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On Density of Some Graphs and their Mycielski Graph

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Abstract. Graph density offers an immediate measure of how tightly a network is connected. This paper gives explicit closed-form formulas for the densities of nine well-known undirected graph, simple, and connected graph families: barbell, friendship, sunlet, banana, lollipop, gear, tadpole, wheel, and fan graphs and their corresponding Mycielski graph. The result supply ready to use expressions that remove the need to build the graphs just to calculate its density. All proofs rely only on counting vertices and edges and substituting them into the definition of density.

2020 Mathematics Subject Classifications: : 05C07, 05C12

Key Words and Phrases: graph density, Mycielski graph, special graph families

1. Introduction

Graphs model pairwise relationship in diverse area such as sociology, biology, and computer science. A fundamental characteristic of any graph is its density which ranges from 0 for an empty graph to 1 for a complete graph. Density influences algorithmic complexity, highlights communication bottlenecks, and guides community-detection heuristics. For example, in social networks, density measures the degree of interaction or connection within a community or group, indicating whether relationships are sparse or highly interconnected [1].

The concept of graph density serves as a fundamental metric in network analysis. A substantial body of research underscores its significance in both theoretical and applied contexts, ranging from sociology, computer science, and data mining to bioinformatics and communication systems. A dense graph often suggests a highly collaborative or interactive environment, whereas a sparse graph may indicate isolated vertices or limited connectivity. A newly introduced parameter, referred to in [2] , has recently been utilized to study the density of various graphs, including corona graphs. In addition to density, centrality measures are key tools used to analyze the relative importance of nodes within a graph.

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These include degree centrality, which counts the number of direct connections a node has; betweenness centrality, which quantifies how often a node appears on the shortest paths between other nodes; closeness centrality, which assesses a node's average distance to all other nodes in the network; and eigenvector centrality, which considers not just the number of a node's connections, but also the importance of the nodes it is connected to. The concept of centralities and measures has also been examined in other works, such as [3], [4] and [5]. These metrics, when used in conjunction with graph density, provide a more comprehensive understanding of network structure, influence, and flow dynamics. Moreover, graph density plays a central role in the study of large-scale and complex networks. In biological systems, for instance, analyzing protein-protein interaction networks with respect to their density can reveal important functional modules or pathways. In transportation and communication networks, understanding how dense certain regions are can influence optimization, reliability, and infrastructure planning. In cybersecurity, analyzing the density of connections can help detect anomalous patterns or vulnerabilities.

By evaluating a network's density, researchers are better equipped to understand and visualize its underlying structure, helping to identify trends, cluster formations, community detection, bottlenecks, and other critical attributes. Density also influences the choice of algorithms for tasks such as traversal, shortest path computation, and centrality measures, and is closely linked to concepts such as graph sparsification and complexity analysis.

This article focuses on nine classical graph that routinely appear in external, labeling, and network analysis problems: barbell, friendship, sunlet, banana, lollipop, gear, tadpole, wheel, and fan graph. For each family we determined closed form expression for $|V(G)|$ and $|E(G)|$, substituting the into $D(G)$, and simplify to obtain an exact formula. Also, their respective Mycielski graphs which are of particular interest due to their ability to increase chromatic number without introducing new triangles are also examined to explore how density evolves under graph transformations.

Future sections of this paper will explore formal definitions of graph density, methods of computation, and comparative results based on the selected graph families.

2. Terminology and Notation

A *graph* G is a finite nonempty set V of objects called *vertices* together with a possibly empty set E of 2-element sets of V called *edge*s. To indicate that a graph G has vertex set V and edge set E, we write $G = (V, E)$. To emphasize that V and E are the vertex set and edge set of a graph G, we often write V as $V(G)$ and E as $E(G)$. Each edge $\{u, v\}$ of G is usually denoted by uv or vu. The number of vertices in a graph G is the *order* of G and the number of edges is the *size* of G. The degree of a vertex v is denoted by $deg(v)$ and the *minimum degree* of G is denoted by $\delta(G)$ and the *maximum degree* of G is denoted by $\Delta(G)$ [6]. An *empty graph* of order n is graph with n vertices where in every pair of distinct vertices are not adjacent [6].

For an integer $n \geq 1$, the path graph P_n is a graph of order n and size $n-1$ whose vertices can be labeled as $v_0, v_1, v_2, ..., v_{n-1}$ and whose edges are v_i, v_{i+1} for $i = 1, 2, 3, ..., n-2$

[6]. For an integer $n \geq 1$, the path graph C_n is a graph of order n and size n whose vertices can be labeled as $v_0, v_1, v_2, ..., v_{n-1}$ and whose edges are v_i, v_{i+1} for $i = 1, 2, 3, ..., n-2$ [6]. For $n \geq 3$, the *fan graph* F_n of order $n+1$ is a graph obtained by connecting a new vertex v to each vertex of the path P_n [7]. *tadpole graph* $(T_{m,n})$ is defined as a graph obtained by combining a vertex of cycle C_m with one of leaf of path P_n [8].

A *complete graph*of order $n \geq 2$, denoted by K_n, is a graph with n vertices where in every pair of distinct vertices are adjacent [6]. *Barbell graph* B_n is a graph obtained by connecting two complete graph K_n by an edge [8]. The $(m,n)-$*lollipop graph* denoted by $L_{m,n}$ is a graph obtained by joining a complete graph K_m to a path graph P_n with a bridge [9].

The *friendship graph*, denoted by Fr_n is a set of n triangle having a common vertex [10]. The *sunlet graph* S_n is the graph on $2n$ vertices obtaining by attaching n pendant edges to a cycle graph C_n [11]. A *banana tree graph* $B_{m,n}$ is a graph of order $mn + 1$ obtained by connecting one leaf on each of m copies of star $K_{1,n-1}$ with a single root vertex v that is distinct from all stars [12].

For $n \geq 3$, the *wheel graph*, W_n of order $n + 1$ is a graph produced from the complete product of an isolated vertex and a cycle C_n [7]. A *gear graph*, denoted by G_n, is obtained from the wheel graph by adding a vertex between every pair of adjacent vertices of the cycle [10].

Consider a graph G with $V(G) = v_1, v_2, v_3, ..., v_n$. Apply the following steps to the graph G:

(i) Take the set of new vertices $U = u_1, u_2, u_3, ..., u_n$ and add edges from each vertex u_i of U to the vertices v_j if the corresponding vertex vi is adjacent to v_j in G.

(ii) Take another new vertex w_0 and add edges joining each element in U .

Here, the new graph obtained is the Mycielski graph, denoted by $\mu(G)$ of graph G [13].

Let G be undirected graph with $|V(G)|$ and $|E(G)|$. The *density* of the graph G, denoted as $D(G)$, is defined as the ratio of the number of edges in the graph to the maximum possible number of edges between the number of vertices [14].
For the undirected graph, the density is given by the formula:

$$D(G) = \frac{2|E(G)|}{|V(G)|(|V(G)| - 1)}.$$

Consider Figure 1, it shows that the number of $|E(G)| = 12$ and $|V(G)| = 11$. clearly,

$$D(G) = \frac{2|E(G)|}{|V(G)|(|V(G)| - 1)} = \frac{2|12|}{11(11 - 1)} = \frac{24}{11(10)} = \frac{26}{110} = 0.218$$

3. Results

Theorem 1. *Let G be a barbell graph (B_n) where $n \geq 3$. Then*

$$D(B_n) = \frac{n^2 - n + 1}{2n^2 - n}.$$

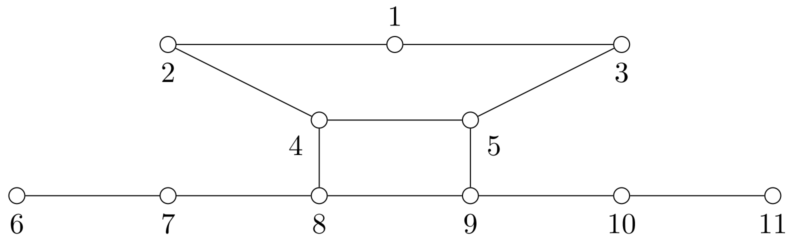

Figure 1: The density of $D(G) = 0.218$

Proof. A barbell graph B_n is constructed by taking two disjoint complete graph K_n and adding exactly one edge that connects one vertex first in K_{n_1} to one vertex in the second K_{n_2}. Thus the number of edges of the barbell graph B_n is:

$$|E(B_n)| = 2\left(\frac{n(n-1)}{2}\right) + 1 = n(n-1) + 1 = n^2 - n + 1.$$

and the number of vertices of the barbell graph B_n is $|V(B_n)| = 2n$. Therefore,

$$G(B_n) = \frac{2|E(B_n)|}{|V(B_n)|(|V(B_n)| - 1)} = \frac{2(n^2 - n + 1)}{2n(2n-1)} = \frac{n^2 - n + 1}{2n^2 - n}.$$

□

Illustration: Consider Figure 2, it shows that the number of edges and vertices of B_4 are $|E| = 13$ and $|V| = 8$ respectively. Clearly,

$$D(B_5) = \frac{2|E(B_5)|}{|V(B_5)|(|V(B_5)| - 1)} \frac{2|13|}{8(8-1)} = \frac{26}{8(7)} = \frac{26}{56} = 0.4642$$

or

$$D(B_5) = \frac{n^2 - n + 1}{2n^2 - n} = \frac{2(4^2 - 2(4) + 2)}{4(4^2) - 2(4)} = \frac{26}{56} = 0.4642.$$

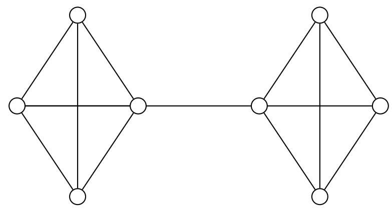

Figure 2: The Barbell Graph B_4

Theorem 2. *Let G be a friendship graph (Fr_n) where $n \geq 2$. Then*

$$D(Fr_n) = \frac{3n}{2n^2 + n}.$$

Proof. a friendship graph Fr_n is constructed by n triangles with a common vertex. Thus the number of edges and vertices of Fr_n are $|E(Fr_n)| = 3n$ and $|V(Fr_n)| = 3n - 2$ respectively. Therefore,

$$D(Fr_n) = \frac{2|E(Fr_n)|}{|V(Fr_n)|(|V(Fr_n)| - 1)} = \frac{2(3n)}{(2n+1)(2n+1-1)} = \frac{6n}{(2n+1)(2n)}$$
$$= \frac{3n}{(2n+1)n} = \frac{3n}{2n^2+n}.$$

□

Illustration: Consider Figure 3, it shows that the number of edges and vertices of Fr_n are $|E(Fr_3)| = 9$ and $|V(Fr_3)| = 7$ respectively. Clearly,

$$D(F_3) = \frac{2|E(Fr_3)|}{|V(Fr_3)|(|V(Fr_3)| - 1)} = \frac{18}{7(6)} = \frac{18}{42} = 0.429$$

or

$$D(F_3) = \frac{3n}{2n^2+n} = \frac{3(3)}{3(3)^2 - 5(3) + 2} = \frac{9}{9 - 15 + 2} = 0.429.$$

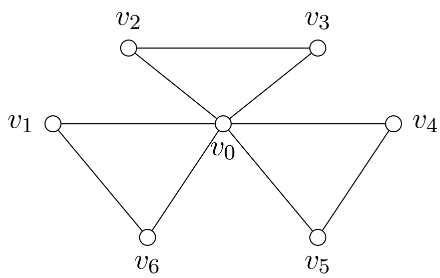

Figure 3: The Friendship Graph Fr_3

Theorem 3. *Let G be a sunlet graph (S_n) where $n \geq 3$. Then*

$$D(S_n) = \frac{2}{2n-1}.$$

Proof. A sunlet graph S_n is constructed by adding n pendant edges to a cycle graph C_n. Thus, the number of edges and vertices of the sunlet graph S_n is $|E(S_n)| = |V(S_n)| = 2n$. Therefore,

$$D(S_n) = \frac{2|E(S_n)|}{|V(S_n)|(|V(S_n)| - 1)} = \frac{2(2n)}{2n(2n-1)} = \frac{2}{2n-1}.$$

□

Illustration: Consider Figure 4, it shows that the number of edges and vertices of S_4 are $|E(S_4)| = 8$ and $|V(S_4)| = 8$ respectively. Clearly,

$$D(S_4) = \frac{2|E(S_4)|}{|V(S_4)|(|V(S_4)| - 1)} = \frac{2|8|}{8(7)} = \frac{16}{56} = \frac{2}{7} = 0.2857$$

or

$$D(S_4) = \frac{2}{2n-1} = \frac{2}{2(4)+1} = \frac{2}{7} = 0.2857.$$

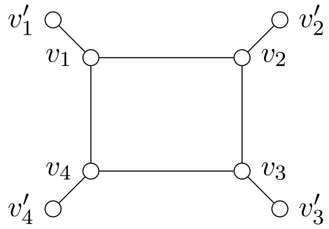

Figure 4: The Sunlet Graph S_4

Theorem 4. *Let G be a banana graph $B_{n,k}$ where $n \geq 2$ and $k \geq 4$. Then*

$$D(B_{n,k}) = \frac{2}{nk+1}.$$

Proof. A banana graph $B_{n,k}$ is constructed by connecting one leaf of each n copies of k-star graph with one root vertex. Thus, the number of edges and vertices of the banana graph $B_{n,k}$ are $|E(B_{n,k})| = nk$ and $|V(B_{n,k})| = nk + 1$ respectively. Therefore,

$$D(B_{n,k}) = \frac{2|E(B_{n,k})|}{|V(B_{n,k})|(|V(B_{n,k})| - 1)} = \frac{2(nk)}{nk+1(nk+1-1)} = \frac{2nk}{(nk+1)nk} = \frac{2}{nk+1}.$$

□

Illustration: Consider Figure 5, it shows that the number of edges and vertices of $B_{3,4}$ are $|E(B_{3,4})| = 12$ and $|V(B_{3,4})| = 13$ respectively. Clearly,

$$D(B_{3,4}) = \frac{2|E(B_{3,4})|}{|V(B_{3,4})|(|V(B_{3,4})| - 1)} = \frac{2|12|}{13(13-1)} = \frac{24}{13(12)} = \frac{24}{156} = 0.153$$

or

$$D(B_{3,4}) = \frac{2}{nk+1} = \frac{2}{3(4)+1} = \frac{2}{13} = 0.153.$$

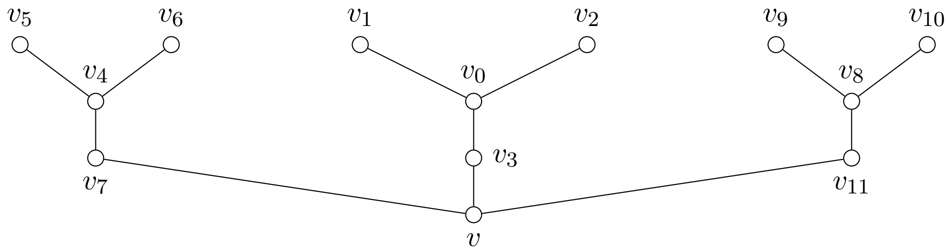

Figure 5: The Banana Graph $B_{3,4}$

Theorem 5. *Let G be a lollipop graph ($L_{m,n}$) where $m \geq 3$ and $n \geq 1$. Then*

$$D(L_{m,n}) = \frac{m(m-1)+2n}{(m+n)(m+n-1)}.$$

Proof. A lollipop graph $L_{m,n}$ is constructed by a complete graph K_n and a path graph P_n adding exactly one edge that connect one vertex first in K_n to one vertex in P_n. Thus, the number of edges and vertices of the lollipop graph $L_{m,n}$ are $|E(L_{m,n})| = \dfrac{m(m-1)+2n}{2}$ and $|V(L_{m,n})| = m+n$ respectively. Therefore,

$$D(L_{m,n}) = \frac{2|E(L_{m,n})|}{|V(L_{m,n})|(|V(L_{m,n})|-1)} = \frac{2\left[\dfrac{m(m-1)+2n}{2}\right]}{(m+n)(m+n+1)} = \frac{m(m-1)+2n}{(m+n)(m+n+1)}.$$

□

Illustration: Consider Figure 6, it shows that the number edges and vertices of $L_{4,3}$ are $|E(L_{4,3})| = 9$ and $|V(L_{4,3})| = 7$. Clearly,

$$D(L_{4,3}) = \frac{2|E(L_{4,3})|}{|V(L_{4,3})|(|V(L_{4,3})|-1)} \frac{2|9|}{7(7-1)} = \frac{18}{7(6)} = \frac{18}{42} = 0.43$$

or

$$D(L_{4,3}) = \frac{m(m-1)+2n}{(m+n)(m+n+1)} = \frac{4(4-1)+2(3)}{(4+3)(4+3-1)} = \frac{18}{42} = 0.43.$$

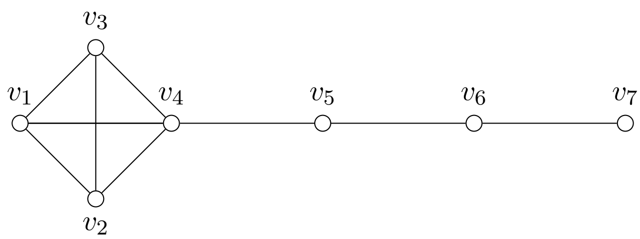

Figure 6: The Lollipop Graph $L_{4,3}$

Theorem 6. *Let G be a tadpole graph ($T_{m,n}$) where $m \geq 3$ and $n \geq 1$. Then*

$$D(T_{m,n}) = \frac{2}{m+n-1}.$$

Proof. A tadpole graph $T_{m,n}$ is constructed by cycle graph C_n and path graph P_n adding exactly one edge that connect one vertex in C_n to one vertex in P_n. Thus, the number of vertices and edges of the tadpole graph $T_{m,n}$ is $|E(T_{m,n})| = |V(T_{m,n})| = m+n$. Therefore,

$$D(T_{m,n}) = \frac{2|E(T_{m,n})|}{|V(T_{m,n})|(|V(T_{m,n})|-1)} = \frac{2(m+n)}{(m+n)(m+n-1)} = \frac{2}{m+n-1}.$$

□

Consider Figure 7, it shows that the number edges and vertices of $T_{4,3}$ are $|E(T_{4,3})| = 7$ and $|V(T_{4,3})| = 7$ respectively. Clearly,

$$D(T_{4,3}) = \frac{2|E(T_{4,3})|}{|V(T_{4,3})|(|V(T_{4,3})|-1)} = \frac{2|7|}{7(7-1)} = \frac{14}{7(6)} = \frac{14}{42} = 0.33$$

or

$$D(L_{4,3}) = \frac{2}{m+n-1} = \frac{2}{(4+3-1))} = \frac{2}{6} = 0.33.$$

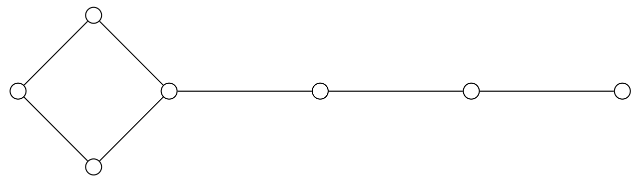

Figure 7: The Tadpole Graph $T_{4,3}$

Theorem 7. *Let G be a gear graph G_n where $n \geq 3$. Then*

$$D(G_n) = \frac{3}{2n+1}.$$

Proof. A gear graph is constructed by wheel graph W_n adding vertex in every pair of adjacent in the outer cycle. Thus, the number of edges and vertices of the gear graph G_n are $|E(G_n)| = 3n$ and the order is $|V(G_n)| = 2n+1$ respectively. Therefore,

$$D(G_n) = \frac{2|E(G_n)|}{|V(G_n)|(|V(G_n)|-1)} = \frac{2(3n)}{(2n+1)(2n+1-1)} = \frac{2(3n)}{(2n+1)2n} = \frac{3}{2n+1}.$$

□

Illustration: Consider Figure 8, it shows that the number of edges and vertices of G_4 are $|E(G_4)| = 16$ and $|V(G_4)| = 9$ respectively. Clearly,

$$D(G_4) = \frac{2|E(G_4)|}{|V(G_4)|(|V(G_4)|-1)} = \frac{2|16|}{9(9-1)} = \frac{32}{9(8)} = \frac{32}{72} = 0.44$$

or

$$D(G_4) = \frac{3}{2n+1} = \frac{4}{2(4)+1} = \frac{4}{9} = 0.44.$$

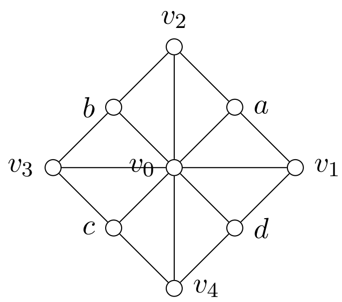

Figure 8: The Gear Graph G_4

Theorem 8. *Let G be a wheel graph (W_n) where $n \geq 4$. Then*

$$D(W_n) = \frac{4}{n}.$$

Proof. A wheel graph W_n is constructed by cycle graph C_{n-1} and consists one central vertex that connects to all vertices in C_{n-1}. Thus, the number of edges and vertices of the wheel graph are $|E(W_n)| = 2n$ and order is $|V(W_n)| = n+1$. Therefore,

$$D(W_n) = \frac{2|E(W_n)|}{|V(W_n)|(|V(W_n)|-1)} = \frac{2(2n)}{(n+1)(n+1-1)} = \frac{4n}{(n+1)n} = \frac{4}{n+1}.$$

□

Illustration: Consider Figure 9, it shows that the number edges and vertices of W_5 are $|E(W_5)| = 10$ and $|V(W_5)| = 6$ respectively. Clearly,

$$D(W_5) = \frac{2|E(W_5)|}{|V(W_5)|(|V(W_5)|-1)} = \frac{2|10|}{6(6-1)} = \frac{20}{6(5)} = \frac{20}{30} = 0.67$$

or

$$D(W_5) = \frac{4}{n+1} = \frac{4}{5+1} = \frac{4}{6} = 0.67.$$

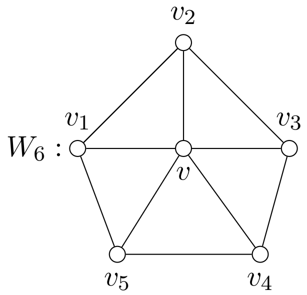

Figure 9: The Graph W_5

Theorem 9. *Let G be a fan graph graph (F_n) where $n \geq 3$. Then*

$$D(F_n) = \frac{4n-2}{(n+1)(n)}.$$

Proof. A fan graph F_n is constructed by a path graph P_n and one central vertex that connects to all vertices in path graph P_n. Thus, the number of edges and vertices of the fan graph F_n are $|E(F_n)| = 2n - 1$ and order $|V(F_n)| = n + 1$ respectively. Therefore,

$$D(F_n) = \frac{2|E(F_n)|}{|V(F_n)|(|V(F_n)| - 1)} = \frac{2m}{(n+1)(n+1-1)} = \frac{2(2n-1)}{n(n-1)} = \frac{4n-2}{(n+1)n}.$$

□

Illustration: Consider Figure 10, it shows that the number of edges and vertices of F_5 are $|E(F_5)| = 9$ and $|V(F_5)| = 6$. Clearly,

$$D(F_5) = \frac{2|E(F_5)|}{|V(F_5)|(|V(F_5)| - 1)} = \frac{2|9|}{6(6-1)} = \frac{18}{6(5)} = \frac{18}{30} = 0.6$$

or

$$D(F_5) = \frac{4n-2}{(n+1)n} = \frac{4(5)-2}{(5+1)5} = \frac{18}{(5)5} = \frac{18}{28} = 0.6.$$

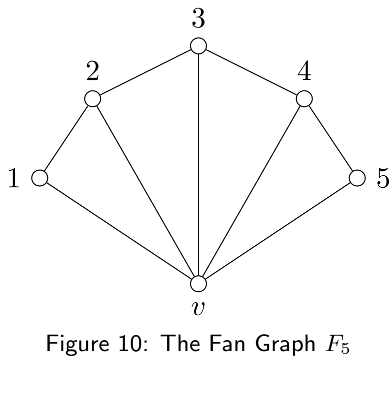

Figure 10: The Fan Graph F_5

By Definition Mycielski Graph, we can easily derived the number of edges and vertices of a Mycielski graph of a connected graph.

Remark 1. *Let G be a nontrivial connected graph. Then $|E(\mu(G))| = 3|E(G)| + |V(G)|$ and $|V(\mu(G))| = 2|V(G)| + 1$.*

Theorem 10. *Let G be a nontrivial connected graph. Then the density of $\mu(G)$ is*

$$D(\mu(G)) = \frac{3|E(G)| + |V(G)|}{|V(G)|(2|V(G)| + 1)}.$$

Proof. By Definition of Mycielski graph and Remark 1,

$$D(\mu(G)) = \frac{2|E(\mu(G))|}{|V(G)|(|V(G)| - 1)} = \frac{2(3|E(G)| + |V(G)|)}{(2|V(G)| + 1)(2|V(G)| + 1 - 1)} = \frac{2(3|E(G)| + |V(G)|)}{(2|V(G)| + 1)2|V(G)|}$$
$$= \frac{3|E(G)| + |V(G)|}{|V(G)|(2|V(G)| + 1)}.$$

□

Corollary 1. *Let G be a path graph (P_n) where $n \geq 2$. Then,*

$$D(\mu(P_n)) = \frac{4n - 3}{2n^2 + n}.$$

Proof. By Theorem 10 and since the number of edges and vertices of P_n are $n - 1$ and n respectively, we have

$$D(\mu(P_n)) = \frac{3|E(P_n)| + |V(P_n)|}{|V(P_n)|(2|V(P_n)| + 1)} = \frac{3(n - 1) + n}{n(2n + 1)} = \frac{3n - 3 + n}{2n^2 + n} = \frac{4n - 3}{2n^2 + n}.$$

□

Corollary 2. *Let G be a cycle graph (C_n) with $n \geq 3$. Then*

$$D(\mu(C_n)) = \frac{4}{2n + 1}.$$

Proof. By Theorem 10 and since the number of edges and vertices of Cn is n, we have

$$D(\mu(C_n)) = \frac{3|E(C_n)| + |V(C_n)|}{|V(C_n)|(2|V(C_n)| + 1)} = \frac{3n + n}{n(2n + 1)} = \frac{4n}{n(2n + 1)} = \frac{4}{2n + 1}.$$

□

Corollary 3. *Let G be a complete graph (K_n) where $n \geq 4$. Then*

$$D(\mu(K_n)) = \frac{3n - 1}{4n + 2}.$$

Proof. By Theorem 10 and since the number of edges and vertices of Kn are $\frac{n(n-1)}{2}$ and n respectively, we have

$$D(\mu(K_n)) = \frac{3|E(K_n)| + |V(K_n)|}{|V(K_n)|(2|V(K_n)|+1)} = \frac{\frac{3(n^2-n)}{2}+n}{2n^2+n} = \frac{\frac{3n^2-3n+2n}{2}}{2n^2+n} = \frac{\frac{3n^2-n}{2}}{2n^2+n}$$
$$= \left(\frac{3n^2-n}{2}\right)\left(\frac{1}{2n^2+n}\right) = \frac{3n^2-n}{4n^2+2n} = \frac{n(3n-1)}{n(4n+2)}$$
$$= \frac{3n-1}{4n+2}.$$

□

Corollary 4. *Let G be a barbell graph (B_n) where $n \geq 3$. Then*

$$D(\mu(B_n)) = \frac{3n^2-n+3}{8n^2+2n}.$$

Proof. By Theorem 10 and since the number of edges and vertices of B_n are n^2-n+1 and $2n$ respectively, we have

$$D(\mu(B_n)) = \frac{3|E(B_n)| + |V(B_n)|}{|V(B_n)|(2|V(B_n)|+1)} = \frac{3(n^2-n+1)+2n}{2n(2\cdot 2n+1)} = \frac{3n^2-3n+3+2n}{8n^2+2n}$$
$$= \frac{3n^2-n+3}{8n^2+2n}.$$

□

Corollary 5. *Let G be a friendship graph (Fr_n) where $n \geq 2$. Then*

$$D(\mu(Fr_n)) = \frac{11n+1}{8n^2+10n+3}.$$

Proof. By Theorem 10 and since the number of edges and vertices of Fr_n are $3n$ and $3n-2$ respectively, we have

$$D(\mu(Fr_n)) = \frac{3|E(Fr_n)| + |V(Fr_n)|}{|V(Fr_n)|(2|V(Fr_n)|+1)} = \frac{3(3n)+2n+1}{(2n+1)[2(2n+1)+1]} = \frac{9n+2n+1}{(2n+1)(4n+3)}$$
$$= \frac{11n+1}{8n^2+10n+3}.$$

□

Corollary 6. *Let G be a sunlet graph (S_n) where $n \geq 3$. Then*

$$D(\mu(S_n)) = \frac{4}{4n+1}.$$

Proof. By Theorem 10 and since the number of edges and vertices of S_n is $2n$, we have

$$D(\mu(S_n)) = \frac{3|E(S_n)| + |V(S_n)|}{|V(S_n)|(2|V(S_n)| + 1)} = \frac{3(2n) + 2n}{2n(2 \cdot 2n + 1)} = \frac{2n(4)}{2n(4n + 1)} = \frac{4}{4n + 1}.$$

□

Corollary 7. *Let G be a banana graph ($B_{n,k}$) where $n \geq 2$ and $k \geq 4$. Then*

$$D(\mu(B_{n,k})) = \frac{4nk + 1}{2n^2k^2 + 5nk + 3}.$$

Proof. By Theorem 10 and since the number of edges and vertices of $B_{n,k}$ are nk and $nk + 1$ respectively, we have

$$\begin{aligned} D(\mu(B_{n,k})) = \frac{3|E(B_{n,k})| + |V(B_{n,k})|}{|V(B_{n,k})|(2|V(B_{n,k})| + 1)} &= \frac{3nk + nk + 1}{(nk + 1)[2(nk + 1) + 1]} = \frac{4nk + 1}{(nk + 1)(2nk + 3)} \\ &= \frac{4nk + 1}{2n^2k^2 + 5nk + 3}. \end{aligned}$$

□

Corollary 8. *Let G be a lollipop graph ($L_{m,n}$) where $m \geq 3$ and $n \geq 1$. Then*

$$D(\mu(L_{m,n})) = \frac{3m^2 - m + 8n}{4m^2 + 8mn + 4n^2 + m + n}.$$

Proof. By Theorem 10 and since the number of edges and vertices of $L_{m,n}$ are $\frac{m(m-1)+2}{2}$ and $m + n$ respectively, we have

$$\begin{aligned} D(\mu(L_{m,n})) = \frac{3|E(L_{m,n})| + |V(L_{m,n})|}{|V(L_{m,n})|(2|V(L_{m,n})| + 1)} &= \frac{3\left(\frac{m(m-1)+2n}{2}\right) + m + n}{(m + n)[2(m + n) + 1]} \\ &= \frac{\frac{3(m^2 - m + 2n)}{2} + m + n}{(m + n)(2m + 2n + 1)} \\ &= \frac{\frac{3m^2 - 3m + 6n + 2m + 2n}{2}}{2m^2 + 4mn + 2n^2 + m + n} \\ &= \frac{\frac{3m^2 - m + 8n}{2}}{2m^2 + 4mn + 2n^2 + m + n} \\ &= \frac{3m^2 - m + 8n}{4m^2 + 8mn + 4n^2 + m + n}. \end{aligned}$$

□

Corollary 9. *Let G be a gear graph (G_n) where $n \geq 3$. Then*

$$D(\mu(G_n)) = \frac{11n+1}{8n^2+10n+3}.$$

Proof. By Theorem 10 and since the number of edges and vertices of G_n are $3n$ and $2n+1$ respectively, we have

$$D(\mu(G_n)) = \frac{3|E(G_n)|+|V(G_n)|}{|V(G_n)|(2|V(G_n)|+1)} = \frac{3(3n)+2n+1}{(2n+1)[2(2n+1)+1]} = \frac{9n+2n+1}{(2n+1)(4n+3)}$$
$$= \frac{11n+1}{8n^2+6n+4n+3} = \frac{11n+1}{8n^2+10n+3}.$$

□

Corollary 10. *Let G be a tadpole graph ($T_{m,n}$) where $m \geq 3$ and $n \geq 1$. Then,*

$$D(\mu(T_{m,n})) = \frac{4}{2m+2n+1}.$$

Proof. By Theorem 10 and since the number of edges and vertices of $T_{m,n}$ is $m+n$, we have

$$D(\mu(T_{m,n})) = \frac{3|E(T_{m,n})|+|V(T_{m,n})|}{|V(T_{m,n})|(2|V(T_{m,n})|+1)} = \frac{3(m+n)+m+n}{(m+n)[2(m+n)+1]} = \frac{4m+4n}{(m+n)(2m+2n+1)}$$
$$= \frac{4(m+n)}{(m+n)(2m+2n+1)} = \frac{4}{2m+2n+1}.$$

□

Corollary 11. *Let G be a wheel graph (W_n) where $n \geq 3$. Then*

$$D(\mu(W_n)) = \frac{7n-6}{2n^2+n}.$$

Proof. By Theorem 10 and since the number of edges and vertices of W_n are $2n-2$ and n respectively, we have

$$D(\mu(W_n)) = \frac{3|E(W_n)|+|V(W_n)|}{|V(W_n)|(2|V(W_n)|+1)} = \frac{3(2n-2)+n}{n(2n+1)} = \frac{6n-6+n}{2n^2+n} = \frac{7n-6}{2n^2+n}.$$

□

Corollary 12. *Let G be a fan graph (F_n) where $n \geq 3$. Then*

$$D(\mu(F_n)) = \frac{7n-2}{2n^2+5n+3}.$$

Proof. By Theorem 10 and since the number of edges and vertices of F_n are $2n-1$ and $n+1$ respectively, we have

$$D(\mu(F_n)) = \frac{3|E(F_n)| + |V(F_n)|}{|V(F_n)|(2|V(F_n)|+1)} = \frac{3(2n-1)+n+1}{(n+1)[2(n+1)+1]} = \frac{6n-3+n+1}{(n+1)(2n+3)}$$
$$= \frac{7n-2}{2n^2+5n+3}.$$

□

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