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Abundant Explicit Non-Traveling Wave Solutions to the $(3+1)$-Dimensional Nonlinear Evolution Equation Using a Generalized Variable Separation Method

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Abstract. Non-traveling wave solutions allow us to characterize more complex natural phenomena across various scientific fields, which may not be easily analyzed using soliton solutions. This study introduces a novel approach for deriving a wide variety of explicit non-traveling wave solutions to the (3 + 1)-dimensional nonlinear evolution equation, utilizing a modified generalized variable separation technique. The newly obtained solutions encompass different non-traveling forms, including periodic solitary waves and soliton-like structures. These results underscore the efficacy of the method in tackling complex nonlinear partial differential equations. The findings presented in this article constitute innovative contributions to the equation modeling the velocity of water waves on the surface of shallow water. Furthermore, our employed technique can also provide a foundation for investigating the stability, dynamics, and practical applications of solutions to other similar problems.

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1. Introduction

Nonlinear partial differential equations are essential in modern mathematics, greatly enhancing our comprehension of intricate natural phenomena [1–11]. Recently, there has been a growing focus on developing efficient methods to obtain analytical and approximate solutions for this specific class of differential equations[12–22]. In recent years, numerous researchers have shown keen interest in advancing methods for obtaining exact wave solutions of the nonlinear models [23–28].

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In this paper, we study a nonlinear (3+1)-dimensional evolution equation [29]

$$3\,u_{xxz} + (2\,u_{xt} + u_{xxxx} - 2\,u_x u_{xx})_y + 2\,(u_{xx} u_y)_x = 0, \tag{1}$$

where $u = u(x, y, z, t)$. This equation models the velocity of water waves on the surface of shallow water.

The (2+1)-dimensional version of the model was proposed by Geng to describe the (2+1)-dimensional interaction between a Riemann wave traveling along the y-axis and a long wave along the x-axis. In his work, the algebraic-geometrical solutions of the model were explicitly expressed in terms of the Riemann theta functions [29]. Wazwaz also successfully derived the dispersive relations and phase shifts for the model, as well as identified multiple soliton solutions for each equation [30]. Additionally, some kink breather soliton and multi-soliton solutions to the model have been discovered in [31]. Moreover, the authors of [32] explored the application of the linear superposition principle to construct resonant multiple wave solutions for a (n+1)(n+1)-dimensional nonlinear evolution equation. They derived two types of resonant solutions through parameterization of wave numbers and frequencies and illustrated the resonance phenomena of multiple waves with figures depicting several sample solutions. This research contributes to our understanding of wave interactions in nonlinear systems. The authors of [33] decomposed a (3+1)-dimensional nonlinear evolution equation (1) into three integrable (1+1)-dimensional models, deriving general Nth-order rational solutions through the Darboux transformation method. They then explored doubly-localized lumps and first-order rogue waves, providing insights into their patterns in various planes. In [34], a comprehensive investigation was carried out to study Eq.(1), focusing on non-singular multi-complexiton waves. They "initially obtained multi-shock waves using linear superposition and then constructed non-singular multi-complexiton waves through symbolic computations. In the work of [35], the Hirota bilinear approach was employed to studying the model. They derived NN-soliton solutions and soliton molecules in various planes, examining diverse hybrid interactions and periodic wave solutions. This research enhances the understanding of the nonlinear dynamics involved, providing a comprehensive graphical analysis of the dynamic attributes of the solutions. In their research, the authors of [36] introduced new solutions to the model, focusing on resonant multiple soliton solutions (RMSSs) obtained through the linear superposition principle and a weight algorithm. They also developed both nonsingular and singular complexiton solutions by introducing conjugate parameter pairs and introduced the bilinear approach to explore complex multiple-soliton solutions.

Nowadays, non-traveling wave solutions are applicable in diverse domains such as physics, engineering, and fluid dynamics, offering enhanced insights into the behavior of complex systems influenced by multiple factors. One of the notable techniques that has contributed to the discovery of non-traveling wave solutions for equations is the extended homoclinic test method, which has been employed to address various equations to date [37–40]. The objective of this research is to introduce a novel generalized variable separation method to obtain a wide range of explicit non-traveling wave solutions for Eq.(1). This article presents a version that is more comprehensive and inclusive than the form considered in [37–39]. To the best of our knowledge, this methodology has yet to be applied to identify non-traveling wave solutions to Eq.(1). This article is organized as follows. The second section introduces the formal structure of the reduction procedure for Eq.(1). Section 3 outlines an alternative scenario to achieve several non-traveling wave solutions to Eq.(1). In Section 4, we present the dynamical analysis of the proposed solutions. Finally, the article concludes with a summary in the last section.

2. The reduction procedure for Eq.(1)

In this paper, we investigate a generalized method for variable separation that enables the derivation of non-traveling wave exact solutions to Eq.(1) within the following specified framework:

$$u(x, y, z, t) = p(y, z, t) \cdot \varphi(\xi, t) + q(y, z, t), \tag{2}$$

where $\xi = \alpha x + \theta(y, z, t)$, and p, q, θ and φ are three unknown functions. This framework represents a broader version of the structures previously found in the literature for $p(y, z, t) = 1$. This new assumption could potentially yield solutions that have not been explored or documented in earlier studies of the equation.

The subsequent theorem outlines the key findings of the paper.

Theorem 1. *Through the application of variable transformation defined as*

$$u(x, y, z, t) = p(y, z, t) \cdot \varphi(\xi, t) + q(y, z, t), \tag{3}$$

in Eq.(1) where $\xi = \alpha x + \theta(y, z, t)$, with unknown functions p, q, θ, and φ, we obtain the following results:
(i) *For any arbitrary continuous two-dimensional functions f_1 and Λ that depend on z and t, if $p(y, z, t) = 1$ holds along with*

$$\theta(y, z, t) = f_1(z, t), \quad q(y, z, t) = -\frac{3f_{1z}(z, t)}{2\alpha} y + \Lambda(z, t), \tag{4}$$

then the solution of Eq.(1) can be expressed as

$$u(x, y, z, t) = \varphi(\alpha x + f_1(z, t), t) - \frac{3\left(\frac{\partial}{\partial z} f_1(z, t)\right) y}{2\alpha} + \Lambda(z, t), \tag{5}$$

where φ is an arbitrary continuous two-variable function.
(ii) *For any arbitrary continuous two-dimensional functions f_1, f_2, and Λ, if $p(y, z, t) = 1$ holds with*

$$\begin{aligned} &\theta(y, z, t) = f_1(z, t) + f_2(y, z), \\ &q(y, z, t) = \frac{-3f_{1z}(z, t)\alpha y - 3\left(\int f_{2z}(y, z)dy\right)\alpha + 2f_2(y, z)f_{1t}(z, t)}{2\alpha^2} + \Lambda(z, t), \end{aligned} \tag{6}$$

then Eq.(1) simplifies to

$$\alpha^3 \varphi_{\xi\xi\xi} - 2\alpha^2 \varphi_\xi^2 + 2\varphi_t = 0. \tag{7}$$

(iii) *For any arbitrary continuous two-dimensional functions* f_1, f_2, *and* Λ, *if it holds that*

$$p(y,z,t) = f_1(z,t), \quad \theta(y,z,t) = f_2(z,t), \quad q(y,z,t) = \left(\frac{3f_{1z}(z,t)}{2\alpha f_1(z,t)} - \frac{3f_{2z}(z,t)}{2\alpha}\right) y + \Lambda(z,t), \tag{8}$$

then the solution of Eq.(1) *is given by*

$$u(x,y,z,t) = f_1(z,t)g(t)e^{-\alpha x - f_2(z,t)} \left(\frac{3f_{1z}(z,t)}{2\alpha f_1(z,t)} - \frac{3f_{2z}(z,t)}{2\alpha}\right) y + \Lambda(z,t). \tag{9}$$

Proof. First, by inserting the symbolic structure Eq.(3) into Eq.(1), it is simplified as

$$\begin{aligned} &\delta_1\varphi_{\xi\xi\xi\xi\xi} + \delta_2\varphi_{\xi\xi\xi\xi} + \delta_3\varphi_{\xi\xi\xi} + \delta_4\varphi_{\xi\xi\xi}\varphi_\xi + \delta_5\varphi_{\xi\xi\xi}\varphi + \delta_6\varphi_{\xi\xi} \\ &\quad + \delta_7\varphi_{\xi\xi}\varphi_\xi + \delta_8\varphi_\xi + \delta_9\varphi_{\xi\xi}^2 + \delta_{10}\varphi_{\xi\, t} + \delta_{11}\varphi_{\xi\xi\, t} = 0, \end{aligned} \tag{10}$$

where

$$\begin{aligned} &\delta_1 = -p\theta_y\alpha^4, \quad \delta_2 = -p_y\alpha^4, \quad \delta_3 = p\alpha\left(2q_y\alpha^2 - 2\theta_y\theta_t + 3\alpha\theta_z\right), \quad \delta_4 = 4p^2\theta_y\alpha^3, \\ &\delta_5 = 2pp_y\alpha^3, \quad \delta_6 = -2p\alpha\theta_{ty} - 2\theta_t\alpha p_y + 3p_z\alpha^2 - 2p_t\alpha\theta_y, \quad \delta_7 = 6\alpha^3 pp_y, \quad \delta_8 = -2\alpha p_{ty}, \\ &\delta_9 = 4p^2\theta_y\alpha^3, \quad \delta_{10} = -2\alpha p_y, \quad \delta_{11} = -2\alpha p\theta_y. \end{aligned} \tag{11}$$

At this point, we are seeking the criteria that would transform Eq.(10) into a more straightforward and solvable form. To accomplish this, we need to analyze the scenario in which the term $\varphi_{\xi\xi\xi}$ in Eq.(10) is removed. This leads us to set $\delta_3 = 0$, resulting in

$$q(y,z,t) = \frac{1}{2\alpha^2}\int (2\theta_y\theta_t - 3\alpha\theta_z)dy + \Lambda(z,t)\,. \tag{12}$$

As the first assumption, we consider the case of $p_y = p_z = p_t = 0$. Without loss of generality, it can be assumed that $p(y,z,t) = 1$. A direct consequence of this assumption is that $\delta_2 = \delta_5 = \delta_7 = \delta_8 = \delta_{10} = 0$.
Further, Eq.(10) reduces to

$$-\theta_y\alpha^3\varphi_{\xi\xi\xi\xi\xi} + 4\alpha^2\theta_y\varphi_{\xi\xi\xi}\varphi_\xi - 2\theta_{ty}\varphi_{\xi\xi} + 4\alpha^2\theta_y\varphi_{\xi\xi}^2 - 2\theta_y\varphi_{\xi\xi t} = 0. \tag{13}$$

By executing a double integration of Eq.(13) with respect to ξ, and then eliminating the resulting integral along with applying some simple algebraic rearrangements, we obtain the following equation

$$\theta_y\left(\alpha^3\varphi_{\xi\xi\xi} - 2\alpha^2\varphi_\xi^2 + 2\varphi_t\right) + 2\theta_{ty}\varphi = 0. \tag{14}$$

(i) By setting $\theta_y = 0$, the entire equation (14) simplifies to zero, resulting in the following conclusion

$$\theta\,(y,z,t) = f_1(z,t)\,. \tag{15}$$

Furthermore, by substituting Eq.(15) into Eq.(12), the solution is simplified to

$$q(y,z,t) = -\frac{3f_{1z}(z,t)}{2\alpha}y + \Lambda(z,t)\,. \tag{16}$$

Thus, a general non-soliton wave solution for Eq.(1) is obtained as follows

$$u(x,y,z,t) = \varphi(\alpha x + f_1(z,t)\,,t) - \frac{3\left(\frac{\partial}{\partial z} f_1(z,t)\right) y}{2\alpha} + \Lambda(z,t)\,, \tag{17}$$

where φ, f_1, and Λ are arbitrary continuous two-variable functions.

(ii) To further simplify Eq.(14), let us consider $\theta_{ty} = 0$, which implies

$$\theta\,(y,z,t) = f_1(z,t) + f_2(y,z)\,. \tag{18}$$

Inserting Eq.(18) into Eq.(12), the solution becomes

$$q(y,z,t) = \frac{-3{f_1}_z(z,t)\,\alpha y - 3\left(\int {f_2}_z(y,z)\,dy\right)\alpha + 2f_2(y,z)\,{f_1}_t(z,t)}{2\alpha^2} + \Lambda(z,t)\,. \tag{19}$$

Furthermore, Eq.(14) is expressed in a simpler form as follows

$$\alpha^3\varphi_{\xi\xi\xi} - 2\alpha^2\varphi_\xi^2 + 2\varphi_t = 0. \tag{20}$$

(iii) Here, we consider another way of simplifying Eq.(10), as $p_y = \theta_y = 0$, or equivalently

$$p(y,z,t) = f_1(z,t), \quad \theta(y,z,t) = f_2(z,t), \tag{21}$$

f_1 and f_2 are arbitrary continuous functions.
Then Eq.(10) reduces to

$$p\left(2q_y\alpha + 3\theta_z\right)\varphi_{\xi\xi\xi} + 3p_z\varphi_{\xi\xi} = 0. \tag{22}$$

Further, we integrate Eq.(22) once with respect to ξ and null the integral constant as

$$p\left(2q_y\alpha + 3\theta_z\right)\varphi_\xi + 3p_z\varphi = 0. \tag{23}$$

Now, if we assume that $p\left(2q_y\alpha + 3\theta_z\right) = 3p_z$ and thanks to (19) and (21), we have

$$q(y,z,t) = \left(\frac{3{f_1}_z(z,t)}{2\alpha f_1(z,t)} - \frac{3{f_2}_z(z,t)}{2\alpha}\right) y + \Lambda(z,t)\,. \tag{24}$$

Taking Eqs.(21), (24) into account in Eq.(23), we derive the following equation

$$\varphi(\xi,t) + \varphi_\xi(\xi,t) = 0. \tag{25}$$

The recent equation clearly possesses the following solution

$$\varphi\,(\xi,t) = g(t)\,\mathrm{e}^{-\xi}, \tag{26}$$

where g is an arbitrary continuous non-zero function.
Thus, a general non-soliton solution for Eq.(1) is obtained as follows

$$u(x,y,z,t) = f_1(z,t)\,g(t)\,\mathrm{e}^{-\alpha x - f_2(z,t)}\left(\frac{3{f_1}_z(z,t)}{2\alpha f_1(z,t)} - \frac{3{f_2}_z(z,t)}{2\alpha}\right) y + \Lambda(z,t)\,. \tag{27}$$

This solution can also be considered a more generalized form of the solution $u_1(x,y,z,t)$ given in the formula (17).

3. Further non-traveling wave solutions to Eq.(1)

As demonstrated in the second part of Theorem 1, under the specified conditions, the main equation is reduced to the Eq.(20). This latter equation is significantly more straightforward and likely easier to solve than the original problem (1). In this section, we aim to obtain analytical solutions for the beta equation, which will subsequently allow us to derive solutions for the original equation using an efficient technique. To this purpose, we introduce the new variable $\varphi(\xi, t) = \varphi(\wp)$ along with

$$\wp = \kappa\xi + \omega t. \tag{28}$$

Applying this transformation to Eq.(20) gives

$$\alpha^2\kappa^3\left(\frac{d^3}{d\wp^3}\varphi\left(\wp\right)\right) + 3\alpha\kappa^2\left(\frac{d}{d\wp}\varphi\left(\wp\right)\right)^2 + \omega\left(\frac{d}{d\wp}\varphi\left(\wp\right)\right) = 0. \tag{29}$$

We now utilize the modified version of the generalized exponential rational function method (abbreviated as mGERFM), which is an innovative analytical approach introduced by Ghanbari in [41]. Based on this method, the solution to Eq.(29) takes the form of the following structure

$$\varphi(\wp) = \varepsilon_0 + \sum_{j=1}^{\mathbf{n}}\varepsilon_j\left(\frac{\Gamma'(\wp)}{\Gamma(\wp)}\right)^j + \sum_{j=1}^{\mathbf{n}}\gamma_j\left(\frac{\Gamma(\wp)}{\Gamma'(\wp)}\right)^j, \tag{30}$$

where

$$\Gamma(\wp) = \frac{\varsigma_1 e^{\vartheta_1\wp} + \varsigma_2 e^{\vartheta_2\wp}}{\varsigma_3 e^{\vartheta_3\wp} + \varsigma_4 e^{\vartheta_4\wp}}, \tag{31}$$

and $\mathbf{n}$ is the balance number of the given equation. By applying the balance rule given in Eq.(29), we may infer that $2(\mathbf{n}+1) = \mathbf{n}+3$, which gives $\mathbf{n} = 1$. Thus, from Eq.(30), we obtain

$$\varphi(\wp) = \varepsilon_0 + \varepsilon_1\left(\frac{\Gamma'(\wp)}{\Gamma(\wp)}\right) + \gamma_1\left(\frac{\Gamma(\wp)}{\Gamma'(\wp)}\right). \tag{32}$$

Upon inserting the expression from Eq.(32) along with Eq.(31) in Eq.(29) and solving the resultant for the unknown parameters, the following solutions are obtained.

Set 1: For $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [2, 0, 1, -1]$ and $[\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4] = [2, 0, 2, 0]$, Eq.(31) reduces to

$$\Gamma(\wp) = \frac{2e^{2\wp}}{e^{2\wp} - 1}. \tag{33}$$

Moreover, the rest of the parameters can be attained as

$$\omega = -2\kappa^3\alpha^3, \varepsilon_1 = 0, \gamma_1 = 3\kappa\alpha, \tag{34}$$

where κ, ε_0 are free-chosen parameters.
Taking the obtained results into account in Eqs.(32) and (33), we have

$$\varphi(\wp) = \frac{(-3\kappa\alpha + \varepsilon_0)\, \mathrm{e}^{2\wp} + 3\kappa\alpha + \varepsilon_0}{\mathrm{e}^{2\wp} + 1}. \tag{35}$$

In light of the recent result along with Eq.(28), the solution for Eq.(20) is obtained as

$$\varphi(\xi, t) = \frac{(-3\kappa\alpha + \varepsilon_0)\, \mathrm{e}^{-4\alpha^3\kappa^3 t + 2\kappa\xi} + 3\kappa\alpha + \varepsilon_0}{\mathrm{e}^{-4\alpha^3\kappa^3 t + 2\kappa\xi} + 1}. \tag{36}$$

As a result, by substituting Eqs.(36), (18), and (19) into (3), we can derive a non-soliton solution for Eq.(1) as

$$\begin{aligned} u(x, y, z, t) = & \frac{(-3\kappa\alpha + \varepsilon_0)\, \mathrm{e}^{-4\alpha^3\kappa^3 t + 2\kappa(\alpha x + f_2(y,z) + f_1(z,t))} + 3\kappa\alpha + \varepsilon_0}{\mathrm{e}^{-4\alpha^3\kappa^3 t + 2\kappa(\alpha x + f_2(y,z) + f_1(z,t))} + 1} \\ & + \frac{f_2(y, z)\left(\frac{\partial}{\partial t} f_1(z, t)\right)}{\alpha^2} - \frac{3y\left(\frac{\partial}{\partial z} f_1(z, t)\right) + 3\left(\int \left(\frac{\partial}{\partial z} f_2(y, z)\right) dy\right)}{2\alpha} + \Lambda(z, t)\,. \end{aligned} \tag{37}$$

Set 2: For $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [1, -1, 2\mathbf{i}, 0]$ and $[\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4] = [2 + \mathbf{i}, 2 - \mathbf{i}, 0, 0]$, Eq.(31) reduces to

$$\Gamma(\wp) = \sin(\wp)\, \mathrm{e}^{2\wp}. \tag{38}$$

Additionally, the remaining parameters can be obtained as follows

$$\omega = 2\kappa^3\alpha^3, \varepsilon_1 = 0, \gamma_1 = 15\kappa\alpha, \tag{39}$$

where κ, ε_0 are free-chosen parameters.
Taking the obtained results into account in Eqs.(32) and (38), one achieves

$$\varphi(\wp) = \frac{(15\kappa\alpha + 2\varepsilon_0)\sin(\wp) + \varepsilon_0\cos(\wp)}{2\sin(\wp) + \cos(\wp)}. \tag{40}$$

In light of the recent result along with Eq.(28), the solution for Eq.(20) is obtained as

$$\varphi(\xi, t) = \frac{(15\kappa\alpha + 2\varepsilon_0)\sin\left(2\alpha^3\kappa^3 t + \kappa\xi\right) + \varepsilon_0\cos\left(2\alpha^3\kappa^3 t + \kappa\xi\right)}{2\sin(2\alpha^3\kappa^3 t + \kappa\xi) + \cos(2\alpha^3\kappa^3 t + \kappa\xi)}. \tag{41}$$

Hence, using Eqs.(41), (18), and (19) in Eq.(3), a non-soliton solution for Eq.(1) can be established in the following manner

$$\begin{aligned} u(x, y, z, t) = & \frac{(15\kappa\alpha + 2\varepsilon_0)\sin\left(2\alpha^3\kappa^3 t + \kappa\xi\right) + \varepsilon_0\cos\left(2\alpha^3\kappa^3 t + \kappa\xi\right)}{2\sin(2\alpha^3\kappa^3 t + \kappa\xi) + \cos(2\alpha^3\kappa^3 t + \kappa\xi)} \\ & + \frac{f_2(y, z)\left(\frac{\partial}{\partial t} f_1(z, t)\right)}{\alpha^2} - \frac{3y\left(\frac{\partial}{\partial z} f_1(z, t)\right) + 3\left(\int \left(\frac{\partial}{\partial z} f_2(y, z)\right) dy\right)}{2\alpha} + \Lambda(z, t)\,, \end{aligned} \tag{42}$$

where $\xi = 2\alpha^3\kappa^3 t + \kappa\left(\alpha x + f_2(y,z) + f_1(z,t)\right)$.

Set 3: For $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [1, 1, 1, 0]$ and $[\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4] = [1, 2, 0, 0]$, Eq.(31) reduces to

$$\Gamma(\wp) = \mathrm{e}^{\wp} + \mathrm{e}^{2\wp}. \tag{43}$$

Moreover, the rest of the parameters can be attained as

$$\omega = -\frac{\kappa^3\alpha^3}{2}, \varepsilon_1 = -3\kappa\alpha, \gamma_1 = 0, \tag{44}$$

where κ, ε_0 are free-chosen parameters.
Taking the obtained results into account in Eqs.(32) and (43), one gets

$$\varphi(\wp) = \frac{\left(-6\kappa\alpha + \varepsilon_0\right)\mathrm{e}^{\wp} - 3\kappa\alpha + \varepsilon_0}{1 + \mathrm{e}^{\wp}}. \tag{45}$$

In light of the recent result along with Eq.(28), the solution for Eq.(20) is obtained as

$$\varphi(\xi, t) = \frac{\left(-6\kappa\alpha + \varepsilon_0\right)\mathrm{e}^{-\frac{1}{2}\alpha^3\kappa^3 t + \kappa\xi} - 3\kappa\alpha + \varepsilon_0}{1 + \mathrm{e}^{-\frac{1}{2}\alpha^3\kappa^3 t + \kappa\xi}}. \tag{46}$$

Consequently, using Eqs.(46), (18) and (19) in Eq.(3), a non-soliton solution for Eq.(1) can be established in the following manner

$$\begin{aligned} u(x,y,z,t) = {} & \frac{\left(-6\kappa\alpha + \varepsilon_0\right)\mathrm{e}^{-\frac{\alpha^3\kappa^3 t}{2} + \kappa(\alpha x + f_2(y,z) + f_1(z,t))} - 3\kappa\alpha + \varepsilon_0}{1 + \mathrm{e}^{-\frac{\alpha^3\kappa^3 t}{2} + \kappa(\alpha x + f_2(y,z) + f_1(z,t))}} \\ & + \frac{f_2(y,z)\left(\frac{\partial}{\partial t} f_1(z,t)\right)}{\alpha^2} - \frac{3y\left(\frac{\partial}{\partial z} f_1(z,t)\right) + 3\left(\int \left(\frac{\partial}{\partial z} f_2(y,z)\right) dy\right)}{2\alpha} + \Lambda(z,t)\,. \end{aligned} \tag{47}$$

Set 4: For $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [2, 0, 1, 1]$ and $[\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4] = [2 + \mathbf{i}, 0, 2\mathbf{i}, 0]$, Eq.(31) reduces to

$$\Gamma(\wp) = \mathrm{e}^{2\xi}\sec(\xi)\,. \tag{48}$$

Additionally, the remaining parameters can be obtained as follows

$$\omega = 2\kappa^3\alpha^3, \varepsilon_1 = 3\kappa\alpha, \gamma_1 = 0\ , \tag{49}$$

where κ, ε_0 are free-chosen parameters.
Taking the obtained results into account in Eqs.(32) and (48), we obtain

$$\varphi(\wp) = 3\alpha\kappa\tan(\wp) + 6\kappa\alpha + \varepsilon_0. \tag{50}$$

In light of the recent result along with Eq.(28), the solution for Eq.(20) is obtained as

$$\varphi(\xi, t) = 3\alpha\kappa\tan\!\left(2\alpha^3\kappa^3 t + \kappa\xi\right) + 6\kappa\alpha + \varepsilon_0. \tag{51}$$

Hence, using Eqs.(51), (18) and (19) in Eq.(3), a non-soliton solution for Eq.(1) is obtained as

$$u(x,y,z,t) = 3\alpha\kappa\tan\left(2\alpha^3\kappa^3 t + \kappa\left(\alpha x + f_2(y,z) + f_1(z,t)\right)\right) + \frac{f_2(y,z)\left(\frac{\partial}{\partial t}f_1(z,t)\right)}{\alpha^2} - \frac{3y\left(\frac{\partial}{\partial z}f_1(z,t)\right) + 3\left(\int\left(\frac{\partial}{\partial z}f_2(y,z)\right)dy\right)}{2\alpha} + \Lambda(z,t)\,. \tag{52}$$

Set 5: For $[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4] = [1,-1,2\mathbf{i},0]$ and $[\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4] = [1+\mathbf{i},1-\mathbf{i},0,0]$, Eq.(31) reduces to

$$\Gamma(\wp) = \mathrm{e}^{\wp}\sin(\wp). \tag{53}$$

Moreover, the rest of the parameters can be attained as

$$\omega = 2\kappa^3\alpha^3, \varepsilon_1 = -3\kappa\alpha, \gamma_1 = 0, \tag{54}$$

where κ, ε_0 are free-chosen parameters.
Taking the obtained results into account in Eqs.(32) and (53), it reads

$$\varphi(\wp) = -3\alpha\kappa\cot(\wp) - 3\kappa\alpha + \varepsilon_0. \tag{55}$$

In light of the recent result along with Eq.(28), the solution for Eq.(20) is obtained as

$$\varphi(\xi,t) = -3\alpha\kappa\cot\left(2\alpha^3\kappa^3 t + \kappa\xi\right) - 3\kappa\alpha + \varepsilon_0. \tag{56}$$

Therefore, by substituting Eqs.(56), (18), and (19) into (3), we derive a non-soliton solution for Eq.(1) as follows

$$u(x,y,z,t) = -3\alpha\kappa\cot\left(2\alpha^3\kappa^3 t + \kappa\left(\alpha x + f_2(y,z) + f_1(z,t)\right)\right) + \frac{f_2(y,z)\left(\frac{\partial}{\partial t}f_1(z,t)\right)}{\alpha^2} - \frac{3y\left(\frac{\partial}{\partial z}f_1(z,t)\right) + 3\left(\int\left(\frac{\partial}{\partial z}f_2(y,z)\right)dy\right)}{2\alpha} + \Lambda(z,t)\,. \tag{57}$$

Remarks 3.1. *All the proposed solutions in this work have been validated using Maple by substituting them back into the original equation.*

4. Dynamical analysis of the proposed solutions

In this section of the article, we will examine the dynamic behaviors of the solutions obtained in the last section.

■ First of all, we investigate the solution given in Eq.(17). Employing a diverse selection of three arbitrary continuous functions φ, f_1, Λ in the formulation of this solution enables the generation of various unique non-traveling exact solutions to Eq.(1). Some of the

derived non-traveling exact solutions from the solution (17) are listed below.
(a): Through the insertion of

$$\varphi(\xi,t)=e^{-\frac{\xi^2+t^2}{10}}\sin(\xi^2+t^2), \tag{58a}$$

$$f_1(z,t)=\cos(z)e^{-\frac{t^2+z^2}{20}}+\sin(2t)(1+z^2)^{0.1}, \tag{58b}$$

$$\Lambda(z,t)=5\tanh(\sin(z)+\cos(t)), \tag{58c}$$

in Eq.(17), we have

$$\begin{aligned} u_1(x,y,z,t) &= 5\tanh(\cos(t)+\sin(z))+\sin(\Delta)\exp\left(-\frac{\Delta}{10}\right) \\ &-\frac{3}{2\alpha}\left(-e^{\frac{1}{20}(-t^2-z^2)}\sin(z)-\frac{1}{10}ze^{\frac{1}{20}(-t^2-z^2)}\cos(z)+\frac{0.2z\sin(2t)}{(z^2+1)^{0.9}}\right)y, \end{aligned} \tag{59}$$

where

$$\Delta=\left(e^{\frac{(-t^2-z^2)}{20}}\cos(z)+\left(z^2+1\right)^{0.1}\sin(2t)+\alpha x\right)^2+t^2.$$

The solution $u_1(x,y,z,t)$ represents a complex, oscillatory medium where wave behavior is modulated by the spatial coordinates x,y,z and the temporal component t. The term φ introduces a smooth decay and oscillation that interacts with the influences of surrounding fields as represented by $f_1(z,t)$ and $\Lambda(z,t)$. Several dynamics of this solution for different values of α are displayed in Fig.1.

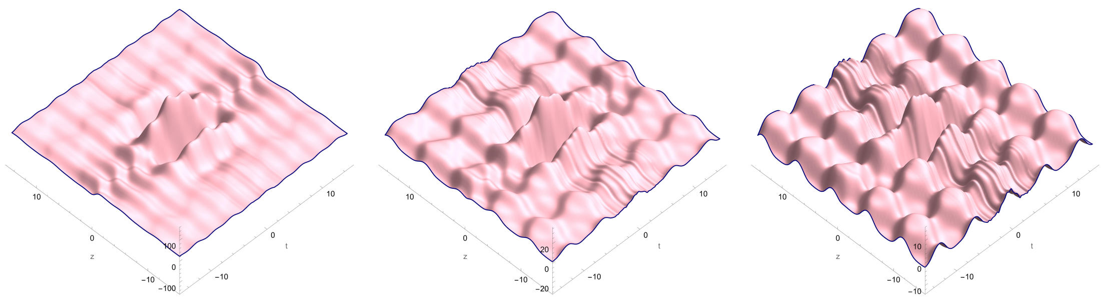

Figure 1: 3D plot of $u_1(1,1,z,t)$ for different values of α .

(b): By incorporating

$$\varphi(\xi,t)=\sin(\xi^2+t)+\cos(2t), \tag{60a}$$

$$f_1(z,t)=\sin(z)e^{-t^2/5}+0.5\cos(3z-t), \tag{60b}$$

$$\Lambda(z,t)=0.5\sin(z)+\cos(t)+e^{-(z^2+t^2)/10}. \tag{60c}$$

in Eq.(17), we have

$$u_2(x,y,z,t)=\sin\left(\left(e^{-\frac{t^2}{5}}\sin(z)+0.5\cos(t-3z)+\alpha x\right)^2+t\right) - \frac{3y\left(e^{-\frac{t^2}{5}}\cos(z)+1.5\sin(t-3z)\right)}{2\alpha}+e^{-\left(t^2+z^2\right)/10}+\cos(t)+\cos(2t)+0.5\sin(z). \tag{61}$$

The recent obtained solution combines oscillatory components to describe a dynamic wave-like field in three-dimensional space. It represents the interaction of spatial oscillations with temporal changes, incorporating the effects of a scalar field dependent on altitude z and time t. The bounded nature of the solution ensures physical constraints are maintained, potentially resembling wave behavior in fluid dynamics or electromagnetic fields. Several dynamics of this solution for different values of α are displayed in Fig.2.

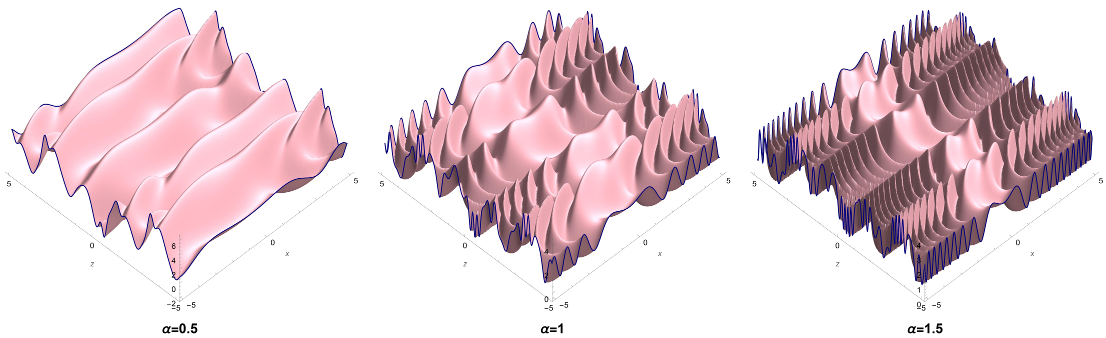

Figure 2: 3D plot of $u_2(x,0.5,z,0.5)$ for different values of α .

(c): Through inserting

$$\varphi(\xi,t)=\sin(\xi)+\cos(2t)+e^{-(\xi^2+t^2)}, \tag{62a}$$

$$f_1(z,t)=0.5z^2+3\sin(z+t)+1, \tag{62b}$$

$$\Lambda(z,t)=2\log(1+z^2)+\sqrt{1+t^2}, \tag{62c}$$

in Eq.(17), we have

$$u_3(x,y,z,t)=e^{-t^2-\left(3\sin(t+z)+\alpha x+0.5z^2+1\right)^2}+\sin\left(3\sin(t+z)+\alpha x+0.5z^2+1\right) - \frac{3y(3\cos(t+z)+z)}{2\alpha}+\cos(2t)+2\log\left(z^2+1\right)+\sqrt{t^2+1}. \tag{63}$$

The formula represents a dynamic system where the spatial variables $x,y,$ and z are modulated by oscillatory components combined with temporal evolution. This functional form may describe phenomena like waves or heat distribution in a medium, illustrating how bounded interactions influence a state variable over time. Several dynamics of this

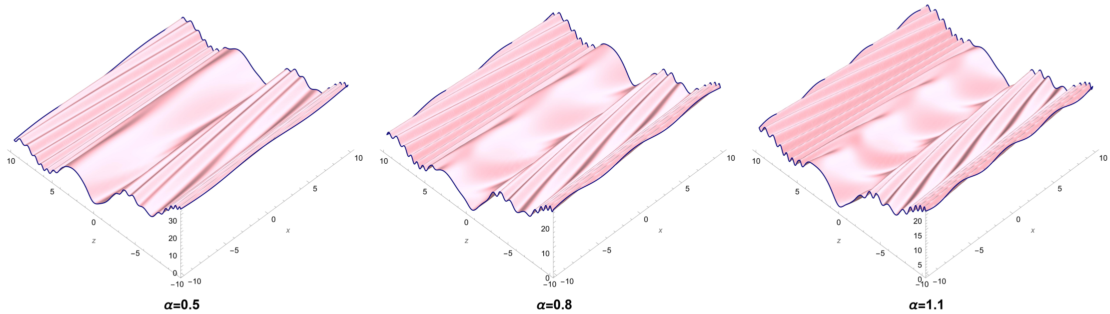

Figure 3: 3D plot of $u_3(x, 0.5, z, 0.2)$ for different values of α .

solution for different values of α are displayed in Fig.3.
(d): Through the insertion of

$$\varphi(\xi, t) = \sin(\xi)e^{-0.1t^2}, \tag{64a}$$

$$f_1(z, t) = \sin(0.5t)\cos(0.5t) + z^2/20, \tag{64b}$$

$$\Lambda(z, t) = (1 + z/10)\sin\left(\sqrt{z^2 + t^2}\right), \tag{64c}$$

in Eq.(17), we have

$$\begin{aligned} u_4(x, y, z, t) = {} & \sin\left(\sin(0.5t)\cos(0.5z) + \alpha x + \frac{z^2}{20}\right) e^{-0.1t^2} \\ & - \frac{3y\left(\frac{z}{10} - 0.5\sin(0.5t)\sin(0.5z)\right)}{2\alpha} + (1 + z/10)\sin\left(\sqrt{z^2 + t^2}\right). \end{aligned} \tag{65}$$

This solution exhibits a dynamic system where the interaction between multiple dimensions generates intricate oscillatory behavior. The contributions from φ create spatial wave patterns, while f_1 and Λ modulate these patterns over time and spatial dimensions, reflecting how different physical phenomena can influence the overall system. Several dynamics of this solution for different values of α are displayed in Fig.4.
(e): By incorporating

$$\varphi(\xi, t) = e^{-x^2}\sin(2\pi t) + \cos(\pi x), \tag{66a}$$

$$f_1(z, t) = \sin(z)\cos(\omega t) + 0.5z, \tag{66b}$$

$$\Lambda(z, t) = t\sin(2\pi z), \tag{66c}$$

in Eq.(17), we have

$$\begin{aligned} u_5(x, y, z, t) = {} & \cos(\pi(\cos(t)\sin(z) + \alpha x + 0.5z)) + \sin(2\pi t)e^{-(\cos(t)\sin(z) + \alpha x + 0.5z)^2} \\ & - \frac{3y(\cos(t)\cos(z) + 0.5)}{2\alpha} + t\sin(2\pi z). \end{aligned} \tag{67}$$

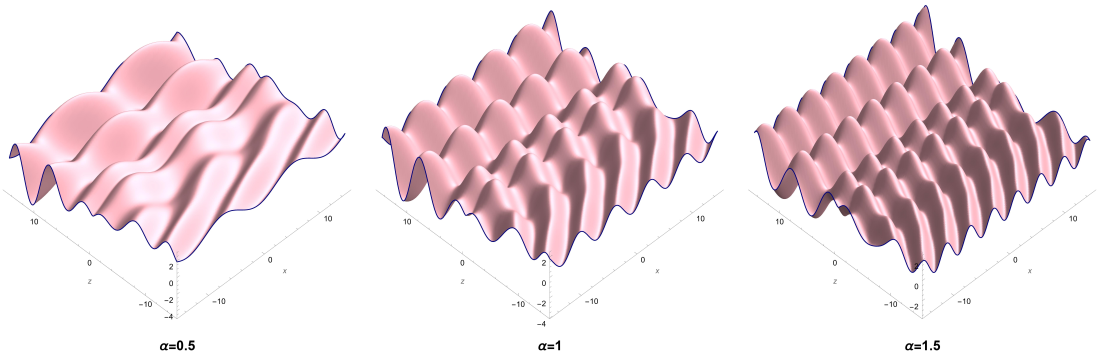

Figure 4: 3D plot of $u_4(x, 0.5, z, 0.2)$ for different values of α .

This solution is used to explain a three-dimensional wave phenomenon where the oscillations and propagation characteristics are influenced by the spatial variables x, y, z and time t. The contributions from φ, f_1, and Λ highlight interactions within the system, showcasing complex patterns and modulations over time while remaining bounded in the specified range, giving insight into various physical systems such as fluid dynamics or wave propagation. Several dynamics of this solution for different values of α are displayed in Fig.5.

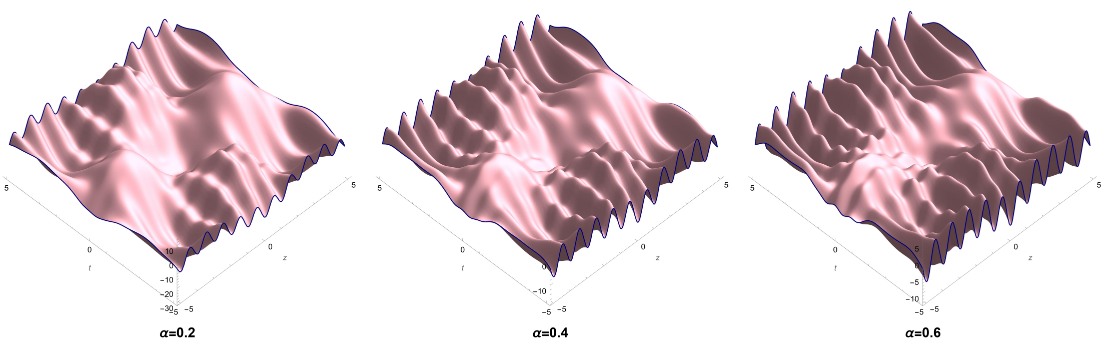

Figure 5: 3D plot of $u_5(1.5, 2.5, z, t)$ for different values of α .

(f): Through inserting

$$\varphi(\xi, t) = \log(1 + x^2)\cos(2\pi t), \tag{68a}$$

$$f_1(z, t) = \frac{\sin(z^2) + \cos(3t)}{1 + z^2}, \tag{68b}$$

$$\Lambda(z, t) = (z^2 + 2)\cos(0.5\pi t), \tag{68c}$$

in Eq.(17), we have

$$
\begin{aligned}
u_6(x,y,z,t) &= \log\left(\left(\left(\frac{\cos(3t)+\sin\left(z^2\right)}{z^2+1}+\alpha x\right)^2+1\right)\cos(2\pi t)\right. \\
&\quad - \frac{3y\left(\frac{2z\cos\left(z^2\right)}{z^2+1}-\frac{2z\left(\cos(3t)+\sin\left(z^2\right)\right)}{\left(z^2+1\right)^2}\right)}{2\alpha} + \left(z^2+2\right)\cos((0.5\pi t).
\end{aligned} \tag{69}
$$

In this solution, we capture complex oscillations with logarithmic variability influenced by both spatial position and temporal cycles. This indicates various potential states of energy or information flowing in a medium, as the higher-order interactions from the compositional functions suggest a rich interplay of forces or fields in a transient environment. Several dynamics of this solution for different values of α are displayed in Fig.6.

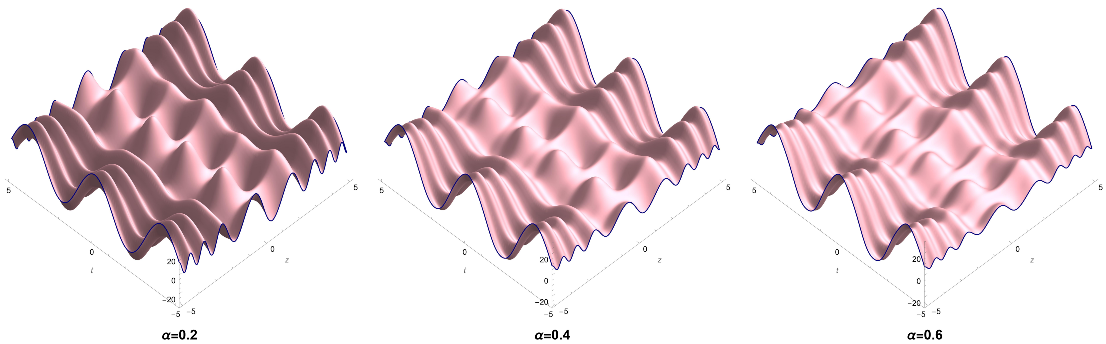

Figure 6: 3D plot of $u_6(x, 0.5, z, 0.2)$ for different values of α .

■ Here, we will explore several non-traveling exact solutions obtained from the solution (27), as follows.

(a): By inserting

$$g(t) = \cos(2\pi t) + 0.5\sin(5\pi t), \quad f_1(z,t) = 1 + 0.5\cos(3z+t), \tag{70a}$$

$$f_2(z,t) = e^{-(0.1z)^2}\sin(4t), \quad \Lambda(z,t) = 1 - e^{-0.05(z+t)}\cos(2z), \tag{70b}$$

in Eq.(27), one gets

$$
\begin{aligned}
u_7(x,y,z,t) &= y(0.5\cos(t+3z)+1)(0.5\sin(5\pi t)+\cos(2\pi t))e^{-\alpha x - e^{-0.01z^2}\sin(4t)} \\
&\times\left(\frac{0.03e^{-0.01z^2}z\sin(4t)}{\alpha} - \frac{2.25\sin(t+3z)}{\alpha(0.5\cos(t+3z)+1)}\right) + 1 - e^{-0.05(t+z)}\cos(2z).
\end{aligned} \tag{71}
$$

The solution $u_7(x,y,z,t)$ models the dynamic system influenced by spatial, temporal, and oscillatory factors. The interplay between the various components represents a wave-like

phenomenon in three dimensions, where f_1 and f_2 contribute spatial complexity while g and Λ introduce time-dependent modulation. The resulting solution encapsulates the behavior of a waveform that evolves in time and space, bounded within specified limits.
(b): By incorporating

$$g(t) = \sin(t) + 0.5t^2, \quad f_1(z,t) = e^{-0.1z^2}\cos(0.5t + z), \tag{72a}$$

$$f_2(z,t) = (1 + \cos(z + \frac{t}{2}))^2, \Lambda(z,t) = \sin(zt) + 0.5z, \tag{72b}$$

in Eq.(27), it reads

$$\begin{aligned} u_8(x,y,z,t) = & y\left(0.5t^2 + \sin(t)\right)\cos(0.5t + z)\, e^{-\alpha x - (\cos(0.5t+z)+1)^2 - 0.1z^2} \\ & \cdot \left(\frac{-3\sin(0.5t + z) - 0.6z\cos(0.5t + z)}{2\alpha\sec(0.5t + z)} + \frac{3\sin(0.5t + z)\left(\cos(0.5t + z) + 1\right)}{\alpha}\right) \\ & + \sin(tz) + 0.5z. \end{aligned} \tag{73}$$

The solution $u_8(x,y,z,t)$ represents a dynamical system influenced by various spatial and temporal factors. The components f_1, f_2, and Λ introduce oscillatory behavior and modulated growth dependent on the dimensions z and time t, while g contributes a smooth, evolving amplitude profile. The derivatives incorporate interactions resulting in shifts of equilibrium state affected by the spatial dimensions and oscillations.
(c): By incorporating

$$g(t) = \sin^2(t) + 0.1, \quad f_1(z,t) = e^{-(z^2+t^2)}\cos(2\pi t) + 0.5, \tag{74a}$$

$$f_2(z,t) = 2e^{-0.4z^2} + \cos(\pi z), \Lambda(z,t) = \sin(0.5z + t) + 1, \tag{74b}$$

in Eq.(27), we obtain

$$\begin{aligned} u_9(x,y,z,t) = & \sin(t + 0.5z) + 1 + y\left(\sin^2(t) + 0.1\right)\left(e^{-t^2-z^2}\cos(2\pi t) + 0.5\right) \\ & \left(-\frac{3ze^{-t^2-z^2}\cos(2\pi t)}{\alpha\left(e^{-t^2-z^2}\cos(2\pi t) + 0.5\right)} - \frac{3\left(-1.6e^{-0.4z^2}z - \pi\sin(\pi z)\right)}{2\alpha}\right) e^{-\alpha x - 2e^{-0.4z^2} - \cos(\pi z)}. \end{aligned} \tag{75}$$

This solution models the dynamic behavior of a physical system that exhibits oscillatory and wave-like characteristics in three-dimensional space. The components of the solution contribute to the spatial and temporal modulation of the system. The presence of exponential and trigonometric terms suggests a phenomenon where waves propagate through a medium, influenced by both spatial coordinates and time.
(d): Through the insertion of

$$g(t) = \sin(t^2) + 0.5\cos\left(\frac{t}{3}\right), \quad f_1(z,t) = e^{-\frac{1}{2}z^2}\sin(2z + t), \tag{76a}$$

$$f_2(z,t) = 0.3z^2 + 0.5\sin(2t) + 0.2\cos(3z), \quad \Lambda(z,t) = 5\sin(z) + 3\cos(t) + \frac{z^2}{10}, \tag{76b}$$

in Eq.(27), we derive

$$u_{10}(x,y,z,t) = y\sin(t+2z)\left(\sin\left(t^2\right) + 0.5\cos\left(\frac{t}{3}\right)\right)e^{-\alpha x - 0.5\sin(2t) - 0.8z^2 - 0.2\cos(3z)}$$
$$\left(\frac{3e^{\frac{z^2}{2}}\csc(t+2z)\left(2e^{-\frac{z^2}{2}}\cos(t+2z) - e^{-\frac{z^2}{2}}z\sin(t+2z)\right)}{2\alpha} - \frac{3(0.6z - 0.6\sin(3z))}{2\alpha}\right)$$
$$+ 3\cos(t) + \frac{z^2}{10} + 5\sin(z). \tag{77}$$

The derived solution $u_{10}(x,y,z,t)$ describes a wave-like phenomenon that varies in space and time, influenced by factors such as external perturbations, dissipation due to the parameter α, and interactions between spatial components captured in f_1 and f_2. The oscillatory behavior of the solution may model various physical systems, such as fluid dynamics or heat transfer, where bounded fluctuations are crucial for stability analysis.

(e): By substituting the selections of

$$g(t) = \sin(t), \quad f_1(z,t) = (1 + 0.2\sin(3z+t)), \tag{78a}$$
$$f_2(z,t) = e^{-0.1z}\sin(0.5t)\cos(1.2z), \quad \Lambda(z,t) = \sin(z+t), \tag{78b}$$

into Eq.(27), we obtain the following solution

$$u_{11}(x,y,z,t) = y\sin(t)(0.2\sin(t+3z)+1)$$
$$\left(\frac{0.9\cos(t+3z)}{\alpha(0.2\sin(t+3z)+1)} - \frac{3\left(-1.2e^{-0.1z}\sin(0.5t)\sin\left(1.2z\right) - 0.1e^{-0.1z}\sin(0.5t)\cos(1.2z)\right)}{2\alpha}\right)$$
$$\times\, e^{-\alpha x - e^{-0.1z}\sin(0.5t)\cos(1.2z)} + \sin(t+z). \tag{79}$$

This solution represents a dynamic field that varies in space and time, influenced by the oscillatory characteristics of its components. The dependence on spatial variables x, y, and z indicates that it models a scenario with wave-like behavior influenced by time-varying parameters. The inclusion of smooth, non-singular functions facilitates a bounded and smooth spatial distribution, keeping values within a controlled range, indicating some form of constraint or equilibrium in the physical context.

(f): By Considering

$$g(t) = e^{\sin(t) + 0.5\cos(2t)}, \quad f_1(z,t) = \sin(z) + 0.5\cos(z+t) + 1, \tag{80a}$$
$$f_2(z,t) = \cos(z)\sin(t), \quad \Lambda(z,t) = 2\sin(z)\cos(t), \tag{80b}$$

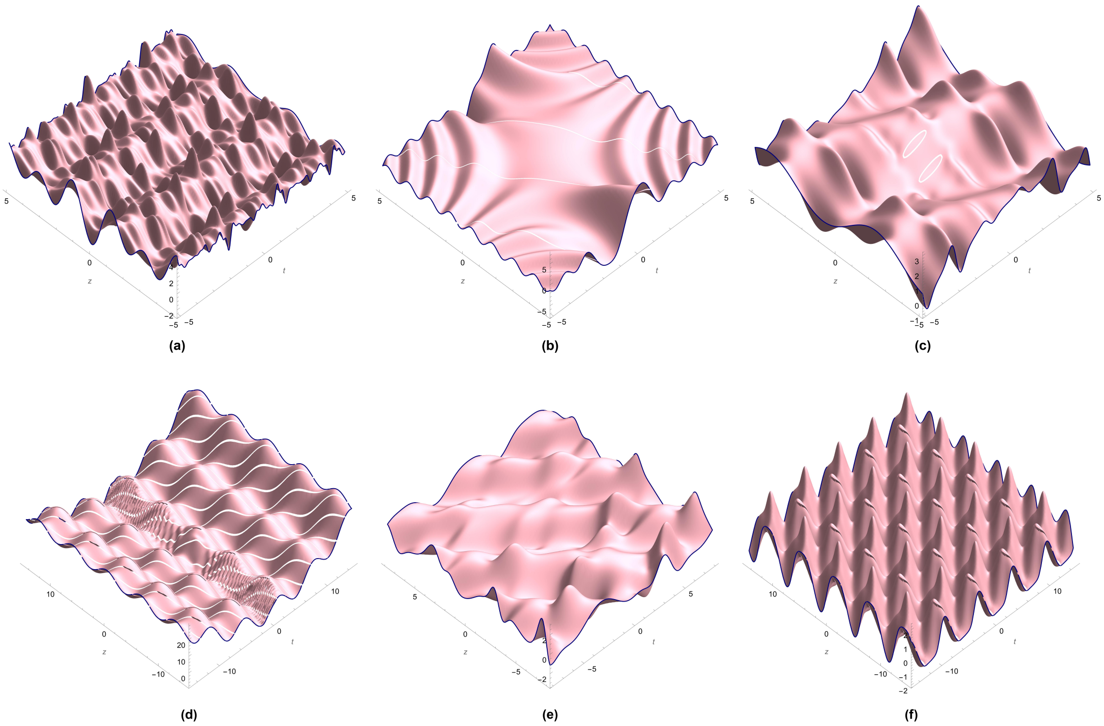

Figure 7: 3D plots of $u_7(1,1,z,t) - u_{12}(1,1,z,t)$ with $\alpha = 1$.

in Eq.(27), we have

$$u_{12}(x,y,z,t) = y(0.5\cos(t+z) + \sin(z) + 1)\left(\frac{3\sin(t)\sin(z)}{2\alpha} + \frac{3(\cos(z) - 0.5\sin(t+z))}{2\alpha(0.5\cos(t+z) + \sin(z) + 1)}\right)$$
$$e^{-\alpha x - \sin(t)\cos(z) + \sin(t) + 0.5\cos(2t)} + 2\cos(t)\sin(z). \tag{81}$$

The solution $u_{12}(x,y,z,t)$ models the dynamic system influenced by spatial, temporal, and oscillatory factors. The interplay between the various components represents a wave-like phenomenon in three dimensions, where f_1 and f_2 contribute spatial complexity while g and Λ introduce time-dependent modulation. The resulting solution encapsulates the behavior of a waveform that evolves in time and space, bounded within specified limits. In Fig.7, we have plotted the 3D dynamics of solutions $u_7(1,1,z,t) - u_{12}(1,1,z,t)$ along with taking $\alpha = 1$.

■ Moreover, some solutions derived from the solution (37) are plotted in Fig.8. In this Figure, we have taken $\alpha = \varepsilon_0 = 1$, corresponding to: **(a)** $f_1(z,t) = \sin(z+t), f_2(y,z) = \cos(y+z), \Lambda(z,t) = (1+e^{z^2+t^2})^{-1}$, **(b)** $f_1(z,t) = \tanh(zt), f_2(y,z) = \sin(z+y), \Lambda(z,t) = (1+e^{z^2+t^2})^{-1}$, **(c)** $f_1(z,t) = \cos(zt), f_2(y,z) = \cos(z+y), \Lambda(z,t) = \tanh(z^2+t^2)$.

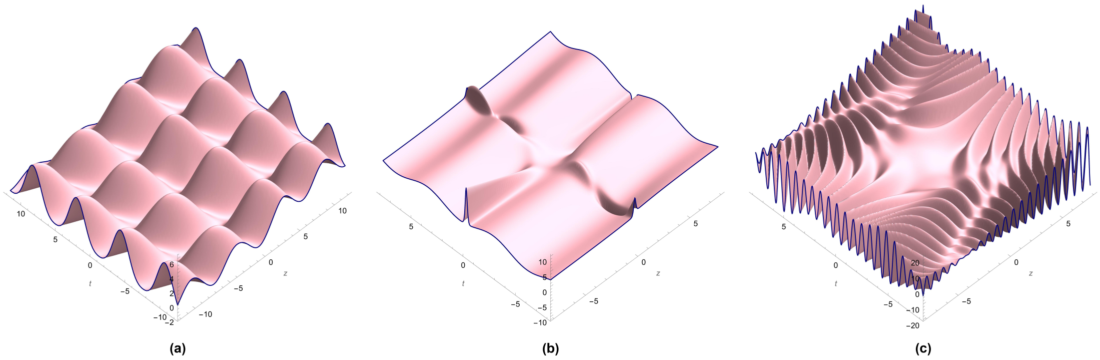

Figure 8: 3D plot of proposed in Eq.(37) with $\alpha = \varepsilon_0 = 1$ along with $\kappa = 0.5$, corresponding to: **(a)** $f_1(z,t) = \sin(z+t), f_2(y,z) = \cos(y+z), \Lambda(z,t) = \frac{1}{1+e^{z^2+t^2}}$, **(b)** $f_1(z,t) = \tanh(zt), f_2(y,z) = \sin(z+y), \Lambda(z,t) = \frac{1}{1+e^{z^2+t^2}}$, **(c)** $f_1(z,t) = \cos(zt), f_2(y,z) = \cos(z+y), \Lambda(z,t) = \tanh(z^2+t^2)$.

5. Conclusion

This paper successfully presents a novel approach for deriving a variety of explicit non-traveling wave solutions to the $(3+1)$-dimensional nonlinear evolution equation using a modified generalized variable separation technique. The solutions obtained not only encompass a richer diversity of non-travelling forms, including periodic solitary waves and soliton-like structures but also demonstrate the effectiveness of the proposed method in addressing complex nonlinear partial differential equations. Each solution's validity was rigorously confirmed through symbolic computational methods using Maple, reinforcing the reliability of these findings. The proposed method offers several advantages, including its novelty and versatility in deriving a wide variety of non-traveling wave solutions, such as periodic solitary waves and soliton-like structures, for the (3+1)-dimensional nonlinear evolution equation. Its effectiveness in addressing complex nonlinear partial differential equations is demonstrated through the obtained solutions, which are rigorously validated using symbolic computational tools like Maple. However, the method also presents certain limitations. The complexity of the mathematical transformations and assumptions involved may restrict its accessibility to researchers without a strong mathematical background. Additionally, the approach is specifically tailored to the (3+1)-dimensional nonlinear evolution equation, and its applicability to other types of equations remains to be explored. Furthermore, the reliance on symbolic computation tools suggests that the method may be computationally intensive for certain cases. Despite these limitations, the method represents a significant contribution to the field, providing a foundation for future research in nonlinear dynamics and wave phenomena. The results obtained in this article for the equation can be considered innovative findings for the main equation, which have not been presented in previous literature. The findings of this paper provide new insights into non-traveling wave phenomena. These insights establish a foundation for future research in nonlinear dynamics.

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