EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 5984
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Independent Double Roman Domination Stability in Graphs

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Abstract. An independent double Roman dominating function (IDRD-function) on a graph G is a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that (i) if $f(v) = 0$, then the vertex v must have at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then the vertex v must have at least one neighbor w with $f(w) \geq 2$, and (ii) the subgraph induced by the vertices with positive weight under f is edgeless. The weight of an IDRD-function is the sum of its function values over all vertices, and the independent double Roman domination number (IDRD-number) $i_{dR}(G)$ is the minimum weight of an IDRD-function on G. The i_{dR}-stability (i^-_{dR}-stability, i^+_{dR}-stability) of G, denoted by $\mathrm{st}_{i_{dR}}(G)$ ($\mathrm{st}^-_{i_{dR}}(G)$, $\mathrm{st}^+_{i_{dR}}(G)$), is defined as the minimum size of a set of vertices whose removal changes (decreases, increases) the independent double Roman domination number. In this paper, we first determine the exact values on the i_{dR}-stability of some special classes of graphs, and then present some bounds on $\mathrm{st}_{i_{dR}}(G)$. In addition, for a tree T with maximum degree Δ, we show that $\mathrm{st}_{i_{dR}}(T) = 1$ and $\mathrm{st}^-_{i_{dR}}(T) \leq \Delta$, and characterize the trees that achieve the upper bound.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Roman domination, double Roman domination, independent Roman domination, independent double Roman domination, independent double Roman domination stability

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.5984

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1. Introduction

Roman domination, introduced by Cockayne et al. [1] in 2004 and inspired by earlier work of Stewart [2] and ReVelle et al. [3], has since seen various extensions. One notable variant is the $[k]$-Roman domination [4], which generalizes the original Roman domination number $\gamma_R(G)$, recovered when $k = 1$ [5–7]. Beeler et al. [8] further refined this concept through double Roman domination, establishing bounds and complexity results, later expanded by Abdollahzadeh Ahangar et al. [9]. To incorporate independence, Maimani et al. [10] introduced independent double Roman domination, providing bounds and relationships to related parameters, and inspiring studies on I$[k]$RD for $k = 1, 2$ [11–14].

The concept of domination stability in graphs was introduced by Bauer et al. [15] in 1983. This notion has since been extended to various domination parameters. In 2016, Rad et al. [16] advanced this line of research by proving that the γ-stability problem is NP-hard, even when restricted to bipartite graphs. Continuing this direction, Zhuang *et al.* [17] determined the exact values of the γ_{dR}-stability number for several special classes of graphs and established bounds on $\mathrm{st}_{\gamma_{dR}}(G)$. In particular, for a tree T with maximum degree Δ, they showed that $\mathrm{st}^-_{\gamma_{dR}}(T) \leq \Delta$, and characterized the trees that attain this bound.

Motivated by these developments, this paper explores the independent double Roman domination stability of graphs. We determine exact values for special graph classes, establish bounds on $\mathrm{st}_{i_{dR}}(G)$, and characterize extremal cases. For trees, we show that $\mathrm{st}_{i_{dR}}(T) = 1$ and $\mathrm{st}^-_{i_{dR}}(T) \leq \Delta$, fully characterizing trees attaining this bound.

2. Terminology and Notation

All graphs considered in this article are finite, undirected, and simple. Let $G = (V, E)$ be a graph of order $|V(G)| = n$. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. We denote the degree of a vertex v in a graph G by $\deg_G(v)$, or simply by $\deg(v)$ if the graph G is clear from the context. Let $\delta(G)$ and $\Delta(G)$ denote the *minimum* and *maximum degrees*, respectively, of vertices in G. We call a vertex of degree one a *leaf*, and its (unique) neighbor a support vertex. A support vertex is said to be *strong* if it has at least two leaf neighbors, otherwise, it is called *weak*.

A *complete graph* on n vertices is denoted by K_n, while a *complete bipartite graph* with partite sets of size p and q is denoted by $K_{p,q}$. We write P_n for the *path* of order n, C_n for the *cycle* of length n, and $\overline{K_n}$ for the graph with n vertices and no edges. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u–v path in G, while the *diameter*, $\mathrm{diam}(G)$, is the maximum distance among all pairs of vertices in G. A *tree* is an acyclic connected graph. A *star* is the graph $K_{1,m}$, where $m \geq 1$; the vertex of degree m is called the *center* of the star. A *double star* $S_{r,t}$ is formed from two disjoint stars $K_{1,r}$ and $K_{1,t}$ by adding an edge joining their center vertices. A *rooted tree* T distinguishes one vertex r, called the *root*. For each vertex $v \neq r$ in T, the *parent* of v is the neighbor of v on the unique r–v path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex $u \neq v$ such that the unique r–u path contains v. Thus, every child of v is a descendant of v. Let $D(v)$ denote the set of descendants of v, and let $D[v] = D(v) \cup \{v\}$. The *depth* of v, denoted $\mathrm{depth}(v)$, is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at u is the subtree of T induced by $D[u]$, and is denoted by T_u.

Let k be a positive integer, and let $f : V(G) \to \{0, 1, 2, \dots, k+1\}$ be a function that assigns labels from the set $\{0, 1, \dots, k+1\}$ to the vertices of a graph G. The *active neighborhood* $AN(v)$ of a vertex $v \in V(G)$ with respect to f is the set of all vertices $w \in N(v)$ such that $f(w) \geq 1$. Let $AN[v] = \{v\} \cup AN(v)$. A $[k]$-*Roman dominating function*, abbreviated as $[k]$RDF, is a function $f : V(G) \to \{0, 1, \dots, k+1\}$ satisfying the condition that for any vertex $v \in V(G)$ with $f(v) < k$, it holds that $\sum_{u \in N[v]} f(u) \geq |AN(v)| + k$. The weight of a $[k]$RDF is defined as $\omega(f) = \sum_{v \in V(G)} f(v)$, and the $[k]$-*Roman domination number* $\gamma_{[kR]}(G)$ of G is the minimum weight of a $[k]$RDF on G. A function f achieving this minimum is called a $\gamma_{[kR]}(G)$-function. For a $[k]$RDF f on G, let $V_i^f = \{v \in V(G) \mid f(v) = i\}$ for all $i \in \{0, 1, \dots, k+1\}$. Consequently, any $[k]$RDF f can be represented by the tuple $(V_0^f, V_1^f, \dots, V_{k+1}^f)$, where the superscript f may be omitted from V_i^f when no confusion arises.

A *double Roman dominating function* (DRDF) on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must have at least two neighbors assigned 2 under f or one neighbor u with $f(u) = 3$, and if $f(v) = 1$, then the vertex v must have at least one neighbor u with $f(u) \geq 2$. The *weight* of a DRDF is the sum of its function values over all vertices, and the *double Roman domination number* $\gamma_{dR}(G)$ is the minimum weight of a DRDF on G. The *double Roman domination stability*, or just γ_{dR}-*stability*, of a graph G is the minimum size of a set of vertices whose removal changes the double Roman domination number. We denote the γ_{dR}-stability of G by $st_{\gamma_{dR}}(G)$. The *decreasing* γ_{dR}-*stability* of G, denoted by $st^-_{\gamma_{dR}}(G)$, is defined as the minimum size of a set of vertices whose removal *decreases* the double Roman domination number. For the null graph N_0, which is the unique graph having no vertices and hence has order zero, we let $st^-_{\gamma_{dR}}(N_0) = 0$. With this consideration, the decreasing γ_{dR}-stability of a non-null graph is always defined. For example, $st^-_{\gamma_{dR}}(K_1) = 1$. The *increasing* γ_{dR}-*stability* of G, denoted by $st^+_{\gamma_{dR}}(G)$, is defined as the minimum size of a set of vertices whose removal *increases* the double Roman domination number, if such a set exists. Clearly, $st_{\gamma_{dR}}(G) = \min\left\{st^-_{\gamma_{dR}}(G),\, st^+_{\gamma_{dR}}(G)\right\}$.

An *independent* $[k]$-*Roman dominating function*, abbreviated as I$[k]$RDF, is a $[k]$-Roman dominating function f such that the subgraph induced by the vertices with positive weight under f is edgeless. The minimum weight of an I$[k]$RDF on a graph G is called the *independent* $[k]$-*Roman domination number*, denoted by $i_{[k]R}(G)$, which we refer to as the

I$[k]$*RD-number*. By definition, we have

$$\gamma_{[k]R}(G) \leq i_{[k]R}(G). \tag{1}$$

An *independent double Roman dominating function* (IDRD-function) on a graph G is a function $f : V(G) \to \{0, 1, 2, 3\}$ having the properties that (i) if $f(v) = 0$, then the vertex v must have at least two neighbors assigned 2 under f, or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then the vertex v must have at least one neighbor w with $f(w) \geq 2$; and (ii) the subgraph induced by the vertices with positive weight under f is edgeless. The *weight* of an IDRD-function is the sum of its function values over all vertices, and the *independent double Roman domination stability*, or simply the i_{dR}-stability, of a graph G is the minimum size of a set of vertices whose removal changes the *independent double Roman domination number*. We denote the i_{dR}-stability of G by $\mathrm{st}_{i_{dR}}(G)$. The i^-_{dR}-stability of G, denoted by $\mathrm{st}^-_{i_{dR}}(G)$, is defined as the minimum size of a set of vertices whose removal *decreases the independent double Roman domination number*, and the i^+_{dR}-stability of G, denoted by $\mathrm{st}^+_{i_{dR}}(G)$, is defined as the minimum size of a set of vertices whose removal *increases the independent double Roman domination number*, if such a set exists. If there is no set of vertices in G whose removal increases the independent double Roman domination number, then we set $\mathrm{st}^+_{i_{dR}}(G) = \infty$. Clearly, $\mathrm{st}_{i_{dR}}(G) = \min\{\mathrm{st}^-_{i_{dR}}(G), \mathrm{st}^+_{i_{dR}}(G)\}$.

3. Preliminary Results

In this section we will investigate simple results.

Remark 1. *Let G be a nontrivial connected graph with $\gamma_{dR}(G) = i_{dR}(G)$. Then $\mathrm{st}^-_{\gamma_{dR}}(G) \leq \mathrm{st}^-_{i_{dR}}(G)$. Moreover, If $\mathrm{st}^+_{\gamma_{dR}}(G) < \infty$, then $\mathrm{st}^+_{i_{dR}}(G) \leq \mathrm{st}^+_{\gamma_{dR}}(G)$.*

Maimani *et al.* [10] observed that for any graph G and any $i_{dR}(G)$-function $f = (V_0, V_1, V_2, V_3)$ we have $V_1 = \emptyset$.

Proposition 1. *Let G be a graph and v be a vertex of G. If G' is obtained from G by adding a star $K_{1,t}$ with $t \geq 2$ and joining v to a leaf of $K_{1,t}$, then $i_{dR}(G') = i_{dR}(G) + 3$.*

Proof. Let $V(K_{1,t}) = \{u, u_1, u_2, \ldots, u_t\}$, where u is the center of the star and $u_1, \ldots, u_t$ are its leaves. Suppose $v \in V(G)$ is connected to the leaf u_1 in G', i.e., $vu_1 \in E(G')$.

First, we show that $i_{dR}(G') \leq i_{dR}(G)+3$. Let f be an IDRD-function on G of minimum weight, i.e., an $i_{dR}(G)$-function. Define an extension f' on G' by:

$$f'(x) = \begin{cases} f(x), & \text{if } x \in V(G), \\ 3, & \text{if } x = u, \\ 0, & \text{if } x \in \{u_1, u_2, \ldots, u_t\}. \end{cases}$$

Clearly, f' is a valid IDRD-function on G', and its weight is $i_{dR}(G) + 3$. Thus, $i_{dR}(G') \leq i_{dR}(G) + 3$.

Now we show that $i_{dR}(G') \geq i_{dR}(G)+3$. Let $f = (V_0, \varnothing, V_2, V_3)$ be an $i_{dR}(G')$-function. If $f(u_1) = 0$, then to double Roman dominate the vertices u and $u_2, \dots u_t$ we must have $f(u) + f(u_2) + \cdots + f(u_t) = 3$. On the other hand, the function f restricted to G is an IDRD-function of G implying that $i_{dR}(G') \geq i_{dR}(G) + 3$. Assume that $f(u_1) \geq 2$. Then we must have $f(u) = 0$ and $f(u_i) = 2$ for $2 \leq i \leq t$. If $f(u_1) = 2$, then to double Roman dominate of v, it must have a neighbor w with $f(w) \geq 2$ and the function g defined on G by $g(w) = \min\{3, f(w) + 1\}$ and $g(x) = f(x)$ for other vertices, is an IDRD-function of G of weight at most $i_{dR}(G')-3$ leading to $i_{dR}(G') \geq i_{dR}(G)+3$. Assume that $f(u_1) = 3$. If v has a neighbor assigned at least two under f, then as before we get $i_{dR}(G') \geq i_{dR}(G)+3$. Hence let $f(N[v] - \{u_1\}) = 0$. Then the function g defined on G by $g(v) = 2$ and $g(x) = f(x)$ for other vertices, is an IDRD-function of G of weight at most $i_{dR}(G') - 3$, leading to $i_{dR}(G') \geq i_{dR}(G) + 3$. Thus, in all cases, we have that $i_{dR}(G') \geq i_{dR}(G) + 3$, and since the reverse inequality was already established, we have $i_{dR}(G') = i_{dR}(G) + 3$.

4. Exact values and bounds

In this section, we obtain the independent double Roman domination stability for some classes of graphs and present various bounds for this parameters. The proof of the next Propositions can be found in [10].

Proposition 2. For $n \geq 1$, $i_{dR}(P_n) = \gamma_{dR}(P_n) = \begin{cases} n & \text{if} \quad n \equiv 0 \pmod 3 \\ n+1 & \text{if} \quad n \equiv 1, 2 \pmod 3. \end{cases}$

Proposition 3. *For* $n \geq 3$, $i_{dR}(C_n) = \gamma_{dR}(C_n) = \begin{cases} n & \text{if} \quad n \equiv 0, 2, 3, 4 \pmod 6 \\ n+1 & \text{if} \quad n \equiv 1, 5 \pmod 6. \end{cases}$

Also Zhuang [17] determined the double Roman domination stability of paths and cycles as follows.

Proposition 4. *For* $n \geq 2$, $\mathrm{st}^{-}_{\gamma_{dR}}(P_n) = \begin{cases} 1 & \text{if} \quad n \equiv 1, 2 \pmod 3 \\ 2 & \text{if} \quad n \equiv 0 \pmod 3. \end{cases}$

Proposition 5. *For* $n \geq 2$, $\mathrm{st}^{+}_{\gamma_{dR}}(P_n) = \begin{cases} \infty & \text{if} \quad n \equiv 1, 2 \pmod 3 \\ 1 & \text{if} \quad n \equiv 0 \pmod 3. \end{cases}$

Proposition 6. *For* $n \geq 3$, $\mathrm{st}^{-}_{\gamma_{dR}}(C_n) = \begin{cases} 1 & \text{if } n \equiv 1, 4, 5 \pmod 6 \\ 2 & \text{otherwise.} \end{cases}$

Proposition 7. *For* $n \geq 3$, $\mathrm{st}^{+}_{\gamma_{dR}}(C_n) = \infty$.

We first determine the i_{dR}-stability for paths.

Proposition 8. *For* $n \geq 2$, $\mathrm{st}^{-}_{i_{dR}}(P_n) = \begin{cases} 1 & \text{if} \quad n \equiv 1, 2 \pmod 3 \\ 2 & \text{if} \quad n \equiv 0 \pmod 3. \end{cases}$

Proof. The result is trivial for $n \leq 3$, so we assume that $n \geq 4$. Let $P = [v_1, v_2, \ldots, v_n]$ be a path on n vertices. If $n \equiv 1, 2 \pmod 3$, then using Proposition 2 we have $i_{dR}(P_n - v_n) = i_{dR}(P_{n-1}) \leq n < i_{dR}(P_n)$. Hence, $\mathrm{st}^-_{i_{dR}}(P_n) = 1$. Now assume that $n \equiv 0 \pmod 3$. First we show that $\mathrm{st}^-_{i_{dR}}(P_n) \geq 2$. By Proposition 4, we have $\mathrm{st}^-_{\gamma_{dR}}(P_n) = 2$. It follows from Remark 1 that $\mathrm{st}^-_{i_{dR}}(P_n) \geq 2$. On the other hand, by Proposition 2 we have $i_{dR}(P_n - \{v_n, v_{n-1}\}) = i_{dR}(P_{n-2}) = n - 1 < i_{dR}(P_n)$ that yields $\mathrm{st}^-_{i_{dR}}(P_n) \leq 2$. Thus $\mathrm{st}^-_{i_{dR}}(P_n) = 2$ and the proof is complete. □

Proposition 9. *For $n \geq 2$,* $\mathrm{st}^+_{i_{dR}}(P_n) = \begin{cases} \infty & \text{if} \quad n \equiv 1, 2 \pmod 3 \\ 1 & \text{if} \quad n \equiv 0 \pmod 3. \end{cases}$

Proof. The result is trivial for $n = 2$, so we consider the case for $n \geq 3$. Let $P = [v_1, v_2, \ldots, v_n]$ be a path on n vertices. If $n \equiv 0 \pmod 3$, then by Proposition 2, we have $i_{dR}(P_n - v_2) = i_{dR}(P_1) + i_{dR}(P_{n-2}) = 2 + (n - 2 + 1) = n + 1 > i_{dR}(P_n)$, and so $\mathrm{st}^+_{i_{dR}}(P_n) = 1$.

Assume that $n \equiv r \pmod 3$ where $r \in \{1, 2\}$. By contradiction, we may assume that there exists a $n' \equiv r \pmod 3$ such that $\mathrm{st}^+_{i_{dR}}(P_{n'})$ is an integer m. Let S be a set of vertices such that $i_{dR}(P_{n'}) < i_{dR}(P_{n'} - S)$. Let $P_{n_1}, P_{n_2}, \ldots, P_{n_k}$ be the components of $P_{n'} - S$. Then by Proposition 2, we have

$$\gamma_{dR}(P_{n'}) = i_{dR}(P_{n'}) < i_{dR}(P_{n'} - S) = \sum_{i=1}^{k} i_{dR}(P_{n_i}) = \sum_{i=1}^{k} \gamma_{dR}(P_{n_i}) = \gamma_{dR}(P_{n'} - S),$$

a contradiction with Proposition 5. Thus $\mathrm{st}^+_{i_{dR}}(P_n) = \infty$ when $n \equiv r \pmod 3$ where $r \in \{1, 2\}$. □

The following result is an immediate consequence of Propositions 8 and 9.

Corollary 1. *For $n \geq 2$,* $\mathrm{st}_{i_{dR}}(P_n) = 1$.

Next we determine the independent double Roman domination stability of cycles.

Theorem 1. *For $n \geq 3$,* $\mathrm{st}^-_{i_{dR}}(C_n) = \begin{cases} 1 & \text{if } n \equiv 1, 4, 5 \pmod 6 \\ 2 & \text{otherwise.} \end{cases}$

Proof. Since $i_{dR}(C_n) = \gamma_{dR}(C_n)$ for $n \geq 3$ and $i_{dR}(P_n) = \gamma_{dR}(P_n)$ for $n \geq 1$, we deduce from Proposition 6 that $st^-_{i_{dR}}(C_n) = 1$ when $n \equiv 1, 4, 5 \pmod 6$. Assume that $n \equiv r \pmod 6$ where $r \in \{0, 2, 3\}$. By Proposition 3, we have $i_{dR}(C_n) = n$. It follows from Proposition 6 and Remark 1 that $\mathrm{st}^-_{i_{dR}}(C_n) \geq 2$. On the other hand, we note that $C_n - \{v_1, v_5\}$ is a disjoint union of two paths P_3 and P_{n-5} and Proposition 2 leads to $i_{dR}(C_n - \{v_1, v_5\}) = 3 + i_{dR}(P_{n-5}) \leq 3 + (n - 5) + 1 < n = i_{dR}(C_n)$. Thus $\mathrm{st}^-_{i_{dR}}(C_n) = 2$ and the proof is complete. □

Theorem 2. *For $n \geq 3$,* $\mathrm{st}^+_{i_{dR}}(C_n) = \infty$.

Proof. By contradiction, we may assume that there exists an integer $n' \geq 3$ such that $\mathrm{st}^{+}_{i_{dR}}(C_{n'})$ is an integer m. Let S be a set of vertices such that $i_{dR}(C_{n'}) < i_{dR}(C_{n'} - S)$. If $P_{n_1}, P_{n_2}, \ldots, P_{n_k}$ are the components of $C_{n'} - S$, then by Proposition 3 we have

$$\gamma_{dR}(C_{n'}) = i_{dR}(C_{n'}) < i_{dR}(C_{n'} - S) = \sum_{i=1}^{k} i_{dR}(P_{n_i}) = \sum_{i=1}^{k} \gamma_{dR}(P_{n_i}) = \gamma_{dR}(C_{n'} - S),$$

a contradiction with Proposition 7. Thus $\mathrm{st}^{+}_{i_{dR}}(C_n) = \infty$. $\square$

Corollary 2. *For $n \geq 3$, $\mathrm{st}_{i_{dR}}(C_n) = \mathrm{st}^{-}_{i_{dR}}(C_n)$.*

One can observe that for $n \geq 2$, $i_{dR}(K_n) = i_{dR}(K_{1,n-1}) = 3$ and for $1 \leq r \leq t$,

$$i_{dR}(S_{r,t}) = \begin{cases} 5 & \text{if } r = 1 \\ 3 + 2r & \text{if } r \geq 2. \end{cases}$$

From the above Observations, we can easily obtain the following conclusions.

Corollary 3. *For $n \geq 2$, $\mathrm{st}_{i_{dR}}(K_n) = \mathrm{st}^{-}_{i_{dR}}(K_n) = n - 1$ and $\mathrm{st}^{+}_{i_{dR}}(K_n) = \infty$.*

Corollary 4. *For $n \geq 3$, $\mathrm{st}_{i_{dR}}(K_{1,n-1}) = \mathrm{st}^{+}_{i_{dR}}(K_{1,n-1}) = 1$ and $\mathrm{st}^{-}_{i_{dR}}(K_{1,n-1}) = n - 1$.*

Corollary 5. *For $1 \leq r \leq t$ with $t \geq 2$, $\mathrm{st}_{i_{dR}}(S_{r,t}) = \mathrm{st}^{-}_{i_{dR}}(S_{r,t}) = 1$ and $\mathrm{st}^{+}_{i_{dR}}(S_{r,t}) = 1$ when $r < t$ and $\mathrm{st}^{+}_{i_{dR}}(S_{r,t}) = 2$ when $r = t$.*

In what follows we give some bounds for the IDRD-stability of a graph. Since for any graph G of order at least 2, $i_{dR}(G) \geq 3$ with equality if and only if $\Delta(G) = n - 1$, the proof of this is trivial.

Remark 2. *If G is a graph of order $n \geq 2$, then $\mathrm{st}^{-}_{i_{dR}}(G) \leq n - 1$ with equality if and only if $\Delta(G) = n - 1$.*

Proposition 10. *If $G \neq K_n$ is a connected graph of order $n \geq 3$ with $i_{dR}(G) \geq 4$, then $\mathrm{st}^{-}_{i_{dR}}(G) \leq n - \omega(G) - 1$ where $\omega(G)$ is the clique number of G.*

Proof. Let S be a maximum clique in G. Since G is connected and $G \neq K_n$, there exists a vertex $x \in V(G) \setminus S$ such that x is adjacent to a vertex in S, say y. Now the function f defined on $G[S \cup \{x\}]$ by $f(x) = 3$ and $f(z) = 0$ for the remaining vertices, is an IDRD-function of $G[S \cup \{x\}]$. This implies that $\mathrm{st}^{-}_{i_{dR}}(G) \leq n - (|S| + 1) = n - \omega(G) - 1$, as desired. $\square$

Proposition 11. *Let G be a graph of order $n \geq 2$. Then*

$$\mathrm{st}_{i_{dR}}(G) \leq \delta(G) + 1.$$

In particular, this bound is sharp for graphs with isolated vertices.

Proof. Let x be a vertex of G with minimum degree $\delta(G)$. Assume that $G' = G - N(x)$ and $G'' = G - N[x]$ and f is an $i_{dR}(G)$-function. If $\deg(x) = 0$, then $f(x) = 2$ and we have $i_{dR}(G'') = i_{dR}(G) - 2$. Thus, $st_{i_{dR}}(G) \leq \delta(G) + 1$. So we assume that $\deg(x) \geq 1$. If $i_{dR}(G') \neq i_{dR}(G)$, then $st_{i_{dR}}(G) \leq \delta(G) < \delta(G) + 1$. Let $i_{dR}(G') = i_{dR}(G)$ and g is an $i_{dR}(G')$-function. Since x is an isolated vertex in G', $g(x) = 2$, and we have $i_{dR}(G') = i_{dR}(G'') + 2$, that is, $i_{dR}(G'') = i_{dR}(G) - 2$, and so $st_{i_{dR}}(G) \leq \delta(G) + 1$. □

Proposition 12. *Let G be a connected graph of order n with $i_{dR}(G) \geq 4$, then*

$$\mathrm{st}_{i_{dR}}(G) \leq n - \Delta(G) - 1.$$

Proof. If $\Delta(G) = 2$, then G is the path P_n or a cycle C_n. By Corollary 1 or Corollary 2, we are done. Let $\Delta(G) \geq 3$ and x be a vertex with maximum degree $\Delta(G)$. Since $i_{dR}(G) \geq 4$, we have $X = V(G) - N[x] \neq \varnothing$. Clearly, the function g defined on $G - X$ by $g(x) = 3$ and $g(y) = 0$ for the remaining vertices, is an IDRD-function of $G - X$, and so $\mathrm{st}_{i_{dR}}(G) \leq n - \Delta(G) - 1$. □

Combining Propositions 11 and 12, the following result follows.

Corollary 6. *Let G be a graph with $i_{dR}(G) \geq 4$, then*

$$\mathrm{st}_{i_{dR}}(G) \leq \min\{\delta(G) + 1, n - \Delta(G) - 1\}.$$

5. Graphs G with large i_{dR}-stability

In this section we characterize graphs G with $\mathrm{st}_{i_{dR}}(G) \in \{n-1, n-2, n-3, n-4\}$.

Proposition 13. *Let G be a connected graph of order $n \geq 2$. Then $st_{i_{dR}}(G) = n - 1$ if and only if $G = K_n$.*

Proof. If $G = K_n$, then clearly $st_{i_{dR}}(G) = n - 1$. Now, we prove the necessity. Let G be a connected graph of order $n \geq 2$ with $st_{i_{dR}}(G) = n - 1$. By Proposition 11, we have $n - 1 = st_{i_{dR}}(G) \leq \delta(G) + 1$, that is $\delta(G) \geq n - 2$. If $\delta(G) = n - 1$, then G is the complete graph K_n, as desired. Assume that $\delta(G) = n - 2$. If $i_{dR}(G) \geq 4$, then by Proposition 12, we obtain $\Delta(G) \leq n - 1 - st_{i_{dR}}(G) = 0$ contradicting the connectivity of G. Thus, $i_{dR}(G) = 3$. Since $\delta(G) = n - 2$, G has two non-adjacent vertices u and v and it follows from $i_{dR}(G[\{u, v\}]) = 4$ that $n - 1 = st_{i_{dR}}(G) \leq n - 2$, a contradiction. This completes the proof. □

Proposition 14. *Let $G \neq K_n$ be a connected graph of order $n \geq 3$. Then $\mathrm{st}_{i_{dR}}(G) = n - 2$ if and only if $G = K_n - e$.*

Proof. If $G = K_n - e$, then clearly $\mathrm{st}_{i_{dR}}(G) = n - 2$. Now, we prove the necessity. Let $G \neq K_n$ be a connected graph of order $n \geq 3$ with $\mathrm{st}_{i_{dR}}(G) = n - 2$. If $i_{dR}(G) \geq 4$, then by Proposition 12, we have $\Delta(G) \leq 1$ which contradicts the connectivity of G. So $i_{dR}(G) = 3$. If G has two pair of non-adjacent vertices u, v and x, y, then $i_{dR}(G[x, y, u, v]) \geq 5$ if $|\{x, y, u, v\}| = 3$ and $i_{dR}(G[x, y, u, v]) \geq 4$ if $|\{x, y, u, v\}| = 4$. This leads to the contradiction $n - 2 = \mathrm{st}_{i_{dR}}(G) \leq n - 3$. Therefore, by Proposition 13, we are done. □

Proposition 15. *Let G be a connected graph of order $n \geq 4$ and $G \notin \{K_n, K_n - e\}$. Then* $\mathrm{st}_{i_{dR}}(G) = n-3$ *if and only if* $G \in \{P_4, C_4, K_{1,3}, K_{1,3}+e, 3K_1 \vee K_{n-3}, K_{n-3} \vee (K_2 \cup K_1)\}$.

Proof. The sufficiency is straightforward to check. To prove the necessity, let G be a connected graph of order $n \geq 4$ such that $G \notin \{K_n, K_n - e\}$ and $\mathrm{st}_{i_{dR}}(G) = n-3$. Obviously, $\Delta(G) \geq 2$. If $\mathrm{st}_{i_{dR}}(G) = 1$, then $n = 4$. It is easy to verify that $G \in \{P_4, C_4, K_{1,3}, K_{1,3}+e\}$ as desired. Hence, we assume that $\mathrm{st}_{i_{dR}}(G) \geq 2$. If $i_{dR}(G) \geq 4$, then Proposition 12 and our earlier assumption leads to $\Delta(G) = 2$. Combining this with the condition that G is connected, we have that G is a path or a cycle. Since $st_{i_{dR}}(P_n) = 1$, it follows from $st_{i_{dR}}(G) \geq 2$ that G is a cycle. Combining Corollary 2 and the condition $st_{i_{dR}}(G) = n-3$, we obtain that $n = 5$ which is a contradiction. Hence, we assume that $i_{dR}(G) = 3$. It follows from this and the fact $st_{i_{dR}}(G) = n-3$ that G has exactly $n-3$ universal vertices. Let x, y, z be the vertices of G that are not universal. It follows that $G[\{x,y,z\}] = 3K_1$ or $G[\{x,y,z\}] = K_2 \cup K_1$. Thus, $G = 3K_1 \vee K_{n-3}$ or $G = K_{n-3} \vee (K_2 \cup K_1)$. This completes the proof. □

Proposition 16. *Let G be a connected graph of order $n \geq 6$. Then* $\mathrm{st}_{i_{dR}}(G) = n-4$ *if and only if* $G \in \{P_6, C_6, K_{n-4} \vee H\}$ *where* $H \in \{P_4, C_4, 2K_2, 4K_1, K_2 \cup 2K_1, P_3 \cup K_1, K_3 \cup K_1\}$.

Proof. The sufficiency is straightforward to check. To prove the necessity, let G be a connected graph of order $n \geq 6$ with $\mathrm{st}_{i_{dR}}(G) = n-4$. Clearly $\Delta(G) \geq 2$ and $\mathrm{st}_{i_{dR}}(G) \geq 2$. First let $i_{dR}(G) \geq 4$. It follows from Proposition 11 that $n-4 = \mathrm{st}_{i_{dR}}(G) \leq \delta + 1$ and so $\delta \geq n-5$. Combining this with Proposition 12, we obtain

$$n - 5 \leq \delta \leq \Delta \leq 3. \tag{2}$$

If $\Delta(G) = 2$, then $n \in \{6, 7\}$ and G is a path or a cycle of order n. It follows from Corollaries 1 and 2 that $G \in \{P_6, C_6\}$. Henceforth we assume that $\Delta(G) = 3$. By (2) we obtain $n \in \{6, 7, 8\}$. Let $v \in V(G)$ be a vertex with maximum degree 3 with $N(v) = \{v_1, v_2, v_3\}$ and let $f = (V_0, \varnothing, V_2, V_3)$ be an $i_{dR}(G)$-function. Since G is a connected graph, we assume, without loss of generality that $u \in N(v_1) - N[v]$. If $i_{dR}(G) \geq 6$, then the function g defined on $G[N[v] \cup \{u\}]$ with $f(v) = 3$, $f(u) = 2$, $f(v_1) = f(v_2) = f(v_3) = 0$ is an IDRDF of weight less that $\omega(f)$ and so $n - 4 = \mathrm{st}_{i_{dR}}(G) \leq n - \Delta(G) - 2$ which leads to the contradiction $\Delta(G) \leq 2$. Thus, we have $i_{dR}(G) \in \{4, 5\}$. Then either $|V_2| = 2$ or $|V_2| = |V_3| = 1$. Since each vertex in V_0 must be adjacent to a vertex with wight 3 or two vertices with weight 2, we have certainly $\Delta(G) \geq 4$ which is a contradiction.

Assume now that $i_{dR}(G) = 3$. It follows from this and the fact $st_{i_{dR}}(G) = n-4$ that G has exactly $n-4$ universal vertices. Let x, y, z, w be the vertices of G that are not universal vertex, that is $\Delta(G[\{x,y,z,w\}]) \leq 2$. There are seven graphs of order 4 with maximum degree at most two, that is $G[\{x,y,z,w\}] \in \{P_4, C_4, 2K_2, 4K_1, K_2 \cup 2K_1, P_3 \cup K_1, K_3 \cup K_1\}$. Thus $G = K_{n-4} \vee H$ where $H \in \{P_4, C_4, 2K_2, 4K_1, K_2 \cup 2K_1, P_3 \cup K_1, K_3 \cup K_1\}$ and the proof is complete. □

At the end of this section, we present a Nordhaus-Gaddum type inequality for the sum of the independent double Roman domination stability of a graph G and its complement $\overline{G}$.

Theorem 3. *Let G be a graph of order $n \geq 2$. Then $st_{i_{dR}}(G) + st_{i_{dR}}(\overline{G}) \leq n$.*

Proof. Since $n \geq 2$, we have $\min\{i_{dR}(G), i_{dR}(\overline{G})\} \geq 3$. If $i_{dR}(G) = 3$ (the case $i_{dR}(\overline{G}) = 3$ is similar), then G has a universal vertex, and so $\overline{G}$ has an isolated vertex. Using Remark 2 and noting that $st_{i_{dR}}(\overline{G}) = 1$, we obtain $st_{i_{dR}}(G) + st_{i_{dR}}(\overline{G}) \leq n$. Now suppose that $\min\{i_{dR}(G), i_{dR}(\overline{G})\} \geq 4$. Since $\Delta(G) + \Delta(\overline{G}) \geq n - 1$, we may assume, without loss of generality that $\Delta(G) \geq (n-1)/2$. Applying Propositions 11 and 12, we obtain

$$\begin{aligned} st_{i_{dR}}(G) + st_{i_{dR}}(\overline{G}) &\leq (n - \Delta(G) - 1) + (\delta(\overline{G}) + 1) \\ &\leq (n - \Delta(G) - 1) + (n - \Delta(G)) \\ &= 2n - 2\Delta - 1 \leq n, \end{aligned}$$

as desired. □

6. Trees

In this section, we determine the $i_{dR}(T)$-stability, the ${i_{dR}}^{+}(T)$-stability and the ${i_{dR}}^{-}(T)$-stability for trees. From Proposition 9, we know that $\mathrm{st}^{+}_{i_{dR}}(T)$ cannot be bounded.

Theorem 4. *For every tree T of order $n \geq 2$, $st_{i_{dR}}(T) = 1$.*

Proof. If $\mathrm{diam}(T) \leq 2$, then T is a star $K_{1,n-1}$, and we have $st_{i_{dR}}(T) = 1$. If $\mathrm{diam}(T) = 3$, then T is a double star $S_{r,t}$ for some $1 \leq r \leq t$ and one can easily deduce from $i_{dR}(S_{r,t}) = 3 + 2r$ that $st_{i_{dR}}(S_{r,t}) = 1$. If $\Delta = 2$, then $T = P_n$ and we are done by Corollary 1. Hence we may assume that $\mathrm{diam}(T) \geq 4$ and $\Delta \geq 3$. By contradiction, we assume that there exists a tree T such that $st_{i_{dR}}(T) \geq 2$. We choose such a tree with smallest order. First, we claim that T has no strong support vertex. Let T has a strong support vertex y with leaf neighbors $y_1, y_2, \ldots, y_k$. Then the vertices $y_1, y_2, \ldots, y_k$ are isolated vertices in $T' = T - y$ and any $i_{dR}(T')$-function certainly assigns 2 to each y_i. Now reassigning $y_1, y_2, \ldots, y_k$ the value 0 and y the value 3 provides an IDRD-function of T of weight less than $i_{dR}(T')$ and this leads to a contradiction. Henceforth, we may assume that T has no strong support vertex. Let $P = x_1x_2 \ldots x_t$ be a longest path in T and root the tree T at the vertex x_t. Let $f = (V_0, \varnothing, V_2, V_3)$ be an $i_{dR}(T)$-function such that $f(x_3)$ is maximized. Since T has no strong support vertex, $\deg(x_2) = 2$ and each child of x_3 with depth 1 has degree 2. We consider the cases:

Case 1. x_3 has a child z with depth 0, that is z is a leaf neighbor of x_3.
It is easy to verify that $f(x_1)+f(x_2)+f(x_3)+f(z) \geq 5$. We have two situations. First note that if $f(x_4) = 0$, then we may assume that $f(x_3) = 3$, $f(x_1) = 2$ and $f(z) = f(x_2) = 0$. This implies that $i_{dR}(T - x_1) < i_{dR}(T)$, which leads to a contradiction. Now, let $f(x_4) \geq 2$, then we may assume that $f(x_2) = 3$, $f(z) = 2$ and $f(x_3) = f(x_1) = 0$. This implies that $i_{dR}(T - z) < i_{dR}(T)$, which leads to a contradiction.

Case 2. x_3 has a child $u_2 \neq x_2$ with depth 1.
Let u_1 be the leaf neighbor of u_2. It is easy to verify that $f(x_1) + f(x_2) + f(x_3) +$

$f(u_2) + f(u_1) \geq 6$. We consider two cases. If $f(x_4) = 0$, then we may assume that $f(x_3) \geq 2$, $f(x_1) = f(u_1) = 2$ and $f(u_2) = f(x_2) = 0$. Then the function g defined on $T - x_1$ by $g(x_3) = 3$, $g(v) = f(v)$ for each vertex $v \in V(T - x_1) - \{x_3\}$, is an IDRD-function of $T - x_1$ of weight at most $\omega(f) - 1$ and so $i_{dR}(T - x_1) < i_{dR}(T)$ leading to a contradiction. Now, if $f(x_4) \geq 2$, then we may assume that $f(x_1) = f(x_3) = f(u_1) = 0$ and $f(u_2) = f(x_2) = 3$. Then the function g defined on $T - x_1$ by $g(x_2) = 2$, $g(v) = f(v)$ for each vertex $v \in V(T - x_1) - \{x_2\}$, is an IDRD-function of $T - x_1$ of weight at most $\omega(f) - 1$ and so $i_{dR}(T - x_1) < i_{dR}(T)$ leading to a contradiction.

Case 3. $\deg(x_3) = 2$.

Now, we take any vertex $w \in V(T) - \{x_1, x_2, x_3\}$, and let $T_1, T_2, \ldots, T_k$ be the components of $T - w$. Without loss of generality, assume that the pendent star $x_1x_2x_3$ is contained in T_1 (Note that if $w = x_4$, then T_1 is a path P_3). Denote $T' = T - \{x_1, x_2, x_3\}$ and $T'_1 = T_1 - \{x_1, x_2, x_3\}$. By Proposition 1, we have $i_{dR}(T) = i_{dR}(T') + 3$ and $i_{dR}(T_1) = i_{dR}(T'_1) + 3$. It conclude that $i_{dR}(T' - w) = i_{dR}(T'_1) + i_{dR}(T_2) + \cdots + i_{dR}(T_k) = i_{dR}(T_1) + i_{dR}(T_2) + \cdots + i_{dR}(T_k) - 3 = i_{dR}(T - w) - 3 = i_{dR}(T) - 3 = i_{dR}(T')$, that is, $st_{i_{dR}}(T') \geq 2$, contradicting the choice of T. Therefore, $st_{i_{dR}}(T) = 1$ for any tree of order $n \geq 2$. This completes the proof. □

Finally, we will establish an upper bound on the $st^{-}_{i_{dR}}(T)$ of a tree. Moreover, we characterize the trees that achieve the upper bound. For this purpose, we define two families of trees as follows.

For integers $k \geq 1$ and $\Delta \geq 2$, let $T_{k,\Delta}$ be a tree obtained from the k copies of the star $K_{1,\Delta}$, say $S_1, S_2, \ldots, S_k$, by adding $k - 1$ edges between the leaves of these stars, so that the resulting graph is a connected graph with maximum degree Δ. Let $\mathcal{T}_{k,\Delta}$ be the family of all such trees $T_{k,\Delta}$, and $\mathcal{T}_\Delta = \bigcup_{k\geq 1} \mathcal{T}_{k,\Delta}$.

Moreover, let $L_{k,\Delta}$ be a graph obtained from a tree $T_{k,\Delta}$ and a path P_2 by joining a vertex of the P_2 to a leaf of some star S_i ($i \in \{1, 2, \ldots, k\}$) of $T_{k,\Delta}$, so that the resulting graph is a tree with maximum degree Δ. Let $\mathcal{L}_{k,\Delta}$ be the family of all such trees $L_{k,\Delta}$, and $\mathcal{L}_\Delta = \bigcup_{k\geq 1} \mathcal{L}_{k,\Delta}$.

It is observed in [17] that if $T \in \mathcal{T}_{k,\Delta}$, then $\gamma_{dR}(T) = 3k$ and T has a unique $\gamma_{dR}(T)$-function f that assigns 3 to the central vertex of each S_i, and 0 to the leaves of each S_i for $i = 1, 2, \ldots, k$. Also, it is shown in [17] that if $T \in \mathcal{L}_{k,\Delta}$, then $\gamma_{dR}(T) = 3k + 3$.

Lemma 1. *If $T \in \mathcal{T}_{k,\Delta}$, then $i_{dR}(T) = 3k$. Furthermore, T has a unique $i_{dR}(T)$-function f that assigns 3 to the central vertex of each S_i, and 0 to the leaves of each S_i for $i = 1, 2, \ldots, k$.*

Proof. First note that $i_{dR}(T) \geq \gamma_{dR}(T) = 3k$. On the other hand, the unique $\gamma_{dR}(T)$-function f is an independent double Roman dominating function of T and therefore $i_{dR}(T) \leq \gamma_{dR}(T) = 3k$. Thus, $i_{dR}(T) = 3k$. It follows from $i_{dR}(T) = 3k$ that any $i_{dR}(T)$-function is a γ_{dR}-function of T and since T has a unique $\gamma_{dR}(T)$-function, we deduce that f is the unique $i_{dR}(T)$-function. □

Lemma 2. *If $T \in \mathcal{L}_{k,\Delta}$, then $i_{dR}(T) = 3k + 3$.*

Proof. First note that $i_{dR}(T) \geq \gamma_{dR}(T) = 3k + 3$. On the other hand, assigning 3 to the center of stars $S_1, \dots, S_k$ and a vertex of P_2 and 0 to other vertices, provides an IDRD-function of weight $3k+3$ leading to $i_{dR}(T) \leq \gamma_{dR}(T) = 3k+3$. Thus, $i_{dR}(T) = 3k+3$. □

Lemma 3. *If* $T \in \mathcal{T}_{\Delta}$*, then* $\mathrm{st}^{-}_{i_{dR}}(T) = \Delta$.

Proof. For $k \geq 1$ and $\Delta \geq 2$, let $T \in \mathcal{T}_{k,\Delta}$. We proceed by induction on the number k. If $k = 1$, then $T \cong K_{1,\Delta}$ and the result follows from Corollary 4. This establishes the base case. Now, let $k \geq 2$. If $\Delta = 2$, then $T \cong P_{3k}$ and we are done by Proposition 8. Assume that $\Delta \geq 3$ and that for any tree $T' \in \mathcal{T}_{k',\Delta}$ with $1 \leq k' < k$ we have $\mathrm{st}^{-}_{i_{dR}}(T') = \Delta$. Since $T \in \mathcal{T}_{k,\Delta}$, T is the graph obtained from the k copies of the star $K_{1,\Delta}$, say $S_1, S_2, \dots, S_k$, by adding $k-1$ edges between the leaves of these stars so that the resulting graph is a connected graph with maximum degree Δ. Let x_i be the central vertex of the star S_i for each $i \in \{1, 2, \dots, k\}$. By Lemma 1, $i_{dR}(T) = 3k$.

By the construction of T, there is one of S_i, say S_k, such that $\Delta - 1$ of its leaves are the leaves of T, and the remaining leaf, say u, has degree two in T. Clearly, u is adjacent to the central vertex x_k of S_k, and a leaf, say v, of another star S_i, say S_{k-1}. To show that $\mathrm{st}^{-}_{i_{dR}}(T) \leq \Delta$, let $T_1 = T - (V(S_k) - \{u\})$, and define a function $f_1 : V(T_1) \to \{0, 1, 2, 3\}$ by $f_1(u) = 2$, $f_1(x_j) = 3$ for $j \in \{1, 2, 3, \dots, k-1\}$, and $f_1(x) = 0$ for $x \in V(T_1) - \{u, x_1, x_2, \dots, x_{k-1}\}$. Clearly, f_1 is an IDRD-function of T_1 with weight $3k - 1$, so we have $\mathrm{st}^{-}_{i_{dR}}(T) \leq \Delta$.

Now, we show that $\mathrm{st}^{-}_{i_{dR}}(T) \geq \Delta$. Let S be a vertex set of T such that $i_{dR}(T - S) \leq 3k - 1$. Among all such sets, we choose S to have minimum cardinality. We show that $|S| \geq \Delta$. Suppose, to the contrary, that $|S| \leq \Delta - 1$. Let $f = (V_0, \varnothing, V_2, V_3)$ be a $i_{dR}(T-S)$-function, and so $\omega(f) \leq 3k - 1$. Choose f such that $f(x_k)$ is as large as possible when $x_k \notin S$. Let $T' = T - V(S_k)$, $S' = S \cap V(T')$ and $W_i = S \cap V(S_i)$ for $i \in \{1, 2, \dots, k\}$. Since $T' \in \mathcal{T}_{k-1,\Delta}$, by Lemma 1, $i_{dR}(T') = 3k - 3$ and from the induction hypothesis we have $\mathrm{st}^{-}_{i_{dR}}(T') = \Delta$.

Since $|S| \leq \Delta - 1$, $|V(S_k) - W_k| \geq 2$. To independent double Roman dominate the vertices in $V(S_k) - W_k$, we have $\sum_{x \in V(S_k) - W_k} f(x) \geq 2$. First let $\sum_{x \in V(S_k) - W_k} f(x) = 2$. Then it is easy to verify that $|V(S_k) - W_k| = 2$, that is, $S = W_k$. Moreover, $V(S_k) - W_k$ consists of u and x_k, or u and a leaf-neighbour of x_k other than u. If $V(S_k) - W_k$ consists of u and x_k, then $T - S \in \mathcal{L}_{k-1,\Delta}$ and by Lemma 2, we have $i_{dR}(T - S) = 3(k-1) + 3$ which is a contradiction. Hence, we assume that $V(S_k) - W_k$ consists of u and a leaf-neighbour y of x_k. Then we must have $f(y) = 2$, $f(u) = 0$ and $f(v) = 3$. Then the restriction of f to $V(T')$, say f', is an IDRD-function of T' which is not an $i_{dR}(T')$-function by Lemma 1. It follows that $i_{dR}(T - S) = \omega(f) = \omega(f') + 2 \geq (3(k-1) + 1) + 2 = 3k$ leading to a contradiction.

Assume that $\sum_{x \in V(S_k) - W_k} f(x) \geq 3$. If $|S \cap \{u, v\}| \geq 1$ or $f(v) \geq 2$ or $f(u) = 0$, then the restriction of f on $V(T' - S')$ is an IDRD-function of $T' - S'$ with weight at most $3k - 4$ and so $i_{dR}(T' - S') \leq 3k - 4$. On the other hand, since $|S'| \leq |S| \leq \Delta - 1$ and $st^{-}_{i_{dR}}(T') = \Delta$, we have $i_{dR}(T' - S') \leq 3k - 4$, a contradiction. So we may assume that $|S \cap \{u, v\}| = 0$, $f(v) = 0$ and $f(u) \geq 2$. By definition, we must have $f(v) = 0$. We distinguish two cases.

Case 1. $f(u) = 2$.
It follows from $\sum_{x\in V(S_k)-W_k} f(x) \geq 3$ and the fact $V_1 = \emptyset$ that $\sum_{x\in V(S_k)-W_k} f(x) \geq 4$. Since $f(v) = 0$, v has a neighbor w with $f(w) \geq 2$. Define the function g on $T' - S'$ by $g(w) = 3$ and $g(x) = f(x)$ for otherwise, is an IDRD-function $T' - S'$ of weight at most $3k - 4$ which leads contradiction.

Case 2. $f(u) = 3$.
If $\sum_{x\in V(S_k)-W_k} f(x) = 3$, then we must have $S = W_k$ and $V(S_k) - S$ consists of u and x_k. But then $T' - S' = T - S \in \mathcal{L}_{k-1,\Delta}$ and f is an IDRD-function of $T' - S'$ of weight $3k - 1$, a contradiction with Lemma 2. Assume that $\sum_{x\in V(S_k)-W_k} f(x) \geq 4$. It follows from $V_1 = \varnothing$ that $\sum_{x\in V(S_k)-W_k} f(x) \geq 5$. If v has a neighbor w with $f(w) \geq 2$, then define the function g on $T' - S'$ by $g(w) = 3$ and $g(x) = f(x)$ otherwise. Clearly, g is an IDRD-function $T' - S'$ of weight $3k - 5$, a contradiction. Assume that all neighbors of v assigned 0 under f. Then the function g on $T' - S'$ by $f(v) = 2$ and $g(x) = f(x)$ otherwise, is an IDRD-function $T' - S'$ of weight $3k - 4$, a contradiction again. $\square$

Theorem 5. *For every tree T of order $n \geq 3$ with maximum degree Δ, $st^-_{i_{dR}}(T) \leq \Delta$ with equality if and only if $T \in \mathcal{T}_\Delta$.*

Proof. If $\mathrm{diam}(T) = 2$, then T is the star $K_{1,\Delta}$ and by Corollary 4, we have $st^-_{i_{dR}}(T) = \Delta$. If $\Delta = 2$, then T is the path P_n and the result is true by Proposition 8. Assume that $\mathrm{diam}(T) \geq 3$ and $\Delta \geq 3$. Let $x_1x_2\ldots x_d$ be a diametral path in G and root T at x_d. We have $d(x_2) \leq \Delta$. Let $f = (V_0, \varnothing, V_2, V_3)$ be an $i_{dR}(T)$-function. Clearly, $f(N[x_2]) \geq 3$. If $f(x_3) \in V_2 \cup V_3$, then $f(x_2) = 0$ and $f(x_1) = 2$. The function g defined on $T - x_1$ by $g(x_3) = \max\{3, f(x_3)\}$ and $g(x) = f(x)$ otherwise, is an IDRD-function of the tree $T - x_1$ of weight at most $\omega(f) - 1$. So, $st^-_{i_{dR}}(T) = 1$. Now, assume that $f(x_3) = 0$. Then we have $f(N[x_2] - \{x_3\}) = 3$. If x_3 has a neighbor $u \neq x_2$ with $f(u) \geq 2$, then the function g on $T - T_{x_2}$ by $g(u) = \min\{3, f(u)+1\}$ and $g(x) = f(x)$ otherwise, is an IDRD-function of the tree $T - T_{x_2}$ of weight at most $\omega(f) - 1$ and so $st^-_{i_{dR}}(T) \leq d(x_2) \leq \Delta$. Now, let $f(x) = 0$ for each $x \in N[x_3] - \{x_2\}$. Then the function g defined on $T - (N[x_2] - \{x_3\})$ by $g(x_3) = 2$ and $g(x) = f(x)$ otherwise, is an IDRD-function of the tree $T - (N[x_2] - \{x_3\})$ of weight $\omega(f) - 1$ and so $st^-_{i_{dR}}(T) \leq d(x_2) \leq \Delta$. This proves the bound.

Now we show that $st^-_{i_{dR}}(T) = \Delta$ if and only if $T \in \mathcal{T}_\Delta$. The sufficiency follows from Lemma 3. To prove the necessity, assume that $st^-_{i_{dR}}(T) = \Delta$. We proceed by induction on n. If $\mathrm{diam}(T) = 2$, then T is the star $K_{1,\Delta}$ and clearly $T \in \mathcal{T}_\Delta$. If $\Delta = 2$, then T is the path P_{3k} and the result is true by Proposition 8. This proves the base case. Suppose that for any tree T' of order $3 \leq n' < n$ with $st^-_{i_{dR}}(T) = \Delta$, we have $T' \in \mathcal{T}_\Delta$. Let T be a tree of order n with $st^-_{i_{dR}}(T) = \Delta$. As before, we can assume that $\mathrm{diam}(T) \geq 3$ and $\Delta \geq 3$. Corollary 5 implies that $\mathrm{diam}(T) \geq 4$. Let f be an i_{dR}-function of T such that there is no vertex assigned 1 under f. Let $[x_1, x_2, \ldots, x_d]$ be a diametral path in G and root T at x_d. Using the above argument and the fact $st^-_{i_{dR}}(T) = \Delta$, we must have $d(x_2) = \Delta$. If $f(x_2) = 0$, then clearly $st^-_{i_{dR}}(T) = 1$ which is contradiction. Since f is an IDRD-function, it follows that $f(x_2) = 3$, and thus $f(x_3) = 0$. We claim that $d(x_3) = 2$. By contradiction, assume that $d(x_3) \geq 3$. Let w be a neighbor of x_3 different from x_2

and x_4. Clearly, w is either a leaf or a support vertex. In the former case, it follows from $f(x_3) = 0$ that $f(w) = 2$. In the latter case, by a similar argument as in above, we have that w has degree Δ and $f(w) = 3$. In either case, remove all leaf-neighbor of x_2 and denote the resulting tree by T'. Then reassigning 2 to x_2 provides an IDRD-function of T' leading to $st^-_{i_{dR}}(T) \leq \Delta - 1$ which is contradiction. Thus, $d(x_3) = 2$.

By symmetry, we have $d(x_{d-1}) = \Delta$ and $d(x_{d-2}) = 2$. If $x_3 = x_{d-2}$, then it is easy to see that $i_{dR}(T) = 6$, and obviously $i_{dR}(T - L(x_2)) = 5$ and so $st^-_{i_{dR}}(T) \leq \Delta - 1$, a contradiction. Hence $x_3 \neq x_{d-2}$. Let $T' = T - T_{x_3}$. By Proposition 2, we have $i_{dR}(T) = i_{dR}(T')+3$. We claim that $st^-_{i_{dR}}(T') = \Delta$. By contradiction, assume that $st^-_{i_{dR}}(T') \leq \Delta-1$ and let S' be a $st^-_{i_{dR}}(T')$-set. Clearly, any $i_{dR}(T' - S')$-function can be extended to an IDRD-function of $T - S'$ by assigning a 3 to x_2 and 0 to the neighbors of x_2 and so $i_{dR}(T - S') \leq i_{dR}(T' - S) + 3 < i_{dR}(T)$ which contradicts the assumption $st^-_{i_{dR}}(T) = \Delta$. Hence, $st^-_{i_{dR}}(T') = \Delta$ and by the induction hypothesis we have $T' \in \mathcal{T}_\Delta$. Thus, T' is a tree obtained from the k' copies of the star $K_{1,\Delta}$, say $S_1, S_2, \ldots, S_{k'}$, by adding $k' - 1$ edges between the leaves of these stars so that the resulting graph is a connected graph with maximum degree Δ. It follows from $\Delta \geq \deg_T(x_4) = \deg_{T'}(x_4) + 1$ that x_4 is a leaf of some star S_i and so $T \in \mathcal{T}_\Delta$ and the proof is completed. □

7. Open questions and problems

We conclude this paper by mentioning some questions and problems suggested by this research.

Problem 1. Characterize the connected graphs G of order n with $i_{dR}(G) \geq 4$ and $\mathrm{st}_{i_{dR}}(G) = n - \Delta(G) - 1$.

Problem 2. Is there a connected graph G of order $n \geq 2$ such that $\mathrm{st}_{i_{dR}}(G) = \delta(G) + 1$.

Problem 3. Determine the independent double Roman domination stability of generalized Petersen graphs and Sierpinski graphs.

8. Conclusion

In this paper, we have studied the independent double Roman domination stability. We determined exact values of the independent double Roman domination stability for special classes of graphs. Additionally, we established bounds on the i_{dR}-stability for general graphs. For trees, we proved that the i_{dR}-stability is always equal to 1, while the i^{-}_{dR}-stability is bounded above by the maximum degree Δ of the tree. We also provided a complete characterization of the trees that attain this upper bound. These results contribute to the growing body of knowledge on Roman domination parameters and open avenues for further research on stability measures in more complex graph structures.

Acknowledgements

We sincerely thank the reviewers for their valuable comments and suggestions, which have significantly improved the quality of this paper.

We acknowledge Ho Chi Minh City University of Technology (HCMUT), VNU-HCM for supporting this study.

In addition, we extend our heartfelt appreciation to Mindanao State University – Tawi-Tawi College of Technology and Oceanography (MSU-TCTO) and Mindanao State University-Iligan Institute of Technology (MSU - IIT) for generously funding this work.

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