operatorname{diam}(G) + \left\lceil \frac{3\operatorname{diam}(G)+6}{8} \right\rceil + \left\lceil \frac{3\operatorname{diam}(G)}{8} \right\rceil, & \text{if } \operatorname{diam}(G) \equiv 2, 4 \pmod 8, \\ n + 1 + \frac{3-\operatorname{diam}(G)}{4}, & \text{if } \operatorname{diam}(G) \equiv 7 \pmod 8. \end{cases}$$

*Proof.* Let $d = \operatorname{diam}(G) + 1$ and $P = [x_1, x_2, \ldots, x_d]$ be a diametral path in $G$. Let $f$ be a $\gamma_{hrs}$-function of $P$ where $s \in \{2, 3\}$, and define the function $g : V(G) \to \mathcal{P}(\{1, \ldots, s\})$ by $g(x) = f(x)$ for $x \in V(P)$ and $g(x) = 1$ for $x \in V(G) \setminus V(P)$. Clearly, $g$ is an H$s$RD-function of $G$ of weight $\omega(f) + n - d - 1$. This implies that $\gamma_{hrs}(G) \leq \omega(f) + n - d - 1$. Thus, the corollary follows by Theorem 2. □

Using Theorem 9 we obtain the next result.

**Corollary 4.** *For every positive integer $a$, there exists a connected graph $G$ such that $\gamma_{hr2}(G) = \gamma_{r2}(G) = a$.*

*Proof.* Clearly, $\gamma_{hr2}(K_a) = \gamma_{r2}(K_a) = a$ for $a \in \{1, 2\}$. Assume that $a \geq 3$. If $a$ is odd and $a = 2m + 1$, then by Theorems 8 and 9 we have that $\gamma_{hr2}(C_{4m+1}) = \gamma_{r2}(C_{4m+1}) = 2m + 1 = a$. Assume that $a$ is even and let $a = 2m$ for some $m \geq 2$. Then again by Theorems 8 and 9 we have $\gamma_{hr2}(C_{4(m-1)+3}) = \gamma_{r2}(C_{4(m-1)+3}) = 2m = a$. □

**Proposition 4.** *For any non-negative integer $a$, there is a connected graph $G$ such that*

$$\gamma_{r2}(G) - \gamma_{hr2}(G) = a.$$

*Proof.* Let $G$ be the graph obtained from a star $K_{1,a+4}$ by subdividing each edge twice. It is not hard to see that $\gamma_{r2}(G) = 2a + 8$ and $\gamma_{hr2}(G) = a + 8$. Therefore, $\gamma_{r2}(G) - \gamma_{hr2}(G) = a$. □

**Proposition 5.** *For any positive integer $a$, there is a connected graph $G$ such that*

$$\gamma_{hr2}(G) - \gamma_{r2}(G) = a.$$

*Proof.* Let $G$ be the graph obtained from a star $K_{1,a+1}$ by first subdividing each edge twice and then adding a new pendant edge at each support vertex. It is easy to verify that $\gamma_{hr2}(G) = 2(a + 1) + 2$ and $\gamma_{r2}(G) = a + 4$. Therfore, $\gamma_{hr2}(G) - \gamma_{r2}(G) = a$. □

**Remark 3.** *Let $G$ be a connected graph. Then the hop $k$-rainbow domination and the $k$-rainbow domination parameters are incomparable.*

*Proof.* By Corollary 4, there exists a connected graph $G$ such that $\gamma_{hr2}(G) = \gamma_{r2}(G)$. By Proposition 4, there exists a connected graph $G'$ such that $\gamma_{r2}(G') > \gamma_{hr2}(G')$. Similarly, by Proposition 5, there exists a connected graph $G''$ such that $\gamma_{hr2}(G'') > \gamma_{r2}(G'')$. These results demonstrate that neither $\gamma_{r2}(G) \leq \gamma_{hr2}(G)$ nor $\gamma_{hr2}(G) \leq \gamma_{r2}(G)$ hold universally. Thus, $\gamma_{r2}(G)$ and $\gamma_{hr2}(G)$ are incomparable. □

## 5. Graphs with $\gamma_{hr2}(G) = n$

In the next theorem we characterize all graphs $G$ with $\gamma_{hr2}(G) = n$. For this purpose, consider a family of graphs defined as follows.

Define $\mathcal{F}$ be the family of graphs $G$ such that $G$ can be constructed from a sequence $H_0, H_1, \ldots, H_t, (t \geq 1)$, of graphs, where $H_0$ is a complete graph as demonstrated in Figure 2 or $H_0 = \overline{K_2}$ as demonstrated in Figure 3, $G = H_t$ and, if $t \geq 1$, then $H_{i+1}$ can be obtained recursively from $H_i$ by adding 2 new vertices and joining each of the new vertices to all vertices in $H_i$.

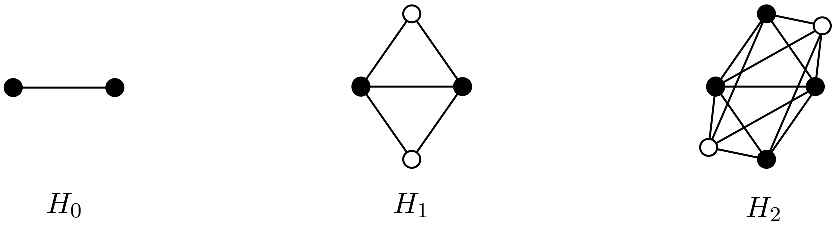


Figure 2: Example of graphs in the family $\mathcal{F}$ if $H_0 = K_2$

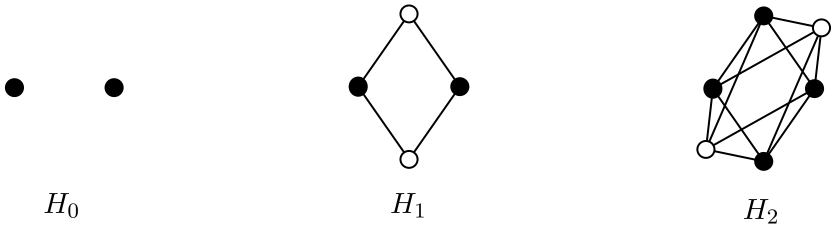


Figure 3: Example of graphs in the family $\mathcal{F}$ if $H_0 = \overline{K_2}$

This family of graphs was introduced by Shabani et al. in [16] for characterizing the graphs $G$ of order $n$ with $\gamma_{hR}(G) = n$ (see Proposition 6).

In what follows is a useful result:

**Proposition 6.** [16] *If $G$ is a connected graph of order $n$, then $\gamma_{hR}(G) = n$ if and only if $G \in \mathcal{F} \cup \{P_4\}$.*

**Theorem 10.** *Let $k \leq 2$ be an integer and $G$ be a connected graph of order $n \geq 1$. Then $\gamma_{hr2}(G) = n$ if and only if $G \in \mathcal{F} \cup \{P_4\}$.*

*Proof.* Let $G$ be a graph of order $n \geq 1$ with $\gamma_{hr2}(G) = n$. We deduce from $n = \gamma_{hr2}(G) \leq \gamma_{hR}(G) \leq n$ that $\gamma_{hR}(G) = n$ and Proposition 6 leads to $G \in \mathcal{F} \cup \{P_4\}$.

Conversely, assume that $G \in \mathcal{F} \cup \{P_4\}$. If $G = P_4$, then clearly $\gamma_{hr2}(G) = 4$. Assume that $G \in \mathcal{F}$. By the construction of $G$ we have $\delta(G) \geq n - 2$. Then by Proposition 4, we have $\gamma_{hr2}(G) = n$. $\square$

## 6. Open questions and problems

We conclude this paper by mentioning some questions and problems suggested by this research.

Using the construction introduced by Shabani et al. in [16], we characterize all connected graph $G$ of order $n$ with $\gamma_{hr2}(G) = n$. Shabani et al. in [16], also characterize all connected graph $G$ of order $n$ with $\gamma_{hR}(G) = n-1$. We think that using theirs construction one can characterize all connected graph $G$ of order $n$ with $\gamma_{hr2}(G) = n - 1$.

**Problem 1.** Characterize all connected graphs $G$ of order $n$ such that $\gamma_{hr2}(G) = n - 1$.

**Problem 2.** For positive integer $k \geq 3$, characterize the graphs $G$ of order $n$ such that $\gamma_{hrk}(G) = n$.

In Theorem 3, we observed that for any positive integer $k \geq 2$ and every graph $G$ of order $n \geq k+2$ we have $\gamma_{hrk}(G) \geq k+1$. To see the sharpness, let $G$ be a graph obtained from complete graph $K_{k+1}$ with vertex set $v_1, \ldots, v_{k+1}$ by first adding $r \geq 0$ new vertices and connecting them to $v_1$ and then adding $s \geq 0$ new vertices and connecting them to $v_{k+1}$. Clearly assigning $\{1\}$ to $v_1, v_{k+1}$, $\{i\}$ to vertex $v_i$ for $i \in \{2, \ldots, k\}$ and $\varnothing$ to the remaining vertices, if any, provides a hop $k$-rainbow dominating function on $G$ of weight $k+1$ and so $\gamma_{hrk}(G) \geq k+1$. Thus, $\gamma_{hrk}(G) = k+1$. This example demonstrate that the bound of Theorem 4 is sharp. Hence, we pose the following problem.

**Problem 3.** For positive integer $k \geq 2$, characterize the graphs $G$ of order $n$ such that $\gamma_{hrk}(G) = k+1$.

Applying the bounds presented in Theorem 4, for any positive integer $k \geq 1$ and every graph $G$ of order $n$, we have that

$$2 \min\{n, k+1\} \leq \gamma_{hrk}(G) + \gamma_{hrk}(\overline{G}) \leq 2n.$$

In particular, if $n \leq k+1$, then we have $\gamma_{hrk}(G) + \gamma_{hrk}(\overline{G}) \leq 2n$. So, finding Nordhaus-Gaddum type results for graphs $G$ of order $n \geq k+2$ is of interest.

**Problem 4.** For graphs $G$ of order $n \geq k+2$, determine Nordhaus-Gaddum type results for $\gamma_{hrk}(G)$.

**Problem 5.** Design an algorithm for computing the value of $\gamma_{hrk}(G)$ for any tree $T$ and $k \geq 3$.

## 7. Conclusion

In this paper, we have introduced and analyzed the concept of hop $k$-rainbow domination in graphs. We established fundamental properties, derived bounds, and determined exact values of $\gamma_{hrk}(G)$ for several graph classes. Additionally, we identified graphs where $\gamma_{hrk}(G) = n$ and showed that the hop $k$-rainbow domination and the $k$-rainbow domination parameters are incomparable. These results lay the groundwork for further exploration of hop $k$-rainbow domination and its applications in graph theory.

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