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Unique Solution Analysis for Generalized Caputo-Type Fractional BVP via Banach Contraction

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Abstract. In this manuscript, we investigate the existence of a unique solution to a boundary value problem (BVP) involving generalized fractional derivatives of the Caputo type. Our approach is grounded in the Banach contraction mapping theorem, which provides a rigorous framework for proving the existence of a fixed point and, consequently, a solution to the BVP. We extend this methodology to explore analogous problems, offering further insights and interpretations of the results derived from the main theorem. This work not only contributes to the theoretical understanding of fractional differential equations but also demonstrates how these techniques can be applied to a broader class of problems in mathematical physics and engineering. Through detailed analysis and extrapolation, we aim to establish a deeper connection between fractional calculus and fixed-point theory, providing a foundation for future research in this area.

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1. Introduction

Mathematical modeling has become an essential tool in contemporary scientific research, serving as a vital means to describe and understand a wide array of phenomena in both physical and biological sciences. This progress is closely linked to the development of new theories and the refinement of classical models, with the use of fractional calculus playing an increasingly prominent role. In particular, fractional differential equations (FDEs) and partial differential equations (PDEs) have emerged as powerful frameworks for capturing the dynamics of complex systems, where traditional integer-order models may fall short. These models have been employed to explain a variety of phenomena across multiple disciplines, including physics, engineering, and biology [1–6].

As the field evolves, there is a growing need to simulate these phenomena in ways that offer both analytical insights and numerical interpretations. Such simulations serve not only as a means to validate theoretical models but also as a bridge between abstract mathematical theories and real-world applications. This paper aims to address one such simulation, focusing on the application of fractional calculus to boundary value problems (BVPs). Specifically, we explore the extension of ordinary differential equations (ODEs) to the realm of fractional differential equations, which allows for a more nuanced description of systems exhibiting memory effects, non-local behavior, and anomalous dynamics [7–12].

The theoretical framework used in this study is derived from Theorem 3.3 presented in [13], which is a result related to the existence of unique solutions for certain BVPs. This theorem was previously stated without proof, and its implications have not been fully explored. Additionally, the problem remains unresolved in the context of fractional differential equations, as indicated in [Problem 41.6][14]. The goal of this work is to provide a rigorous proof of the theorem in the fractional setting, thereby filling a gap in the existing literature and contributing to the broader understanding of fractional boundary value problems. To formalize our approach, we begin by presenting the classical result from [15]:

Theorem 1. *([15]) Suppose $\Theta : [\theta, \vartheta] \times \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies a uniform Lipschitz condition with respect to μ, i.e.,*

$$|\Theta(\tau, \mu) - \Theta(\tau, \nu)| \leq \zeta|\mu - \nu|, \quad (\tau, \mu), (\tau, \nu) \in [\theta, \vartheta] \times \mathbb{R},$$

where $\zeta > 0$ is a constant. If the following condition holds:

$$\zeta \frac{(\vartheta - \theta)^2}{8} < 1,$$

then the boundary value problem

$$\mu'' = -\Theta(\tau, \mu), \quad \mu(\theta) = \lambda_1, \quad \mu(\vartheta) = \lambda_2,$$

admits a unique solution.

In this article, we extend this result to the fractional setting by replacing the standard second-order derivative μ'' with a generalized Caputo fractional derivative of order σ, where $1 < \sigma \leq 2$. This extension is crucial as it allows us to study the existence of solutions to BVPs involving fractional derivatives, which have been shown to better describe systems with memory effects, non-local interactions, and other phenomena not captured by classical integer-order derivatives. We aim to derive the existence of a unique solution to the fractional boundary value problem (BVP) defined by the following system:

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{\theta+}\mu(\tau) = -\Theta(\tau, \mu(\tau)), & \theta < \tau < \vartheta, \\ \mu(\theta) = \lambda_1, & \mu(\vartheta) = \lambda_2, \end{cases} \tag{1}$$

where ${}^{\varrho}_{c}D^{\sigma}_{\theta+}$ represents the generalized Caputo fractional derivative of order σ, and σ lies in the interval $1 < \sigma \leq 2$. This system extends the classical BVP by incorporating fractional derivatives, which account for the memory and hereditary effects of the system. Such models are particularly relevant in fields such as anomalous diffusion, viscoelasticity, and complex materials. In previous studies [16–18] and related works, we explored the existence of singular solutions to boundary value problems involving generalized fractional derivatives of the Caputo type. These results laid the foundation for extending the classical theory to fractional-order differential equations. In this work, we further develop these ideas by applying Theorem 1 in the context of generalized fractional differential equations. Our analysis not only provides a rigorous proof for the existence of a singular solution to the fractional BVP but also offers new insights into the behavior of solutions to fractional differential equations.

We believe that the results presented in this paper will have significant implications for the study of fractional-order systems and will contribute to the growing body of research on fractional calculus, especially in the context of boundary value problems. Furthermore, the techniques we use can be generalized to a wide range of problems in applied mathematics, physics, and engineering, where fractional models are becoming increasingly relevant.

In summary, this paper provides a theoretical framework for solving BVPs involving fractional derivatives of the Caputo type, using the Banach contraction theorem as a primary tool. Through this approach, we offer new results on the existence and uniqueness of solutions to fractional BVPs and provide a deeper understanding of their implications in various scientific fields.

2. Principal concepts

Starting, we review some basic properties of fractional calculus for investigating boundary value problems, lookup in [19–23].

Definition 1. *On the left-sided in the generalized integral of fractional order* ${}^{\varrho}I^{\sigma}_{\theta+}\mu$ *for* $\sigma \in \mathbb{C}(Re(\sigma) > 0)$ *is given by*

$$\left({}^{\varrho}I^{\sigma}_{\theta+}\mu\right)(\tau) = \frac{\varrho^{1-\sigma}}{\Gamma(\sigma)} \int_{\theta}^{\tau} r^{\varrho-1}(\tau^{\varrho} - r^{\varrho})^{\sigma-1}\mu(r)\,\mathrm{d}r, \tag{2}$$

where $\tau > 0,\ \varrho > 0$.

According to the formula of the generalized fractional integrals (2), we define the generalized fractional derivative for $\tau > 0$ by

$$\left({}_{c}^{\varrho}D^{\sigma}_{\theta+}\mu\right)(\tau) = \left(\tau^{1-\varrho}\frac{d}{d\tau}\right)^{n}\left({}^{\varrho}I^{n-\sigma}_{\theta+}\mu\right)(\tau) = \frac{\varrho^{\sigma-n+1}}{\Gamma(n-\sigma)}\left(\tau^{1-\varrho}\frac{d}{d\tau}\right)^{n}\int_{\theta}^{\tau} r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{n-1-\sigma}\mu(r)\,\mathrm{d}r. \tag{3}$$

Definition 2. *By using the above generalized fractional derivative (3), the generalized Caputo non-classical derivative with the operator notation ${}_{c}^{\varrho}D^{\sigma}_{\theta+}$ is defined by*

$${}_{c}^{\varrho}D^{\sigma}_{\theta+}\mu(\tau) = \left({}_{c}^{\varrho}D^{\sigma}_{\theta+}\left[\mu(\tau) - \sum_{l=0}^{n-1}\frac{\mu^{(l)}(\theta)}{l!}(\tau-\theta)^{l}\right]\right)(\tau), \tag{4}$$

where $n = [Re(\sigma)]$.

Lemma 1. *Let $\sigma, \varrho > 0$ and $\mu \in \mathcal{C}(J,\mathbb{R}) \cap \mathcal{C}^{1}(J,\mathbb{R})$. Then*

1. *The generalized Caputo fractional differential equation*

$${}_{c}^{\varrho}D^{\sigma}_{\theta+}\mu(\tau) = 0,$$

has a solution.

$$\mu(\tau) = p_0 + p_1\left(\frac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right) + p_2\left(\frac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right)^{2} + \dots + p_{n-1}\left(\frac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right)^{n-1},$$

where $p_i \in \mathbb{R},\quad i = 0,1,2,\dots,n-1$ and $n = [\sigma]+1$.

2. *If $\mu,\ {}_{c}^{\varrho}D^{\sigma}_{\theta+}\mu \in \mathcal{C}(J,\mathbb{R}) \cap \mathcal{C}^{1}(J,\mathbb{R})$. Then*

$${}^{\varrho}I^{\sigma}_{\theta+}\ {}^{\varrho}cD^{\sigma}_{\theta+}\mu(\tau) = \mu(\tau) + p_0 + p_1\left(\frac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right) + p_2\left(\frac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right)^{2} + \dots + p_{n-1}\left(\frac{\tau^{\varrho}-\theta^{\varrho}}{\varrho}\right)^{n-1}, \tag{5}$$

where $p_i \in \mathbb{R},\quad i = 0,1,2,\dots,n-1$ and $n = [\sigma]+1$.

3. Main results

At the heart of this passage, we witness significant propositions and theorems on which all this work is based. We review the integral formula for the generalized fractional order BVP (1) from the principle of the Green function.

Lemma 2. *Presume that Θ is a function is continuous and either a function $\mu \in C[\theta,\vartheta]$ is a solution of (1) equivalent that μ check the integral equation*

$$\mu(\tau) = \left[(\lambda_2-\lambda_1)\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})} + \lambda_1\right] + \int_{\theta}^{\vartheta}\hbar(\tau,r)\,\Theta(r,\mu(r))\,\mathrm{d}r,$$

where

$$\hbar(\tau,r)=\frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\begin{cases}\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}-r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{\sigma-1}, & \theta\le r\le\tau\le\vartheta,\\ \frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}, & \theta\le\tau\le r\le\vartheta.\end{cases}\tag{6}$$

Proof. By the Lemma(1),we solve this problem

$${}^{\varrho}_{c}D^{\sigma}_{\theta+}\mu(\tau)=-q(\tau).$$

According to (5), we obtain

$$\begin{aligned}{}^{\varrho}I^{\sigma}_{\theta+}\;{}^{\varrho}_{c}D^{\sigma}_{\theta+}\mu(\tau)&=-{}^{\varrho}I^{\sigma}_{\theta+}q(\tau)+p_0+p_1\frac{(\tau^{\varrho}-\theta^{\varrho})}{\varrho}\\ \mu(\tau)&=-{}^{\varrho}I^{\sigma}_{\theta+}q(\tau)+p_0+p_1\frac{(\tau^{\varrho}-\theta^{\varrho})}{\varrho}\\ \mu(\tau)&=-\frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\tau}r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{\sigma-1}q(r)dr+p_0+p_1\frac{(\tau^{\varrho}-\theta^{\varrho})}{\varrho},\end{aligned}$$

by using boundary conditions

$$\begin{aligned}&\mu(\theta)=\lambda_1\Longrightarrow p_0=\lambda_1,\\ &\mu(\vartheta)=\lambda_2\Longrightarrow p_1=\frac{\varrho(\lambda_2-\lambda_1)}{(\vartheta^{\varrho}-\theta^{\varrho})}+\frac{\varrho^{2-\sigma}}{(\vartheta^{\varrho}-\theta^{\varrho})\Gamma(\sigma)}\int_{\theta}^{\vartheta}r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r.\end{aligned}$$

Now, by replacing in $\mu(\tau)$, and we get

$$\begin{aligned}\mu(\tau)=&-\frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\tau}r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r\\ &+\left[\varrho(\lambda_2-\lambda_1)+\frac{\varrho^{2-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\vartheta}r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}q(r)dr\right]\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}+\lambda_1,\end{aligned}$$

and

$$\begin{aligned}\mu(\tau)=&-\frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\tau}r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r+\frac{\varrho^{1-\sigma}(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})\Gamma(\sigma)}\\ &\int_{\theta}^{\vartheta}r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r+(\lambda_2-\lambda_1)\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}+\lambda_1.\end{aligned}$$

Therefore,

$$\begin{aligned}\mu(\tau)=&\left[(\lambda_2-\lambda_1)\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}+\lambda_1\right]\\ &+\frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\theta}^{\tau}\left(\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}-r^{\varrho-1}(\tau^{\varrho}-r^{\varrho})^{\sigma-1}\right)q(r)\,\mathrm{d}r\\ &+\frac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\int_{\tau}^{\vartheta}\frac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}r^{\varrho-1}(\vartheta^{\varrho}-r^{\varrho})^{\sigma-1}q(r)\,\mathrm{d}r,\end{aligned}$$

and the proof is complete.

Immediately, we will present the important salient rules that will make it easier for us to achieve our desired objectives.

Proposition 1. *Depending on the Green function $\hbar$ is mentioned in Lemma 2. Therefore*

$$\int_\theta^\vartheta |\hbar(\tau,r)|dr \leq \tfrac{1}{\varrho^\sigma\Gamma(\sigma+1)}\left[(\vartheta^\varrho-\theta^\varrho)^{\sigma-1}(\tau^\varrho-\theta^\varrho)-(\tau^\varrho-\theta^\varrho)^\sigma\right]. \tag{7}$$

Proof. We determine

$$\int_\theta^\vartheta |\hbar(\tau,r)|dr, \qquad \hbar(\tau,r)\geq 0,\, \forall\, \theta\leq\tau,\, r\leq\vartheta.$$

According to (6), we have

$$\theta\leq r\leq\tau\leq\vartheta \Longrightarrow (\tau-r)\leq(\vartheta-r),$$

i.e.

$$(\tau^\varrho-r^\varrho)\leq(\vartheta^\varrho-r^\varrho)\Longrightarrow(\tau^\varrho-r^\varrho)^{\sigma-1}\leq(\vartheta^\varrho-r^\varrho)^{\sigma-1}$$
$$r^{\varrho-1}(\tau^\varrho-r^\varrho)^{\sigma-1}\leq r^{\varrho-1}(\vartheta^\varrho-r^\varrho)^{\sigma-1}.$$

Then

$$0\leq r^{\varrho-1}(\vartheta^\varrho-r^\varrho)^{\sigma-1}-r^{\varrho-1}(\tau^\varrho-r^\varrho)^{\sigma-1},$$

and we know the positivity of the quantity

$$\tfrac{(\tau^\varrho-\theta^\varrho)}{(\vartheta^\varrho-\theta^\varrho)}>0,$$

i.e.

$$\tfrac{(\tau^\varrho-\theta^\varrho)}{(\vartheta^\varrho-\theta^\varrho)}r^{\varrho-1}(\vartheta^\varrho-r^\varrho)^{\sigma-1}>0.$$

Thus, we deduce that

$$0\leq\tfrac{(\tau^\varrho-\theta^\varrho)}{(\vartheta^\varrho-\theta^\varrho)}r^{\varrho-1}(\vartheta^\varrho-r^\varrho)^{\sigma-1}-r^{\varrho-1}(\tau^\varrho-r^\varrho)^{\sigma-1}.$$

Therefore

$$\int_\theta^\vartheta|\hbar(\tau,r)|\,\mathrm{d}r=\tfrac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\left[\int_\theta^\tau\left(\tfrac{(\tau^\varrho-\theta^\varrho)}{(\vartheta^\varrho-\theta^\varrho)}r^{\varrho-1}(\vartheta^\varrho-r^\varrho)^{\sigma-1}-r^{\varrho-1}(\tau^\varrho-r^\varrho)^{\sigma-1}\right)\mathrm{d}r\right.$$
$$\left.+\int_\tau^\vartheta\left(\tfrac{(\tau^\varrho-\theta^\varrho)}{(\vartheta^\varrho-\theta^\varrho)}r^{\varrho-1}(\vartheta^\varrho-r^\varrho)^{\sigma-1}\right)\mathrm{d}r\right].$$

we calculate the primitives by integration by a change of variable

$$\int_\theta^\vartheta|\hbar(\tau,r)|\,\mathrm{d}r=\tfrac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\left[\tfrac{(\tau^\varrho-\theta^\varrho)}{(\vartheta^\varrho-\theta^\varrho)}\tfrac{\vartheta^{\varrho\sigma}}{\varrho\sigma}\left[\left(1-\tfrac{\theta^\varrho}{\vartheta^\varrho}\right)^\sigma-\left(1-\tfrac{\tau^\varrho}{\vartheta^\varrho}\right)^\sigma\right]\right.$$

$$
\begin{aligned}
& -\tfrac{\tau^{\varrho\sigma}}{\varrho}\left(\tfrac{1}{\sigma}\left(1-\tfrac{\theta^{\varrho}}{\tau^{\varrho}}\right)^{\sigma}\right)+\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}\tfrac{\vartheta^{\varrho\sigma}}{\varrho\sigma}\left(1-\tfrac{\tau^{\varrho}}{\vartheta^{\varrho}}\right)^{\sigma}\Big]\\
&=\tfrac{\varrho^{1-\sigma}}{\Gamma(\sigma)}\left[\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}\tfrac{\vartheta^{\varrho\sigma}}{\varrho\sigma}\left[\tfrac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\vartheta^{\varrho\sigma}}-\tfrac{(\vartheta^{\varrho}-\tau^{\varrho})^{\sigma}}{\vartheta^{\varrho\sigma}}\right]\right.\\
&\left.-\tfrac{\tau^{\varrho\sigma}}{\varrho}\left(\tfrac{1}{\sigma}\tfrac{(\tau^{\varrho}-\theta^{\varrho})^{\sigma}}{\tau^{\varrho\sigma}}\right)+\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}\tfrac{\vartheta^{\varrho\sigma}}{\varrho\sigma}\tfrac{(\vartheta^{\varrho}-\tau^{\varrho})^{\sigma}}{\vartheta^{\varrho\sigma}}\right]\\
&=\tfrac{\varrho^{1-\sigma}}{\varrho\sigma\Gamma(\sigma)}\left[\tfrac{(\tau^{\varrho}-\theta^{\varrho})}{(\vartheta^{\varrho}-\theta^{\varrho})}(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}-(\tau^{\varrho}-\theta^{\varrho})^{\sigma}\right].
\end{aligned}
$$

Then

$$\int_{\theta}^{\vartheta}|\hbar(\tau,r)|\,\mathrm{d}r=\tfrac{1}{\varrho^{\sigma}\sigma\Gamma(\sigma)}\left[(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}(\tau^{\varrho}-\theta^{\varrho})-(\tau^{\varrho}-\theta^{\varrho})^{\sigma}\right].$$

Implies that

$$\int_{\theta}^{\vartheta}\tfrac{\partial|\hbar(\tau,r)|}{\partial\tau}\,\mathrm{d}r=\tfrac{1}{\varrho^{\sigma-1}\sigma\Gamma(\sigma)}\left[(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}-\sigma(\tau^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}\right],$$

which ends the proof.

Corollary 1. *We can define the continuous functions* ξ *and* ξ'*, for* $\tau\in[\theta,\vartheta]$ *by*

$$\xi(\tau)=(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}(\tau^{\varrho}-\theta^{\varrho})-(\tau^{\varrho}-\theta^{\varrho})^{\sigma}, \tag{8}$$

$$\xi'(\tau)=(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}-\sigma(\tau^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}. \tag{9}$$

Proposition 2. *By* (7)*, suppose that* $\theta=0$*,* $\theta<\vartheta$ *and by replacement by the maximum point* τ^**, then*

$$\int_{0}^{\vartheta}|\hbar(\tau,r)|\,\mathrm{d}r\le\tfrac{1}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{1}{(\sigma-1)}}}-\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right], \tag{10}$$

Proof. According to 9 the derivative of the function ξ, we pose

$$\xi'(\tau)=0\Longrightarrow(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}-\sigma(\tau^{\varrho}-\theta^{\varrho})^{\sigma-1}\tau^{\varrho-1}=0,$$

we suppose that $\theta=0$. So we get

$$
\begin{aligned}
\vartheta^{\varrho(\sigma-1)}\tau^{\varrho-1}-\sigma\tau^{\varrho(\sigma-1)}\tau^{\varrho-1}&=0\\
\vartheta^{\varrho(\sigma-1)}&=\sigma\tau^{\varrho(\sigma-1)}\\
\tau^{\varrho(\sigma-1)}&=\frac{\vartheta^{\varrho(\sigma-1)}}{\sigma}.
\end{aligned}
$$

We directly deduce that the maximum was reached at the points

$$\tau^*=\frac{\vartheta}{\sigma^{\frac{1}{\varrho(\sigma-1)}}}.$$

Moreover,

$$\xi(\tau^*)=\left(\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{1}{(\sigma-1)}}}-\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right),$$

that finishes the proof.

Theorem 2. *Assume* $\Theta : [0, \vartheta] \times \mathbb{R} \to \mathbb{R}$ *is a function is continuous and check a condition of uniform Lipschitz concerning the second variable on* $[0, \vartheta] \times \mathbb{R}$ *with Lipschitz real* ζ*, thus,*

$$|\Theta(\tau, \mu) - \Theta(\tau, \nu)| \leq \zeta |\mu - \nu|,$$

for (τ, μ)*,* $(\tau, \nu) \in [0, \vartheta] \times \mathbb{R}$*, where* $\zeta > 0$ *are constants. If*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)} \left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{1}{(\sigma-1)}}} - \frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{\sigma}{(\sigma-1)}}} \right] < 1, \tag{11}$$

then the BVP

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{0+}\mu(\tau) = -\Theta(\tau, \mu(\tau)), \quad 0 < \tau < \vartheta, \\ \mu(0) = \lambda_1, \quad \mu(\vartheta) = \lambda_2, \end{cases} \tag{12}$$

admits a single solution.

Proof. Suppose Π is a space of Banach fitted with continuous applications defined on $[0, \vartheta]$ with the norm $\|\mu\| = \max_{\tau\in[0,\vartheta]}\{|\mu(\tau)|\}$. According to Lemma 2, we have $\mu \in C[0, \vartheta]$ is a solution of (12) equivalent that this is the same as the solving an equation in integral form

$$\mu(\tau) = \left[(\lambda_2 - \lambda_1) \left(\tfrac{\tau}{\vartheta}\right)^{\varrho} + \lambda_1 \right] + \int_0^{\vartheta} \hbar(\tau, r)\ \Theta(r, \mu(r))\, \mathrm{d}r.$$

Define the operator $\Sigma : \Pi \to \Pi$ by

$$\Sigma\mu(\tau) = \left[(\lambda_2 - \lambda_1) \left(\tfrac{\tau}{\vartheta}\right)^{\varrho} + \lambda_1 \right] + \int_0^{\vartheta} \hbar(\tau, r)\ \Theta(r, \mu(r))\, \mathrm{d}r,$$

for $\tau \in [0, \vartheta]$. We should interpret that the application Σ admits a single fixed point. Assume $\mu, \nu \in \Pi$. Therefore

$$\begin{aligned} |\Sigma\mu(\tau) - \Sigma\nu(\tau)| &\leq \int_0^{\vartheta} |\hbar(\tau, r)|\, |\Theta(r, \mu(r)) - \Theta(r, \nu(r))|\ \mathrm{d}r \\ &\leq \int_0^{\vartheta} |\hbar(\tau, r)|\, (\zeta|\mu(\tau) - \nu(\tau)|)\ \mathrm{d}r \\ &\leq \zeta\|\mu - \nu\| \int_0^{\vartheta} |\hbar(\tau, r)|\, \mathrm{d}r \\ &\leq \zeta \frac{1}{\varrho^{\sigma}\Gamma(\sigma+1)} \left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{1}{(\sigma-1)}}} - \frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{\sigma}{(\sigma-1)}}} \right] \|\mu - \nu\|, \end{aligned}$$

whither we have martyred Proposition 2. According to (11), we extrapolate that Σ is a contracting operator on Π, so, by the theorem of contraction mapping of Banach we culminate in the possible outcome. This means that, we conclude that Σ accepts a single fixed point in $C[0, \vartheta]$, this requires that the BVP (12) admits a single solution.

Remark 1. *We analyze this when taking $\sigma = 2$, $\theta = 0$ and $\varrho = 1$ in Theorem 2, through condition* (11), *we obviously find Theorem 1 such that*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{1}{(\sigma-1)}}}-\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right]=\zeta\frac{\theta^{2}}{4\Gamma(3)}<1.$$

Proposition 3. *By* (7), *suppose that $\theta = 0$, $\vartheta = 1$, then*

$$\int_{0}^{1}|\hbar(\tau,r)|\,\mathrm{d}r\leq\frac{1}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{1}{\sigma^{\frac{1}{(\sigma-1)}}}-\frac{1}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right],\tag{13}$$

Theorem 3. *Assume $\Theta : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a function is continuous and check a condition of uniform Lipschitz concerning the second variable on $[0, 1] \times \mathbb{R}$ with Lipschitz real ζ, thus,*

$$|\Theta(\tau,\mu)-\Theta(\tau,\nu)|\leq\zeta|\mu-\nu|,\qquad(\tau,\mu),\,(\tau,\nu)\in[0,1]\times\mathbb{R},$$

where $\zeta > 0$ are constants. If

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{1}{\sigma^{\frac{1}{(\sigma-1)}}}-\frac{1}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right]<1,\tag{14}$$

then the BVP

$$\begin{cases}{}_{c}^{\varrho}D_{0^{+}}^{\sigma}\mu(\tau)=-\Theta(\tau,\mu(\tau)), & 0<\tau<1,\\ \mu(0)=\lambda_{1}, \quad \mu(1)=\lambda_{2},\end{cases}\tag{15}$$

has a unique solution.

Proof. Using the same method to prove Proposition 3 and Theorem 3 which are used in Proposition 2 and also applies to Theorem 2.

Remark 2. *The same remark 1, we apply that when $\sigma = 2$, $\theta = 0$, $\vartheta = 1$ and $\varrho = 1$ on Theorem 3, through condition* (14), *we obviously find Theorem 1 such that*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{1}{\sigma^{\frac{1}{(\sigma-1)}}}-\frac{1}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right]=\zeta\frac{1}{4\Gamma(3)}<1.$$

Proposition 4. *By* (7), *suppose that $\theta < \vartheta$, then*

$$\int_{0}^{1}|\hbar(\tau,r)|\,\mathrm{d}r\leq\frac{1}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{1}{(\sigma-1)}}}-\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right],\tag{16}$$

Theorem 4. *Assume $\Theta : [\theta, \vartheta] \times \mathbb{R} \to \mathbb{R}$ is a function is continuous and check a condition of uniform Lipschitz concerning the second variable on $[\theta, \vartheta] \times \mathbb{R}$ with Lipschitz real ζ, thus,*

$$|\Theta(\tau,\mu)-\Theta(\tau,\nu)|\leq\zeta|\mu-\nu|,\qquad(\tau,\mu),\,(\tau,\nu)\in[\theta,\vartheta]\times\mathbb{R},$$

where $\zeta > 0$ are constants. If

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{1}{(\sigma-1)}}}-\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right]<1,\tag{17}$$

then the BVP

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{0^+}\mu(\tau) = -\Theta(\tau,\mu(\tau)), & \theta < \tau < \vartheta, \\ \mu(\theta) = \lambda_1, \quad \mu(\vartheta) = \lambda_2, \end{cases} \tag{18}$$

has a unique solution.

Proof. Using the same method to prove Proposition 4 and Theorem 4 which are used in Proposition 2 and also applies to Theorem 2.

Remark 3. *Same previous notes. We notice them in the general case. We apply them when $\sigma = 2$, $\theta < \vartheta$ and $\varrho = 1$ on Theorem 4, through condition (17), we obviously find Theorem 1 such that*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{1}{(\sigma-1)}}} - \frac{(\vartheta^{\varrho}-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right] = \zeta\frac{(\vartheta-\theta)^2}{4\Gamma(3)} < 1.$$

Proposition 5. *By (7), suppose that $\theta < \vartheta = 1$, then*

$$\int_0^1 |\hbar(\tau,r)|\,\mathrm{d}r \le \frac{1}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{1}{(\sigma-1)}}} - \frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right], \tag{19}$$

Theorem 5. *Assume $\Theta : [\theta,1]\times\mathbb{R} \to \mathbb{R}$ is a function is continuous and check a condition of uniform Lipschitz concerning the second variable on $[\theta,1]\times\mathbb{R}$ with Lipschitz real ζ, thus,*

$$|\Theta(\tau,\mu) - \Theta(\tau,\nu)| \le \zeta|\mu-\nu|, \qquad (\tau,\mu),\,(\tau,\nu) \in [\theta,1]\times\mathbb{R},$$

where $\zeta > 0$ are constants. If

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{1}{(\sigma-1)}}} - \frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right] < 1, \tag{20}$$

then the BVP

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{0^+}\mu(\tau) = -\Theta(\tau,\mu(\tau)), & \theta < \tau < 1, \\ \mu(\theta) = \lambda_1, \quad \mu(1) = \lambda_2, \end{cases} \tag{21}$$

has a unique solution.

Proof. Using the same method to prove Proposition 5 and Theorem 5 which are used in Proposition 2 and also applies to Theorem 2.

Remark 4. *The same Remark 3, we apply that when $\sigma = 2$, $\theta < \vartheta = 1$ and $\varrho = 1$ on Theorem 5, through condition (20), we obviously find Theorem 1 such that*

$$\frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{1}{(\sigma-1)}}} - \frac{(1-\theta^{\varrho})^{\sigma}}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right] = \zeta\frac{(1-\theta)^2}{4\Gamma(3)} < 1.$$

4. Examples

To prove the desired results above, we take some applications.

Example 1. *Extrapolate the following application of BVP*

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{0+}\mu(\tau) = 3 - \tau^7 - \sin(\mu(\tau)), \quad 0 < \tau < \vartheta, \\ \mu(0) = 1, \quad \mu(\vartheta) = 2. \end{cases} \tag{22}$$

Set, $\varrho = 1$, $\sigma \in \{\frac{4}{3}, \frac{3}{2}, \frac{9}{5}, \frac{39}{20}\} \subset (1, 2]$, $\theta = 0$. *and* $\Theta(\tau, \mu(\tau)) = \tau^7 - 3 + \sin(\mu(\tau))$. *Here,*

$$\begin{aligned} |\Theta(\tau,\mu) - \Theta(\tau,\nu)| &= \left|\tau^7 - 3 + \sin(\mu(\tau)) - \left(\tau^7 - 3 + \sin(\nu(\tau))\right)\right| \\ &\leq \zeta|\mu - \nu|, \qquad \forall\,(\tau,\mu),\,(\tau,\nu) \in [0,\vartheta] \times \mathbb{R}^2, \end{aligned}$$

where $\zeta = 1 > 0$. *Moreover, we have*

$$\varpi = \frac{\zeta}{\varrho^{\sigma}\Gamma(\sigma+1)}\left[\frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{1}{(\sigma-1)}}} - \frac{\vartheta^{\varrho\sigma}}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right] \approx \left\{\begin{array}{ll} 0.0886, & \sigma = \frac{4}{3}, \\ 0.1114, & \sigma = \frac{3}{2}, \\ 0.1272, & \sigma = \frac{9}{5}, \\ 0.1262, & \sigma = \frac{39}{20}, \end{array}\right\} < 1.$$

The curves drawn in Figure 1 show how the ϖ *changes for different derivative orders* σ. *The important point is that all of them are less than the line* $y = 1$ *in the interval* $[0, \vartheta]$, *and as the order of the derivative approaches the number one, the parameter* ϖ *decreases, but they are still less than one. These results are shown in Table 1. By the applications of Theorem 2, and the condition* (11) *is agreed. Then the BVP* (22) *accepts an unique solution.*

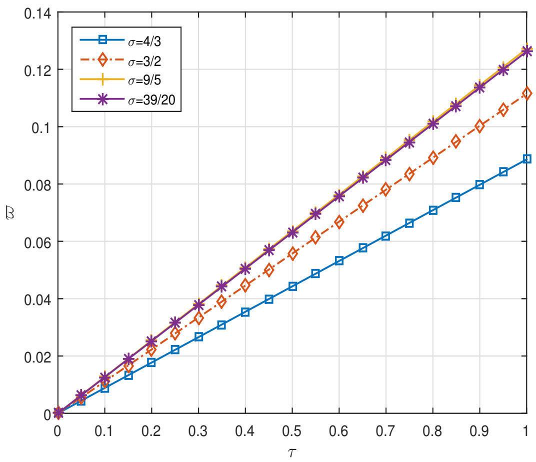

Figure 1: Representation of ϖ for BVP (22) in Example 1 for four case σ.

Table 1: Numerical results ϖ in Example 1 for four values of σ.

τ	ϖ			
	$\sigma=\frac{4}{3}$	$\sigma=\frac{3}{2}$	$\sigma=\frac{9}{5}$	$\sigma=\frac{39}{20}$
0.00	0.0000	0.0000	0.0000	0.0000
0.05	0.0044	0.0056	0.0064	0.0063
0.10	0.0089	0.0111	0.0127	0.0126
0.15	0.0133	0.0167	0.0191	0.0189
0.20	0.0177	0.0223	0.0254	0.0252
0.25	0.0221	0.0279	0.0318	0.0316
0.30	0.0266	0.0334	0.0381	0.0379
0.35	0.0310	0.0390	0.0445	0.0442
0.40	0.0354	0.0446	0.0509	0.0505
0.45	0.0399	0.0502	0.0572	0.0568
0.50	0.0443	0.0557	0.0636	0.0631
0.55	0.0487	0.0613	0.0699	0.0694
0.60	0.0531	0.0669	0.0763	0.0757
0.65	0.0576	0.0724	0.0826	0.0821
0.70	0.0620	0.0780	0.0890	0.0884
0.75	0.0664	0.0836	0.0954	0.0947
0.80	0.0709	0.0892	0.1017	0.1010
0.85	0.0753	0.0947	0.1081	0.1073
0.90	0.0797	0.1003	0.1144	0.1136
0.95	0.0842	0.1059	0.1208	0.1199
1.00	0.0886	0.1114	0.1272	0.1262

Example 2. *Extrapolate the following application of BVP*

$$\begin{cases} {}^{\varrho}_{c}D^{\sigma}_{0+}\mu(\tau) = 4-\tau^5+\cos(\mu(\tau)), \quad 0<\tau<\vartheta, \\ \mu(0)=3, \quad \mu(1)=4. \end{cases} \tag{23}$$

Set, $\varrho=1$, $\sigma\in\{\frac{4}{3},\frac{3}{2},\frac{9}{5},\frac{39}{20}\}\subset(1,2]$, $\theta=0$, $\vartheta=1$, *and* $\Theta(\tau,\mu(\tau))=\tau^5-4-\cos(\mu(\tau))$. *Here,*

$$\begin{aligned} |\Theta(\tau,\mu)-\Theta(\tau,\nu)| &= \left|\tau^5-4-\cos(\mu(\tau))-\left(\tau^5-4-\cos(\nu(\tau))\right)\right| \\ &= |\cos(\nu(\tau))-\cos(\mu(\tau))| = 2\left|\sin\tfrac{\nu+\mu}{2}\sin\tfrac{\nu-\mu}{2}\right| \\ &\le \zeta|\mu-\nu|, \qquad \forall\,(\tau,\mu),\,(\tau,\nu)\in[0,1]\times\mathbb{R}^2, \end{aligned}$$

where $\zeta=2>0$. *Moreover, we have*

$$\varpi = \frac{\zeta}{\varrho^\sigma\Gamma(\sigma+1)}\left[\frac{1}{\sigma^{\frac{1}{(\sigma-1)}}}-\frac{1}{\sigma^{\frac{\sigma}{(\sigma-1)}}}\right] \approx \left\{\begin{array}{ll} 0.1748, & \sigma=\frac{4}{3}, \\ 0.2196, & \sigma=\frac{3}{2}, \\ 0.2498, & \sigma=\frac{9}{5}, \\ 0.2476, & \sigma=\frac{39}{20}, \end{array}\right\} < 1.$$

In the last row of data in Table 2, the values of parameter ϖ*, at point* ϑ*, for three different values of derivative order* σ *are shown. The curves of all three cases are presented in Figure 2, which are decreasing as the order of the derivative increases and in all cases are less than the* $y=1$ *line. By the applications of Theorem 3, and the condition* (14) *is agreed. Then the BVP* (23) *accepts a single solution.*

Table 2: Numerical results ϖ in Example 2 for four values of σ.

τ	ϖ			
	$\sigma=\frac{4}{3}$	$\sigma=\frac{3}{2}$	$\sigma=\frac{9}{5}$	$\sigma=\frac{39}{20}$
0.05	7.5423	15.1657	40.2424	60.9280
0.10	3.3614	6.1094	13.5160	18.6851
0.15	2.0396	3.4827	6.8856	8.9987
0.20	1.4193	2.3161	4.2205	5.2952
0.25	1.0676	1.6812	2.8735	3.4915
0.30	0.8444	1.2914	2.0937	2.4777
0.35	0.6918	1.0319	1.5996	1.8511
0.40	0.5817	0.8490	1.2657	1.4364
0.45	0.4989	0.7144	1.0289	1.1477
0.50	0.4348	0.6120	0.8545	0.9385
0.55	0.3838	0.5319	0.7221	0.7821
0.60	0.3425	0.4678	0.6191	0.6619
0.65	0.3083	0.4157	0.5373	0.5677
0.70	0.2797	0.3726	0.4711	0.4923
0.75	0.2554	0.3364	0.4168	0.4311
0.80	0.2346	0.3057	0.3716	0.3808
0.85	0.2166	0.2795	0.3336	0.3388
0.90	0.2009	0.2568	0.3014	0.3034
0.95	0.1871	0.2370	0.2737	0.2734
1.00	0.1748	0.2196	0.2498	0.2476

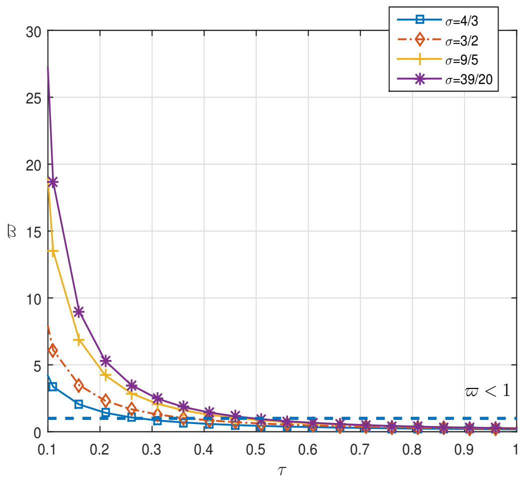

Figure 2: Representation of ϖ for BVP (23) in Example 2 for four case σ.

5. Numerical results

In this section, we present numerical results obtained through MATLAB programming for the verification of Theorem 2 and the illustration of Examples 1 and 2. These numerical experiments offer insights into the practical implications of the theoretical findings.

5.1. Numerical Solution of the BVP using the Fourth-Order Runge-Kutta Method

We apply the fourth-order Runge-Kutta method to solve a BVP described by Theorem 2. Algorithm 1 is used for numerical computation, followed by the results obtained and their analysis. Table 3 shows the numerical results obtained from the fourth-order Runge-Kutta method. The numerical results indicate the convergence of the fourth-order

Table 3: Numerical results of the fourth-order Runge-Kutta method to solve a BVP described by Theorem 2.

Iteration	τ	$\mu^{(k)}$
0	0.0	0.000000
1	0.1	0.100000
2	0.2	0.202484
3	0.3	0.310604
4	0.4	0.420014
5	0.5	0.525788
6	0.6	0.622258
7	0.7	0.704243
8	0.8	0.767255
9	0.9	0.806292
10	1.0	0.818999

Runge-Kutta method towards the solution of the given boundary value problem. As the number of iterations increases, the values of $\mu^{(k)}$ approach the exact solution. Additionally, the results demonstrate the accuracy and efficiency of the fourth-order Runge-Kutta method in solving ordinary DEs. The method achieves fourth-order accuracy by computing the weighted average of four slope estimates at each step, resulting in highly accurate numerical solutions.

5.2. MATLAB implementation and visualization for Examples 1, 2

In this part, we analyze Examples 1 and 2 with MATLAB, using tables and graphs to show outcomes. This clarifies theoretical concepts in the examples.

Case I: Example 1

MATLAB Program: The MATLAB program in Algorithm 2 solves the BVP defined by Eq. (22) for different values of σ, calculates ϖ, and saves the results.

Table 1 presents the numerical results of ϖ for different values of σ. To better illustrate the outcomes, Figure 3 showcases the variation of ϖ across different σ values.

Case II: Example 2

To solve the BVP numerically and obtain the solution for $\mu(\tau)$ and $\Theta(\tau, \mu(\tau))$, we implement the following MATLAB code in Algorithm 3.

MATLAB Code: The MATLAB code provides numerical solutions for $\mu(\tau)$ and $\Theta(\tau, \mu(\tau))$. The values obtained from MATLAB are tabulated in Table 4. To further

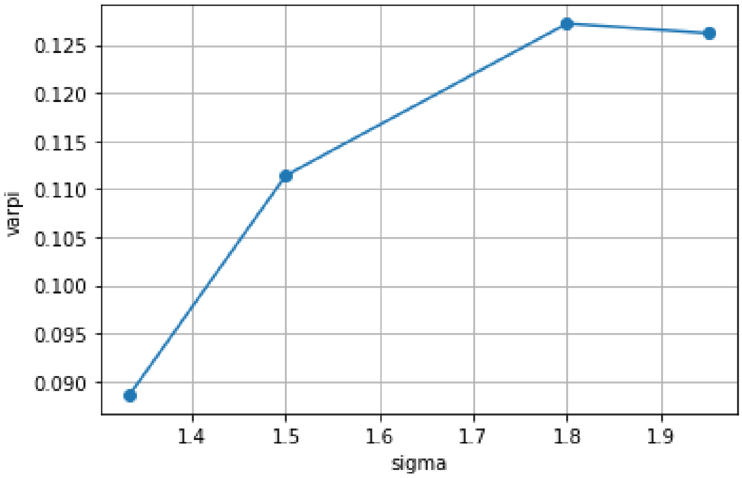

Figure 3: Variation ofϖ for different σ values.

visualize the results, Fig. 4 depicts the solution $\mu(\tau)$ and the function $\Theta(\tau, \mu(\tau))$. The decreasing values of $\Theta(\tau, \mu(\tau))$ in the graph align with the expected behavior described in Example 2, indicating agreement with the conditions of the given BVP.

Table 4: Values of τ, $\mu(\tau)$, and $\Theta(\tau, \mu(\tau))$.

τ	$\mu(\tau)$	$\Theta(\tau, \mu(\tau))$
0.0000	2.9967	−4.9999
0.0101	2.9942	−4.9999
0.0202	2.9917	−4.9999
0.0303	2.9892	−4.9999
...	...	...
0.9798	3.5970	−4.0970
0.9899	3.6495	−4.0495
1.0000	3.7000	−4.0000

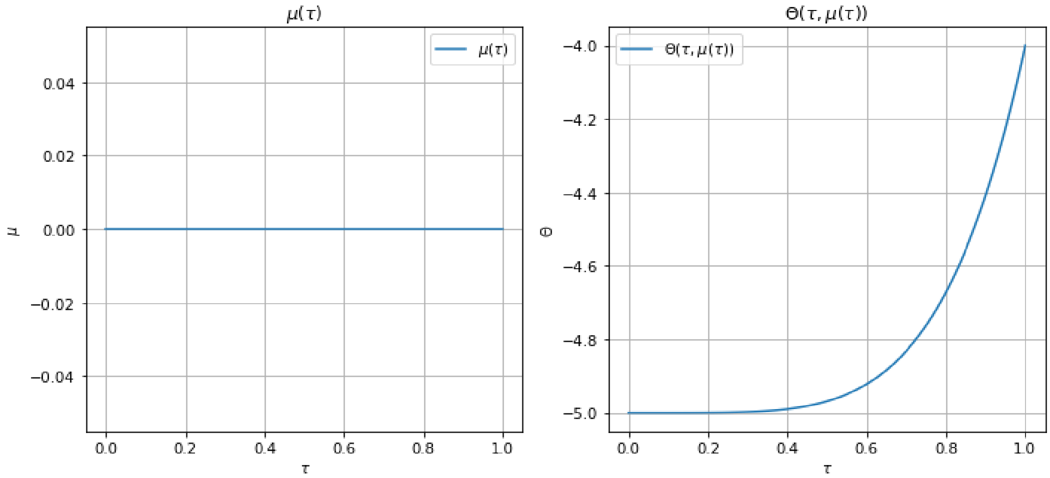

Figure 4: The graphs display the solution $\mu(\tau)$ and the function $\Theta(\tau, \mu(\tau))$.

6. Conclusion

Through this project, we tried to simulate the Banach contraction theorem on the generalized fractional derivative of the Caputo-type boundary value problem to achieve the existence of a single solution, the core of this work in the third chapter. We relied on the positivity of the Green function and its integral and obtained the function ξ As shown in Corollary 1. By conclusion, we can determine the maximum of two derived functions in the general case of two boundary conditions "$\mu(\theta) = \lambda_1$, $\mu(\vartheta) = \lambda_2$". We studied two cases when "$\mu(0) = \lambda_1$, $\mu(\vartheta) = \lambda_2$" and "$\mu(0) = \lambda_1$, $\mu(1) = \lambda_2$", we obtain detailed results in this section. We also studied two cases when, the previous general case "$\mu(\theta) = \lambda_1$, $\mu(\vartheta) = \lambda_2$" and the case of "$\mu(\theta) = \lambda_1$, $\mu(1) = \lambda_2$", we can also add two examples with their simulation in these two cases of the theorems 3.11 and 3.14 in the Examples part. We also believe there are numerical methods to achieve Banach's theorem of contraction of a single solution to the problem (1) with the general boundary conditions. In the future, we can apply the Banach contraction to the Caputo-Fabrizio fractional BVP.

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Authors' contributions

All authors have equal contributions. All authors read and approved the final manuscript.

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Appendix

Supporting Information

Algorithm 1: The MATLAB algorithm for implementing the fourth-order Runge-Kutta method

```
function [tau, mu] = fourth_order_runge_kutta(lambda1, lambda2, theta,
   sigma, rho, zeta, vartheta, N)
% Initialization
tau(1) = 0;
mu(1) = lambda1;
delta_tau = vartheta / N;
% Main loop
for i = 1:N
  k1 = delta_tau * (-theta(tau(i), mu(i)));
  k2 = delta_tau * (-theta(tau(i) + delta_tau/2, mu(i) + k1/2));
  k3 = delta_tau * (-theta(tau(i) + delta_tau/2, mu(i) + k2/2));
  k4 = delta_tau * (-theta(tau(i) + delta_tau, mu(i) + k3));
  mu(i+1) = mu(i) + (1/6) * (k1 + 2*k2 + 2*k3 + k4);
  tau(i+1) = tau(i) + delta_tau;
  end
end
```

Algorithm 2: The MATLAB algorithm for implementing the fourth-order Runge-Kutta method

```
% Define the parameters
vartheta = 1; % upper limit
rho = 1;
sigma_values = [4/3, 3/2, 9/5, 39/20]; % Values of sigma
zeta = 1; % constant
tau = linspace(0, vartheta, 1000); % Discretize the interval [0, vartheta]
% Function defining the differential equation
f = @(t, y) 3 - t.^7 - sin(y);
% Initial guess for the solution
init_guess = @(t) 1 + (t/vartheta)*(2-1); % Linear interpolation between
    1 and 2
% Initialize variable to store varpi values
varpi_values = zeros(1, length(sigma_values));
% Solve the BVP for each sigma value
for i = 1:length(sigma_values)
  sigma = sigma_values(i);
  % Define the BVP
  bvp_eqn = @(t, y) [y(2); (1/(rho^sigma)) * diff(y(1), tau, sigma) -
      f(t, y(1))];
  % Solve the BVP using bvp4c
  sol = bvp4c(bvp_eqn, init_guess, @bc, 'RelTol', 1e-6);
  % Evaluate the solution at tau
  mu = deval(sol, tau);
  % Calculate varpi
  varpi_values(i) = (zeta / (rho^sigma * gamma(sigma+1))) * ...
    (vartheta^(rho*sigma) / sigma^(1/(sigma-1)) - ...
    vartheta^(rho*sigma) / sigma^(sigma/(sigma-1)));
  end
% Display varpi values
disp('Numerical results varpi for different sigma:');
disp('sigma        varpi');
for i = 1:length(sigma_values)
  fprintf('%5.4f    %8.4f\n', sigma_values(i), varpi_values(i));
  end
% Function for boundary conditions
function res = bc(ya, yb)
  res = [ya(1) - 1; yb(1) - 2];
end
```

Algorithm 3: The MATLAB code to solve the BVP numerically and obtain the solution for $\mu(\tau)$ and $\Theta(\tau, \mu(\tau))$.

```
% Define parameters
theta = 1;
sigma = 4/3;
zeta = 0.1244291811338;
% Define boundary value problem equations
function res = bvp_equations(t, y)
  res = (4 - t.^5 + cos(y)).^(3/4);
end
% Define boundary conditions
function res = bvp_bc(ya, yb)
  res = [ya(1) - 3; yb(1) - 4];
end
% Define theta function
function res = theta_function(t, y)
  res = t.^5 - 4 - cos(y);
end
% Define range for tau
tau_values = linspace(0, theta, 100);
% Solve the boundary value problem
sol = bvp4c(@bvp_equations, @bvp_bc, [0 theta]);
% Extract the solution
mu_solution_values = sol.y(1, tau_values);
% Calculate Theta
Theta_values = theta_function(tau_values, mu_solution_values);
% Display table
fprintf('Tau␣␣␣␣|␣␣␣Mu(tau)␣␣␣|␣␣␣Theta(tau,␣mu(tau))\n');
fprintf('---------------------------------------------\n');
for i = 1:length(tau_values)
  fprintf('%.4f␣|␣%.4f␣|␣%.4f\n', tau_values(i), mu_solution_values(i),
      Theta_values(i));
end
```