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A Numerical Solution for Nash Differential Games Based on the Runge Kutta 4th-Order Method

Abd El-Monem A. Megahed[1,*], Nesreen M. Kamel[1], I. M. Hanafy[2], Nora A. Omar[2]

[1] *Department of Basic Science, Faculty of Computers and Informatics, Suez Canal University, Ismailia 41522, Egypt*
[2] *Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Egypt*

Abstract. In this paper, we present a numerical solution for an open-loop Nash differential game modeling competition between two firms. Using the fourth-order Runge-Kutta method, we computed the numerical solution and analyzed the stability of the open-loop Nash equilibrium. Additionally, we examined the uniform convergence of the solution. Finally, an illustrative example is presented to clarify the results, accompanied by figures to illustrate the findings.

2020 Mathematics Subject Classifications: 91A23, 49N05, 49N70, 49N90

Key Words and Phrases: Open-loop Nash differential game, Runge-Kutta fourth-order method, stability, uniform convergence

1. Introduction

Differential games are a branch of game theory that focus on studying and developing optimal control strategies for dynamic systems influenced by the decisions of multiple players [1]. An open-loop Nash differential game is a specific type of differential game in which the strategies of the players depend solely on time (t) rather than the current state of the dynamic system. In other words, players predefine their strategies at the start of the game and do not update them based on subsequent observations or system states. In game theory, differential games are used to model and analyze conflicts, such as competition, within dynamical systems. Differential equations play a key role in many applications in physics, engineering, and the modeling of natural phenomena [2].
An effective approach for solving differential games is through numerical methods. For example, Dehghan Banadaki and Navidi [3] studied open-loop Nash differential games using

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Email addresses: abdelmoneam_mgahed@ci.suez.edu.eg (A. A. Megahed), nesreenmostafakamelbs@ci.suez.edu.eg (N. M. Kamel), ihanafy@hotmail.com (I. M. Hanafy), noraalaa19862013@gmail.com (N. A. Omar)

the Legendre Tau method combined with the fourth-order Runge–Kutta method. Kuvshinov and Osipov [4] investigated Stackelberg solutions for linear positional differential games by applying the polyhedron method. Megahed et al. [5] studied an open-loop Nash differential game and employed the Picard method to approximate the solution of a model describing competition between two firms, incorporating market share, advertising efforts, and advertising costs. Megahed et al. [6] studied a zero-sum differential game for modeling coronavirus dynamics, solving it using the homotopy perturbation method combined with a new iterative approach. The Stackelberg differential game of E-differentiable and E-convex functions was applied to combat terrorism, considering government actions [7]. Hemeda [8] introduced the Integral Iterative Method (IIM), a modification of the Picard method, as a numerical technique for solving nonlinear integro-differential equations and systems. Megahed et al. [9] investigated an open-loop Nash differential game, where they derived the necessary conditions for equilibrium and analyzed the existence and uniform convergence of the solutions using the Picard method. Illustrative figures were provided to demonstrate applications in economic, financial, and industrial contexts.

Kassem et al. [10] discussed a Nash-collaborative approach for a differential game involving multiple governments and terrorist organizations, analyzing government cooperation and the role of each government in counterterrorism. Youness [11] introduced a differential game termed "Nash coalitional," which extends the traditional Nash equilibrium framework by incorporating partial cooperation among players, enabling them to achieve mutually beneficial outcomes while still accounting for individual objectives. Youness et al. [12] examined the necessary conditions for determining optimal strategies in fuzzy continuous differential games under Nash equilibrium. Sun et al. [13] introduced a linear–quadratic stochastic two-person nonzero-sum differential game with both open-loop and closed-loop Nash equilibria. Engwerda [14] analyzed the open-loop Nash equilibrium in a linear-quadratic (LQ) differential game. Min-max differential games with fuzzy objectives and controls, as well as large-scale differential games, have been discussed in [15, 16]. Megahed [17] analyzed terrorism dynamics through a two-player differential game involving the International Terrorism Organization (ITO). Youness et al. [18] examined a min-max differential game whose state trajectory is governed by a Cauchy initial value problem (CIVP). They provided both analytical and approximate numerical solutions using the Picard method, demonstrating the model's effectiveness in dynamic optimization contexts.

In this paper, we present the numerical solution of an open-loop Nash differential game using the fourth-order Runge–Kutta method [19]. The system dynamics, representing competition between two firms in the market, are modeled by differential equations. Each player aims to optimize their objective while accounting for the impact of both their own actions and those of the other player. Section 2 formulates the dynamical system of the problem and derives the necessary conditions for an open-loop Nash equilibrium. Section 3 presents the numerical solution using the fourth-order Runge–Kutta method and discusses its convergence. Section 4 analyzes the stability of the numerical solution, and Section 5 concludes the paper.

2. Problem Formulation

In this section, we discuss the solution of an open-loop Nash differential game between two firms over the time interval $t \in [0, T]$. The system dynamics can be expressed as follows:

$$\frac{dx}{dt} = u_1(t)\big(1 - x(t)\big) - u_2(t)x(t), \tag{1}$$

where $t \in [0, T]$, $\quad x(0) = x_0$, subject to the constraint

$$0 \leq x(t) \leq 1. \tag{2}$$

where $x(t)$ denotes the market share of Firm 1 at time t, and $1 - x(t)$ represents the market share of Firm 2. The control variables $u_1(t)$ and $u_2(t)$ are defined as follows: $u_1(t)$ corresponds to the advertising efforts of Firm 1 at time t, while $u_2(t)$ corresponds to the advertising efforts of Firm 2 at time t. For Firm 1, the number of customers increases due to its own advertising efforts, whereas the advertising efforts of Firm 2 decrease the number of customers of Firm 1. The state dynamics and the payoff functionals are described as follows:

$$J_i\big(u_1(t), u_2(t)\big) = \int_0^T I_i\big(x(t), u_1(t), u_2(t), t\big)\, dt, \quad i = 1, 2. \tag{3}$$

The payoff functionals of the two firms in problems (1) and (2) are defined as follows:

$$\begin{aligned} J_1(u_1(t), u_2(t)) &= \int_0^T e^{-r_1 t}[\phi_1 x(t) - C_1 u_1(t)]\, dt, \\ J_2(u_1(t), u_2(t)) &= \int_0^T e^{-r_2 t}[\phi_2 (1 - x(t)) - C_2 u_2(t)]\, dt. \end{aligned} \tag{4}$$

where r_i is the interest rate of Firm i, ϕ_i is the fractional revenue potential of Firm i, and $C_i(s)$ denotes the advertising cost function. We assume that

$$C_i(s) = \frac{P_i}{2} s^2, \qquad P_i > 0, \quad i = 1, 2.$$

$$\begin{aligned} f(x(t), u_1(t), u_2(t), t) &= u_1(t)\big(1 - x(t)\big) - u_2(t)x(t), \\ I_1(x(t), u_1(t), u_2(t), t) &= \phi_1\, x(t) - C_1\big(u_1(t)\big) = \phi_1\, x(t) - \tfrac{P_1}{2} u_1(t)^2, \\ I_2(x(t), u_1(t), u_2(t), t) &= \phi_2\,(1 - x(t)) - C_2\big(u_2(t)\big) = \phi_2\,(1 - x(t)) - \tfrac{P_2}{2} u_2(t)^2. \end{aligned} \tag{5}$$

Theorem 1. *Let $f(x(t), u_1(t), u_2(t), t)$ and $I_i(x(t), u_1(t), u_2(t), t)$ be continuously differentiable functions on $\mathbb{R}^n$, $i = 1, 2$. Specifically,*

$$f(x(t), u_1(t), u_2(t), t) : \mathbb{R}^n \times \mathbb{R}^s \times [0, T] \to \mathbb{R}, \quad f \in C^1,$$

where $s = \sum_{j=1}^{N} s_j$, $i \neq j$, *and*

$$I_i(x(t), u_1(t), u_2(t), t) : \mathbb{R}^n \times \mathbb{R}^s \times [0, T] \to \mathbb{R}, \quad I_i \in C^1, \quad i = 1, 2.$$

If $u_i^*(t)$, $0 \leq t \leq T$, *is an open-loop Nash equilibrium solution, and* $x^*(t)$, $0 \leq t \leq T$, *is the corresponding state trajectory. Then, there exist two costate vectors* $\lambda_i(t) : [0, T] \to \mathbb{R}^n$ *and two Hamiltonian functions .*

$$H_i\big(\lambda_i(t), x(t), u_1(t), u_2(t), t\big) = I_i\big(x(t), u_1(t), u_2(t), t\big) + \lambda_i(t)^\top f\big(x(t), u_1(t), u_2(t), t\big), \quad i = 1, 2. \tag{6}$$

such that the following necessary conditions hold:

$$\begin{aligned}
\frac{dx^*(t)}{dt} &= f\big(x^*(t), u_1^*(t), u_2^*(t), t\big), \qquad x^*(0) = x_0,\\
\frac{d\lambda_1^*(t)}{dt} &= -\frac{\partial H_1\big(\lambda_1(t), x^*(t), u_1^*(t), u_2^*(t), t\big)}{\partial x},\\
\frac{d\lambda_2^*(t)}{dt} &= -\frac{\partial H_2\big(\lambda_2(t), x^*(t), u_1^*(t), u_2^*(t), t\big)}{\partial x},
\end{aligned} \tag{7}$$

$$\frac{\partial H_1\big(\lambda_1(t), x^*(t), u_1^*(t), u_2^*(t), t\big)}{\partial u_1} = 0,$$

$$\frac{\partial H_2\big(\lambda_2(t), x^*(t), u_1^*(t), u_2^*(t), t\big)}{\partial u_2} = 0.$$

with initial and terminal conditions

$$x^*(0) = x_0, \quad \lambda_1(T) = 0, \quad \lambda_2(T) = 0. \tag{8}$$

The proof of this theorem is presented in [20].

3. The Numerical Solution by Using the Fourth-Order Runge-Kutta Method

3.1. Existence and Convergence of the Solution

Existence of the Solution

After applying the necessary conditions for an open-loop Nash equilibrium differential game, the problem (1)–(5) is reduced to the following Hamiltonian functions.

For Player 1, the Hamiltonian is defined as

$$H_1(\lambda_1(t), x(t), u_1(t), u_2(t), t) = \phi_1 x(t) - \frac{P_1}{2} u_1^2(t) + \lambda_1(t)^T \Big[u_1(t)(1 - x(t)) - u_2(t)x(t)\Big]. \tag{9}$$

And for Player 2, the Hamiltonian is

$$H_2(\lambda_2(t), x(t), u_1(t), u_2(t), t) = \phi_2\big(1-x(t)\big) - \frac{P_2}{2}u_2^2(t) + \lambda_2(t)^T\Big[u_1(t)(1-x(t)) - u_2(t)x(t)\Big]. \tag{10}$$

$$\begin{aligned}
&\frac{dx}{dt} = u_1(t)\big(1-x(t)\big) - u_2(t)x(t), \quad x(0) = x_0,\\
&\frac{d\lambda_1(t)}{dt} = \lambda_1(t)\big[u_1(t) + u_2(t)\big] - \phi_1, \quad \lambda_1(T) = 0,\\
&\frac{d\lambda_2(t)}{dt} = \lambda_2(t)\big[u_1(t) + u_2(t)\big] + \phi_2, \quad \lambda_2(T) = 0,\\
&u_1(t) = \frac{\lambda_1(t)}{P_1}\big(1-x(t)\big),\\
&u_2(t) = -\frac{\lambda_2(t)}{P_2}x(t),\\
&(c_i)_n = \tfrac{p_i}{2}\big(u_i^2\big)_n, \quad i = 1, 2.
\end{aligned} \tag{11}$$

The existence of a solution to system (11) is discussed in [9].

Uniform Convergence of the Solution

Now, we apply the fourth-order Runge–Kutta method to discuss the uniform convergence of the solution. The scheme is given by

$$\begin{aligned}
x_{n+1} &= x_n + \tfrac{h}{6}\,\Phi_1(x_n, \lambda_{1,n}, \lambda_{2,n}, t_n),\\
\lambda_{1,n+1} &= \lambda_{1,n} + \tfrac{h}{6}\,\Phi_2(x_n, \lambda_{1,n}, \lambda_{2,n}, \phi_1, t_n),\\
\lambda_{2,n+1} &= \lambda_{2,n} + \tfrac{h}{6}\,\Phi_3(x_n, \lambda_{1,n}, \lambda_{2,n}, \phi_2, t_n),
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\Phi_1(x_n, \lambda_{1,n}, \lambda_{2,n}, t_n) &= K_{1x} + 2K_{2x} + 2K_{3x} + K_{4x},\\
\Phi_2(x_n, \lambda_{1,n}, \lambda_{2,n}, \phi_1, t_n) &= K_{1\lambda_1} + 2K_{2\lambda_1} + 2K_{3\lambda_1} + K_{4\lambda_1},\\
\Phi_3(x_n, \lambda_{1,n}, \lambda_{2,n}, \phi_2, t_n) &= K_{1\lambda_2} + 2K_{2\lambda_2} + 2K_{3\lambda_2} + K_{4\lambda_2}.
\end{aligned} \tag{13}$$

The Runge-Kutta variables k_i are defined as

$$
\begin{aligned}
K_{1x} &= hF_x(x_n, t_n), & K_{2x} &= hF_x\left(x_n + \frac{h}{2}, t_n + \frac{K_{1x}}{2}\right), \\
K_{3x} &= hF_x\left(x_n + \frac{h}{2}, t_n + \frac{K_{2x}}{2}\right), & K_{4x} &= hF_x\left(x_n + h, t_n + K_{3x}\right), \\
K_{1\lambda_1} &= hF_{\lambda_1}(x_n, t_n), & K_{2\lambda_1} &= hF_{\lambda_1}\left(x_n + \frac{h}{2}, t_n + \frac{K_{1\lambda_1}}{2}\right), \\
K_{3\lambda_1} &= hF_{\lambda_1}\left(x_n + \frac{h}{2}, t_n + \frac{K_{2\lambda_1}}{2}\right), & K_{4\lambda_1} &= hF_{\lambda_1}\left(x_n + h, t_n + K_{3\lambda_1}\right), \\
K_{1\lambda_2} &= hF_{\lambda_2}(x_n, t_n), & K_{2\lambda_2} &= hF_{\lambda_2}\left(x_n + \frac{h}{2}, t_n + \frac{K_{1\lambda_2}}{2}\right), \\
K_{3\lambda_2} &= hF_{\lambda_2}\left(x_n + \frac{h}{2}, t_n + \frac{K_{2\lambda_2}}{2}\right), \quad \text{and} & K_{4\lambda_2} &= hF_{\lambda_2}\left(x_n + h, t_n + K_{3\lambda_2}\right).
\end{aligned} \tag{14}
$$

The sequence $x_{n+1}, \lambda_{1,n+1}$, and $\lambda_{2,n+1}$ can be written as the finite series

$$
\begin{aligned}
x_{n+1} &= x_0 + \sum_{j=0}^{n} \left(x_{j+1} - x_j\right), \\
\lambda_{1,n+1} &= \lambda_{1,0} + \sum_{j=0}^{n} \left(\lambda_{1,j+1} - \lambda_{1,j}\right), \\
\lambda_{2,n+1} &= \lambda_{2,0} + \sum_{j=0}^{n} \left(\lambda_{2,j+1} - \lambda_{2,j}\right).
\end{aligned} \tag{15}
$$

If $x_{n+1}, \lambda_{1,n+1}$ and $\lambda_{2,n+1}$ are convergent, then the infinite series

$$
\sum_{j=0}^{\infty}(x_{j+1} - x_j), \quad \sum_{j=0}^{\infty}\left(\lambda_{1,j+1} - \lambda_{1,j}\right), \quad \sum_{j=0}^{\infty}\left(\left(\lambda_{2,j+1} - \lambda_{2,j}\right)\right)
$$

are convergent, and the solution will be x, λ_1 and λ_2.

$$
\begin{aligned}
\lim_{n\to\infty} x_{n+1} &= x, \\
\lim_{n\to\infty} \lambda_{1,n+1} &= \lambda_1, \\
\lim_{n\to\infty} \lambda_{2,n+1} &= \lambda_2.
\end{aligned} \tag{16}
$$

Assume that

1. The functions

$$
\Phi_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \to \mathbb{R}, \quad i = 1, 2, 3,
$$

are continuous, and there exist positive constants M_i such that

$$|\Phi_i| \leq M_i, \quad i = 1, 2, 3.$$

2. Each function Φ_i satisfies the Lipschitz condition with Lipschitz constants L_i, where $0 < L_i < 1$, $i = 1, 2, 3$, such that

$$|\Phi_i(z_1) - \Phi_i(z_2)| \leq L_i \|z_1 - z_2\|, \quad \forall z_1, z_2 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0, T].$$

In particular,

$$|\Phi_1(x, \lambda_1, \lambda_2, t) - \Phi_1(y, \lambda_1, \lambda_2, t)| < L_1 |x - y|, \tag{17}$$

$$|\Phi_2(x, \lambda_1, \lambda_2, t) - \Phi_2(x, \alpha, \lambda_2, t)| < L_2 |\lambda_1 - \alpha|, \tag{18}$$

$$|\Phi_3(x, \lambda_1, \lambda_2, t) - \Phi_3(x, \lambda_1, \gamma, t)| < L_3 |\lambda_2 - \gamma|. \tag{19}$$

If the three series converge, then the three sequences $x_{n+1}, \lambda_{1,n+1}$ and $\lambda_{2,n+1}$ will converge to x, λ_1 and λ_2, respectively. To discuss the uniform convergence of $x_{n+1}, \lambda_{1,n+1}$ and $\lambda_{2,n+1}$, we consider the following three associated series:

$$\begin{aligned} &\sum_{n=0}^{\infty} (x_{n+1} - x_n), \\ &\sum_{n=0}^{\infty} (\lambda_{1,n+1} - \lambda_{1,n}), \\ &\sum_{n=0}^{\infty} (\lambda_{2,n+1} - \lambda_{2,n}). \end{aligned} \tag{20}$$

$$\begin{aligned} \text{For } n = 0, \quad x_1 - x_0 &= \frac{h}{6} \Phi_1(x_n, \lambda_{1,n}, \lambda_{2,n}, t), \\ |x_1 - x_0| &= \left| \frac{h}{6} \Phi_1(x_n, \lambda_{1,n}, \lambda_{2,n}, t) \right|, \\ |x_1 - x_0| &\leq \frac{h}{6} M_1. \end{aligned} \tag{21}$$

$$\begin{aligned} \text{Similarly,} \quad |\lambda_{1,1} - \lambda_{1,0}| &\leq \frac{h}{6} M_2, \\ |\lambda_{2,1} - \lambda_{2,0}| &\leq \frac{h}{6} M_3. \end{aligned} \tag{22}$$

Now, we will get an estimation for$(x_{n+1} - x_n), (\lambda_{1,n+1} - \lambda_{1,n})$ and $(\lambda_{2,n+1} - \lambda_{2,n})$

$$\begin{aligned}
|x_{n+1}-x_n| &= \left|(x_n-x_{n-1})+\frac{h}{6}\left(\Phi_{1,n}-\Phi_{1,n-1}\right)\right|, \\
|x_{n+1}-x_n| &\le |x_n-x_{n-1}|+\frac{h}{6}\left|\Phi_{1,n}-\Phi_{1,n-1}\right|, \\
|x_{n+1}-x_n| &\le |x_n-x_{n-1}|+\frac{h}{6}\left|L_1(x_n-x_{n-1})\right|, \\
|x_{n+1}-x_n| &\le (1+L_1\frac{h}{6})\,|x_n-x_{n-1}|.
\end{aligned} \tag{23}$$

$$\begin{aligned}
\text{Similarly,}\quad |\lambda_{1,n+1}-\lambda_{1,n}| &\le (1+\frac{h}{6}L_2)\,|\lambda_{1,n}-\lambda_{1,n-1}|, \\
|\lambda_{2,n+1}-\lambda_{2,n}| &\le (1+\frac{h}{6}L_3)\,|\lambda_{2,n}-\lambda_{2,n-1}|.
\end{aligned} \tag{24}$$

$$\begin{aligned}
\text{At } n=1,\quad |x_2-x_1| &\le (1+L_1\frac{h}{6})\,|x_1-x_0| \\
|x_2-x_1| &\le \frac{h}{6}M_1(1+L_1\frac{h}{6}).
\end{aligned} \tag{25}$$

$$\begin{aligned}
\text{At } n=2,\quad |x_3-x_2| &\le (1+L_1\frac{h}{6})\,|x_2-x_1|, \\
|x_3-x_2| &\le \frac{h}{6}M_1(1+L_1\frac{h}{6})^2.
\end{aligned} \tag{26}$$

$$\begin{aligned}
\text{At } n=3,\quad |x_4-x_3| &\le (1+L_1\frac{h}{6})\,|x_3-x_2|, \\
|x_4-x_3| &\le \frac{h}{6}M_1(1+L_1\frac{h}{6})^3.
\end{aligned} \tag{27}$$

$$\text{And so on}\quad |x_{n+1}-x_n| \le \frac{h}{6}M_1(1+L_1\frac{h}{6})^n. \tag{28}$$

$$\begin{aligned}
\text{Similarly,}\quad |\lambda_{1,n+1}-\lambda_{1,n}| &\le \frac{h}{6}M_2(1+L_1\frac{h}{6})^n, \\
|\lambda_{2,n+1}-\lambda_{2,n}| &\le \frac{h}{6}M_2(1+L_1\frac{h}{6})^n.
\end{aligned} \tag{29}$$

Since $L_i \le 1$, $M_i \le 1$ and $h \le 1$ $i = 1, 2, 3$.
Then the series $\sum_{n=0}^{\infty}(x_{n+1}-x_n)$, $\sum_{n=0}^{\infty}(\lambda_{1,n+1}-\lambda_{1,n})$, $\sum_{n=0}^{\infty}(\lambda_{2,n+1}-\lambda_{2,n})$, and the sequences $x_{n+1}, \lambda_{1,n+1}, \lambda_{2,n+1}$ are uniformly convergent .

Since $L_i \le 1$, $M_i \le 1$, and $h \le 1$ for $i = 1, 2, 3$, the series $\sum_{n=0}^{\infty}(x_{n+1}-x_n)$, $\sum_{n=0}^{\infty}(\lambda_{1,n+1}-\lambda_{1,n})$, and $\sum_{n=0}^{\infty}(\lambda_{2,n+1}-\lambda_{2,n})$, as well as the sequences x_{n+1}, $\lambda_{1,n+1}$, and $\lambda_{2,n+1}$, are uniformly convergent.

3.2. Numerical Solution

We now apply the fourth-order Runge–Kutta method to obtain the numerical solution for the problem (11). By substituting the controls u_1 and u_2 into the state and costate equations, we obtain:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{\lambda_1}{P_1}(1-x)^2 + \frac{\lambda_2}{P_2}x^2, \quad x(0) = x_0, \\
\frac{d\lambda_1(t)}{dt} &= \frac{\lambda_1^2}{P_1}(1-x) - \frac{\lambda_1\lambda_2(t)}{P_2}x - \phi_1, \quad \lambda_1(T) = 0, \\
\frac{d\lambda_2(t)}{dt} &= \frac{\lambda_1\lambda_2(t)}{P_1}(1-x) - \frac{\lambda_2^2}{P_2}x + \phi_2, \quad \lambda_2(T) = 0.
\end{aligned} \tag{30}$$

Then, the system becomes :

$$\begin{aligned}
F_x(x_n, t_n) &= \frac{(\lambda_1(t))_n}{P_1}(1-x_n(t))^2 + \frac{(\lambda_2(t))_n}{P_2}x_n^2, \\
F_{\lambda 1}(x_n, t_n) &= \frac{(\lambda_1(t)^2)_n}{P_1}(1-x_n(t)) - \frac{(\lambda_1(t))_n(\lambda_2(t))_n}{P_2}x_n(t) - \phi_1, \\
F_{\lambda 2}(x_n, t_n) &= \frac{(\lambda_1(t))_n(\lambda_2(t))_n}{P_1}(1-x_n(t)) - \frac{(\lambda_2^2(t))_n}{P_2}x_n(t) + \phi_2, \\
(c_i)_{n+1} &= \frac{p_i}{2}(u_i^2)_{n+1}, \quad i = 1, 2.
\end{aligned} \tag{31}$$

with the initial and terminal conditions:

$$x(0) = x_0, \quad (\lambda_1)_0 = 0, \quad (\lambda_2(T))_0 = 0.$$

The general form after applying the Runge-Kutta 4th-order method is as follows,

$$\begin{aligned}
x_{n+1} &= x_n + \frac{h}{6}(K_{1x} + 2K_{2x} + 2K_{3x} + K_{4x}), \\
\lambda_{1,n+1} &= \lambda_{1,n} + \frac{h}{6}(K_{1\lambda_1} + 2K_{2\lambda_1} + 2K_{3\lambda_1} + K_{4\lambda_1}), \\
\lambda_{2,n+1} &= \lambda_{2,n} + \frac{h}{6}(K_{1\lambda_2} + 2K_{2\lambda_2} + 2K_{3\lambda_2} + K_{4\lambda_2}).
\end{aligned} \tag{32}$$

Where

$$
\begin{aligned}
K_{1x} &= hF_x(x_n, t_n), & K_{2x} &= hF_x\left(x_n + \frac{h}{2}, t_n + \frac{K_{1x}}{2}\right),\\
K_{3x} &= hF_x\left(x_n + \frac{h}{2}, t_n + \frac{K_{2x}}{2}\right), & K_{4x} &= hF_x\left(x_n + h, t_n + K_{3x}\right),\\
K_{1\lambda_1} &= hF_{\lambda_1}(x_n, t_n), & K_{2\lambda_1} &= hF_{\lambda_1}\left(x_n + \frac{h}{2}, t_n + \frac{K_{1\lambda_1}}{2}\right),\\
K_{3\lambda_1} &= hF_{\lambda_1}\left(x_n + \frac{h}{2}, t_n + \frac{K_{2\lambda_1}}{2}\right), & K_{4\lambda_1} &= hF_{\lambda_1}\left(x_n + h, t_n + K_{3\lambda_1}\right),\\
K_{1\lambda_2} &= hF_{\lambda_2}(x_n, t_n), & K_{2\lambda_2} &= hF_{\lambda_2}\left(x_n + \frac{h}{2}, t_n + \frac{K_{1\lambda_2}}{2}\right),\\
K_{3\lambda_2} &= hF_{\lambda_2}\left(x_n + \frac{h}{2}, t_n + \frac{K_{2\lambda_2}}{2}\right), \quad \text{and} & K_{4\lambda_2} &= hF_{\lambda_2}\left(x_n + h, t_n + K_{3\lambda_2}\right).
\end{aligned} \tag{33}
$$

At n=0,

$$
x_1 = x_0 + \frac{h}{6}(K_{1x} + 2K_{2x} + 2K_{3x} + K_{4x}) \tag{34}
$$

Where

$$
\begin{aligned}
&K_{1x} = hF_x(x_0, t_0) = h\left[\frac{\lambda_{1,0}(t)}{P_1}(1 - x_0(t))^2 + \frac{(\lambda_2(t))_0}{P_2}x_0^2\right] = 0,\\
&K_{2x} = hF_x\left(x_0 + \frac{h}{2}, t_0 + \frac{K_{1x}}{2}\right) = h\left[\frac{\lambda_1(t_0 + \frac{K_{1x}}{2})}{P_1}(1 - (x_0 + \frac{h}{2}))^2 + \frac{\lambda_2(t_0 + \frac{K_{1x}}{2})}{P_2}(x_0 + \frac{h}{2})^2\right] = 0\\
&K_{3x} = hF_x\left(x_0 + \frac{h}{2}, t_0 + \frac{K_{2x}}{2}\right) = h\left[\frac{\lambda_1(t_0 + \frac{K_{2x}}{2})}{P_1}(1 - (x_0 + \frac{h}{2}))^2 + \frac{\lambda_2(t_0 + \frac{K_{2x}}{2})}{P_2}(x_0 + \frac{h}{2})^2\right] = 0,\\
&K_{4x} = hF_x\left(x_0 + h, t_0 + K_{3x}\right) = h\left[\frac{\lambda_1(t_0 + K_{3x})}{P_1}(1 - (x_0 + h))^2 + \frac{\lambda_2(t_0 + K_{3x})}{P_2}(x_0 + h)^2\right] = 0.
\end{aligned} \tag{35}
$$

Then $X = x_0$.

$$
(\lambda_1)_1 = (\lambda_1)_0 + \frac{h}{6}(K_{1\lambda_1} + 2K_{2\lambda_1} + 2K_{3\lambda_1} + K_{4\lambda_1}) \tag{36}
$$

Where

$$\begin{aligned}
K_{1\lambda_1} &= hF_{\lambda_1}(x_0,t_0) = h\left[\frac{(\lambda_1(t)^2)_0}{P_1}(1-x_0) - \frac{\lambda_{1,0}(t)(\lambda_2(t))_0}{P_2}x_0(t) - \phi_1\right] = -h\phi_1,\\
K_{2\lambda_1} &= hF_{\lambda_1}\left(x_0+\frac{h}{2}, t_0+\frac{K_{1\lambda_1}}{2}\right) = h\left[\frac{\lambda_1\left(t_0+\frac{K_{1\lambda_1}}{2}\right)^2}{P_1}\left(1-\left(x_0+\frac{h}{2}\right)\right)\right.\\
&\quad \left. - \frac{\lambda_1\left(t_0+\frac{K_{1\lambda_1}}{2}\right)\lambda_2\left(t_0+\frac{K_{1\lambda_1}}{2}\right)}{P_2}\left(x_0+\frac{h}{2}\right) - \phi_1\right] = -h\phi_1,\\
K_{3\lambda_1} &= hF_{\lambda_1}\left(x_0+\frac{h}{2}, t_0+\frac{K_{2\lambda_1}}{2}\right) = h\left[\frac{\lambda_1\left(t_0+\frac{K_{2\lambda_1}}{2}\right)^2}{P_1}\left(1-\left(x_0+\frac{h}{2}\right)\right)\right.\\
&\quad \left. - \frac{\lambda_1\left(t_0+\frac{K_{2\lambda_1}}{2}\right)\lambda_2\left(t_0+\frac{K_{2\lambda_1}}{2}\right)}{P_2}\left(x_0+\frac{h}{2}\right) - \phi_1\right] = -h\phi_1,\\
K_{4\lambda_1} &= hF_{\lambda_1}\left(x_0+h, t_0+K_{3\lambda_1}\right) = h\left[\frac{\lambda_1\left(t_0+K_{3\lambda_1}\right)^2}{P_1}\left(1-\left(x_0+h\right)\right)\right.\\
&\quad \left. - \frac{\lambda_1\left(t_0+K_{3\lambda_1}\right)\lambda_2\left(t_0+K_{3\lambda_1}\right)}{P_2}(x_0+h) - \phi_1\right] = -h\phi_1.
\end{aligned} \tag{37}$$

Then $\lambda_{1,1} = -h^2\phi_1$

$$\lambda_{2,n+1} = \lambda_{2,n} + \frac{h}{6}(K_{1\lambda_2} + 2K_{2\lambda_2} + 2K_{3\lambda_2} + K_{4\lambda_2}) \tag{38}$$

$$\begin{aligned}
K_{1\lambda_2} &= hF_{\lambda_2}(x_n,t_n),\\
K_{2\lambda_2} &= hF_{\lambda_2}\left(x_n+\frac{h}{2}, t_n+\frac{K_{1\lambda_2}}{2}\right),\\
K_{3\lambda_2} &= hF_{\lambda_2}\left(x_n+\frac{h}{2}, t_n+\frac{K_{2\lambda_2}}{2}\right),\\
K_{4\lambda_2} &= hF_{\lambda_2}\left(x_n+h, t_n+K_{3\lambda_2}\right).
\end{aligned} \tag{39}$$

Where

$$
\begin{aligned}
K_{1\lambda_2} &= hF_{\lambda_2}(x_0,t_0) = h\left[\frac{\lambda_{1,0}(t)(\lambda_2(t))_0}{P_1}(1-x_0) - \frac{\lambda_{2,0}^2(t)}{P_2}x_0(t)+\phi_1\right] = h\phi_2,\\
K_{2\lambda_2} &= hF_{\lambda_2}\left(x_0+\frac{h}{2}, t_0+\frac{K_{1\lambda_2}}{2}\right) = h\left[\frac{\lambda_1\left(t_0+\frac{K_{1\lambda_2}}{2}\right)\lambda_2\left(t_0+\frac{K_{1\lambda_2}}{2}\right)}{P_1}\left(1-\left(x_0+\frac{h}{2}\right)\right)\right.\\
&\quad\left. - \frac{\lambda_2\left(t_0+\frac{K_{1\lambda_2}}{2}\right)^2}{P_2}\left(x_0+\frac{h}{2}\right)+\phi_1\right] = h\phi_2,\\
K_{3\lambda_2} &= hF_{\lambda_2}\left(x_0+\frac{h}{2}, t_0+\frac{K_{2\lambda_2}}{2}\right) = h\left[\frac{\lambda_1\left(t_0+\frac{K_{2\lambda_2}}{2}\right)\lambda_2\left(t_0+\frac{K_{2\lambda_2}}{2}\right)}{P_1}\left(1-\left(x_0+\frac{h}{2}\right)\right)\right.\\
&\quad\left. - \frac{\lambda_2\left(t_0+\frac{K_{2\lambda_2}}{2}\right)^2}{P_2}\left(x_0+\frac{h}{2}\right)+\phi_1\right] = h\phi_2,\\
K_{4\lambda_2} &= hF_{\lambda_2}\left(x_0+h, t_0+K_{3\lambda_2}\right) = h\left[\frac{\lambda_1\left(t_0+K_{3\lambda_2}\right)\lambda_2\left(t_0+K_{3\lambda_2}\right)}{P_1}\left(1-(x_0+h)\right)\right.\\
&\quad\left. - \frac{\lambda_2\left(t_0+K_{3\lambda_2}\right)^2}{P_2}(x_0+h)+\phi_1\right] = h\phi_2.
\end{aligned}
\tag{40}
$$

Then $\lambda_{2,1} = h^2\phi_2$

$$
\begin{aligned}
u_{1,1} &= \frac{\lambda_{1,1}}{P_1}(1-x_1) = -\frac{h^2\phi_1}{P_1}(1-x_0),\\
u_{2,1} &= -\frac{\lambda_2}{P_2}x = -\frac{h^2\phi_2}{P_2}x,\\
c_{1,1} &= \frac{p_1}{2}u_{1,1}^2 = \frac{p_1}{2}\left(\frac{h^2\phi_1}{P_1}(1-x_0)\right)^2,\\
c_{2,1} &= \frac{p_2}{2}u_{2,1}^2 = \frac{p_2}{2}\left(\frac{h^2\phi_2}{P_2}x\right)^2.
\end{aligned}
\tag{41}
$$

At n=1

$$x_2 = x_1 + \frac{h}{6}(K_{1x}+2K_{2x}+2K_{3x}+K_{4x}) \tag{42}$$

Where

$$\begin{aligned}
K_{1x} &= hF_x(x_1,t_1) = h\Bigg[\frac{(\lambda_1(t))_1}{P_1}(1-x_1(t))^2 + \frac{(\lambda_2(t))_1}{P_2}x_1^2\Bigg],\\
K_{2x} &= hF_x\left(x_1+\frac{h}{2}, t_1+\frac{K_{1x}}{2}\right) = h\Bigg[\frac{\lambda_1\left(t_1+\frac{K_{1x}}{2}\right)}{P_1}\left(1-\left(x_1+\frac{h}{2}\right)\right)^2 + \frac{\lambda_2\left(t_1+\frac{K_{1x}}{2}\right)}{P_2}\left(x_1+\frac{h}{2}\right)^2\Bigg],\\
K_{3x} &= hF_x\left(x_1+\frac{h}{2}, t_1+\frac{K_{2x}}{2}\right) = h\Bigg[\frac{\lambda_1\left(t_1+\frac{K_{2x}}{2}\right)}{P_1}\left(1-\left(x_1+\frac{h}{2}\right)\right)^2 + \frac{\lambda_2\left(t_1+\frac{K_{2x}}{2}\right)}{P_2}\left(x_1+\frac{h}{2}\right)^2\Bigg],\\
K_{4x} &= hF_x\left(x_1+h, t_1+K_{3x}\right) = h\Bigg[\frac{\lambda_1\left(t_1+K_{3x}\right)}{P_1}\left(1-(x_1+h)\right)^2 + \frac{\lambda_2\left(t_1+K_{3x}\right)}{P_2}(x_1+h)^2\Bigg].
\end{aligned} \tag{43}$$

Then

$$x_2 = x_0+\frac{h^4}{6}\left[\frac{\phi_1}{p_1}\left((1-x_0)^2+4(1-x_0-\frac{h}{2})^2+(1-x_0-h)^2\right)-\frac{\phi_2}{p_2}\left((x_0)^2+4(x_0+\frac{h}{2})^2+(x_0+h)^2\right)\right].$$

$$(\lambda_1)_2 = \lambda_{1,1} + \frac{h}{6}(K_{1\lambda_1} + 2K_{2\lambda_1} + 2K_{3\lambda_1} + K_{4\lambda_1}) \tag{44}$$

Where

$$\begin{aligned}
K_{1\lambda_1} &= hF_{\lambda_1}(x_1,t_1) = h\Bigg[\frac{(\lambda_1^2(t))_1}{P_1}(1-x_1) - \frac{(\lambda_1(t))_1(\lambda_2(t))_1}{P_2}x_1(t) - \phi_1\Bigg],\\
K_{2\lambda_1} &= hF_{\lambda_1}\left(x_1+\frac{h}{2}, t_1+\frac{K_{1\lambda_1}}{2}\right) = h\Bigg[\frac{\lambda_1(t_1+\frac{K_{1\lambda_1}}{2})^2}{P_1}(1-(x_1+\frac{h}{2}))\\
&\quad - \frac{\lambda_1(t_1+\frac{K_{1\lambda_1}}{2})\lambda_2(t_1+\frac{K_{1\lambda_1}}{2})}{P_2}(x_1+\frac{h}{2}) - \phi_1\Bigg],\\
K_{3\lambda_1} &= hF_{\lambda_1}\left(x_1+\frac{h}{2}, t_1+\frac{K_{2\lambda_1}}{2}\right) = h\Bigg[\frac{\lambda_1(t_1+\frac{K_{2\lambda_1}}{2})^2}{P_1}(1-(x_1+\frac{h}{2}))\\
&\quad - \frac{\lambda_1(t_1+\frac{K_{2\lambda_1}}{2})\lambda_2(t_1+\frac{K_{2\lambda_1}}{2})}{P_2}(x_1+\frac{h}{2}) - \phi_1\Bigg],\\
K_{4\lambda_1} &= hF_{\lambda_1}\left(x_1+h, t_1+K_{3x}\right) = h\Bigg[\frac{\lambda_1(t_1+K_{3\lambda_1})^2}{P_1}(1-(x_1+h))\\
&\quad - \frac{\lambda_1(t_1+K_{3\lambda_1})\lambda_2(t_1+K_{3\lambda_1})}{P_2}(x_1+h) - \phi_1\Bigg].
\end{aligned} \tag{45}$$

$$\begin{aligned}
\lambda_{1,2} = -h^2\phi_1 + \frac{h^6}{6}\Bigg[&\frac{(\phi_1)^2}{P_1}\big((1-x_0)+4(1-x_0-\frac{h}{2})+(1-x_0-h)\big)\\
&- \frac{\phi_1\phi_2}{P_2}\big(x_0+4(x_0+\frac{h}{2})+(x_0+h)\big)\Bigg] + \phi_1 h^2
\end{aligned} \tag{46}$$

$$
\begin{aligned}
K_{1\lambda_1} &= hF_{\lambda_1}(x_1,t_1) = h\Big[\frac{\lambda_{1,1}(t)}{P_1}(1-x_1) - \frac{\lambda_{1,1}(t)\lambda_{2,1}(t)}{P_2}x_1(t) - \phi_1\Big],\\
K_{2\lambda_1} &= hF_{\lambda_1}\left(x_1+\frac{h}{2}, t_1+\frac{K_{1\lambda_1}}{2}\right) = h\Big[\frac{\lambda_1\big(t_1+\frac{K_{1\lambda_1}}{2}\big)^2}{P_1}\big(1-(x_1+\frac{h}{2})\big)\\
&\quad - \frac{\lambda_1\big(t_1+\frac{K_{1\lambda_1}}{2}\big)\lambda_2\big(t_1+\frac{K_{1\lambda_1}}{2}\big)}{P_2}(x_1+\frac{h}{2}) - \phi_1\Big],\\
K_{3\lambda_1} &= hF_{\lambda_1}\left(x_1+\frac{h}{2}, t_1+\frac{K_{2\lambda_1}}{2}\right) = h\Big[\frac{\lambda_1\big(t_1+\frac{K_{2\lambda_1}}{2}\big)^2}{P_1}\big(1-(x_1+\frac{h}{2})\big)\\
&\quad - \frac{\lambda_1\big(t_1+\frac{K_{2\lambda_1}}{2}\big)\lambda_2\big(t_1+\frac{K_{2\lambda_1}}{2}\big)}{P_2}(x_1+\frac{h}{2}) - \phi_1\Big],\\
K_{4\lambda_1} &= hF_{\lambda_1}(x_1+h, t_1+K_{3\lambda_1}) = h\Big[\frac{\lambda_1\big(t_1+K_{3\lambda_1}\big)^2}{P_1}(1-(x_1+h))\\
&\quad - \frac{\lambda_1\big(t_1+K_{3\lambda_1}\big)\lambda_2\big(t_1+K_{3\lambda_1}\big)}{P_2}(x_1+h) - \phi_1\Big].
\end{aligned} \tag{47}
$$

Similarly,

$$
\begin{aligned}
\lambda_{2,2} &= h^2\phi_2 - \frac{h^6}{6}\Bigg[\frac{\phi_1\phi_2}{P_1}\big((1-x_0)+4(1-x_0-\frac{h}{2})+(1-x_0-h)\big)\\
&\quad + \frac{(\phi_2)^2}{P_2}\big(x_0+4(x_0+\frac{h}{2})+(x_0+h)\big)\Bigg] + \phi_2 h^2.
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
u_{1,2} &= \frac{\lambda_{1,2}}{P_1}(1-x_2) \\
&= \frac{1}{P_1}\Bigg[\Bigg(1-\Bigg(x_0+\frac{h^4}{6}\Bigg[\frac{\phi_1}{p_1}\big((1-x_0)^2+4(1-x_0-\frac{h}{2})^2+(1-x_0-h)^2\big) \\
&\quad -\frac{\phi_2}{p_2}\big(x_0^2+4(x_0+\frac{h}{2})^2+(x_0+h)^2\big)\Bigg]\Bigg)\Bigg)\Bigg] \\
&\quad \cdot\Bigg[h^2\phi_1+\frac{h^6}{6}\Bigg(\frac{(\phi_1)^2}{P_1}\big((1-x_0)+4(1-x_0-\frac{h}{2})+(1-x_0-h)\big) \\
&\quad -\frac{\phi_1\phi_2}{P_2}\big(x_0+4(x_0+\frac{h}{2})+(x_0+h)\big)\Bigg)+\phi_1 h^2\Bigg], \qquad (49) \\
u_{2,2} &= -\frac{\lambda_{2,2}}{P_2}x_2 \\
&= -\frac{1}{P_2}\Bigg[-h^2\phi_2-\frac{h^6}{6}\Bigg(\frac{\phi_1\phi_2}{P_1}\big((1-x_0)+4(1-x_0-\frac{h}{2})+(1-x_0-h)\big) \\
&\quad +\frac{(\phi_2)^2}{P_2}\big(x_0+4(x_0+\frac{h}{2})+(x_0+h)\big)\Bigg)+\phi_2 h^2\Bigg] \\
&\quad \cdot\Bigg[x_1+\frac{h}{6}(K_{1x}+2K_{2x}+2K_{3x}+K_{4x})\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
c_{1,2} &= \frac{p_1}{2}(u_1^2)_2 \\
&= \frac{p_1}{2}\Bigg[\frac{1}{P_1}\Bigg[1-\Bigg(x_0+\frac{h^4}{6}\Bigg[\frac{\phi_1}{p_1}\big((1-x_0)^2+4(1-x_0-\frac{h}{2})^2+(1-x_0-h)^2\big) \\
&\quad -\frac{\phi_2}{p_2}\big(x_0^2+4(x_0+\frac{h}{2})^2+(x_0+h)^2\big)\Bigg]\Bigg)\Bigg] \\
&\quad \cdot\Bigg[h^2\phi_1+\frac{h^6}{6}\Bigg(\frac{(\phi_1)^2}{P_1}\big((1-x_0)+4(1-x_0-\frac{h}{2})+(1-x_0-h)\big) \\
&\quad -\frac{\phi_1\phi_2}{P_2}\big(x_0+4(x_0+\frac{h}{2})+(x_0+h)\big)\Bigg)+\phi_1 h^2\Bigg]\Bigg]^2, \qquad (50) \\
c_{2,2} &= \frac{p_2}{2}(u_2^2)_2 \\
&= \frac{p_2}{2}\Bigg[-\frac{1}{P_2}\Bigg[-h^2\phi_2-\frac{h^6}{6}\Bigg(\frac{\phi_1\phi_2}{P_1}\big((1-x_0)+4(1-x_0-\frac{h}{2})+(1-x_0-h)\big) \\
&\quad +\frac{(\phi_2)^2}{P_2}\big(x_0+4(x_0+\frac{h}{2})+(x_0+h)\big)\Bigg)+\phi_2 h^2\Bigg] \\
&\quad \cdot\Bigg[x_1+\frac{h}{6}(K_{1x}+2K_{2x}+2K_{3x}+K_{4x})\Bigg]\Bigg]^2
\end{aligned}
$$

Suppose that Firm 2 is already present in the market, with its market share represented by $1-x(t)$. Firm 1 enters the market to compete with Firm 2, and its market share at time t is represented by $x(t)$. We assume that at the beginning of the competition, the initial market share of Firm 1 is $x_0 = 0$. The parameters are set as follows: $\phi_1 = 0.1$, $\phi_2 = 0.3$, $P_1 = 0.25$, $P_2 = 0.5$, the time interval $t \in [0, 1.5]$, and the step size $h = 0.5$.

Then we have a comparison between the two firms.

At n=0,

$$
\begin{aligned}
x_0 &= 0, & 1-x_0 &= 1, \\
u_{1,0} &= 0, & u_{2,0} &= 0, \\
c_{1,0} &= 0, & c_{2,0} &= 0, \\
\lambda_{1,0} &= 0, & \lambda_{2,0} &= 0.
\end{aligned} \qquad (51)
$$

This indicates that, initially, Firm 2 was the only firm in the market and did not undertake any advertising efforts.

At n=1

$$
\begin{aligned}
x_1 &= 0, & 1-x_1 &= 1, \\
u_{1,1} &= 0.1, & u_{2,1} &= 0, \\
c_{1,1} &= 0.00125, & c_{2,1} &= 0, \\
\lambda_{1,1} &= 0.25, & \lambda_{2,1} &= -0.075.
\end{aligned} \qquad (52)
$$

This indicates that initially, Firm 1 entered the market without holding any market share. Consequently, it launched advertising campaigns to increase its market share, while Firm 2 did not engage in any advertising. As a result, the number of customers for Firm 1 increased.

At n=2,

$$\begin{aligned} x_2 &= 0.00989583, & 1 - x_2 &= 0.99010417, \\ u_{1,2} &= 0.18533512, & u_{2,2} &= -0.0001182685, \\ c_{1,2} &= 0.004293638, & c_{2,2} &= 3.496865 \times 10^{-9}, \\ \lambda_{1,2} &= 0.04679687, & \lambda_{2,2} &= -0.005976. \end{aligned} \tag{53}$$

This indicates that initially, Firm 1's market share increased, while Firm 2 began to lose market share and experienced a decrease in sales. As a result of these losses, Firm 2 obtained a loan to finance its advertising campaigns.

At n=3,

$$\begin{aligned} x_3 &= 0.0362395, & 1 - x_3 &= 0.96376, \\ u_{1,3} &= 0.283165810, & u_{2,3} &= 0.010028594, \\ c_{1,3} &= 0.005976423, & c_{2,3} &= 0.00446470, \\ \lambda_{1,3} &= 0.0734531, & \lambda_{2,3} &= -0.082457305. \end{aligned} \tag{54}$$

Now, we present a comparison of the obtained numerical solutions for the two firms in the following figures:

Figure 1 shows the market shares x and $1 - x$, representing Firm 1 and Firm 2, respectively, at time t. It is observed that Firm 1's market share increases over time, while Firm 2's market share decreases. Figure 2 illustrates the controls u_1 and u_2, representing the advertising efforts of Firm 1 and Firm 2, respectively. Initially, Firm 1 increased its advertising efforts to attract more customers, while Firm 2 had no advertising. Over time, Firm 2 gradually increased its advertising to reduce Firm 1's market share. Figure 3 presents the advertising cost functions for the two firms, c_1 for Firm 1 and c_2 for Firm 2. Firm 1's cost is noticeably higher due to its stronger advertising intensity, whereas Firm 2 incurs minimal costs during the early stages of the competition. Figure 4 depicts the costate variables λ_1 and λ_2 for Firm 1 and Firm 2, respectively. For Firm 1, the costate variable starts at zero since it is new to the market and initially has no impact. It then rises as Firm 1 gains influence and faces competition, and later decreases as it approaches the optimal strategy, indicating a balanced state. Conversely, λ_2 initially starts at zero and then decreases to negative values as Firm 2 loses market share due to Firm 1's entry. The negative values indicate that maintaining its market share becomes increasingly costly, highlighting the competitive pressure imposed by Firm 1.

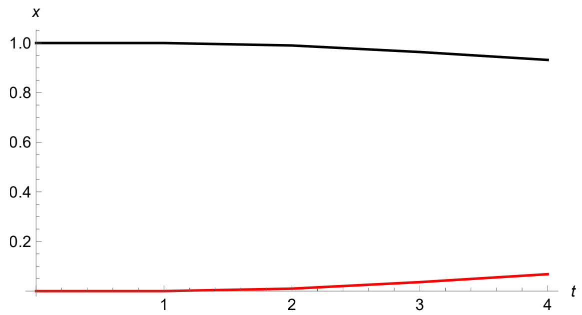

Figure 1: Market shares of Firm 1 and Firm 2 at time t.

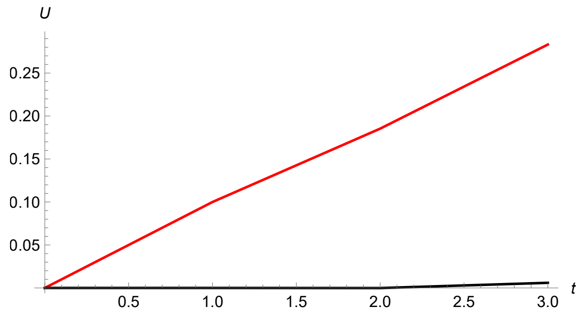

Figure 2: The advertising efforts of Firm 1 and Firm 2 at time t.

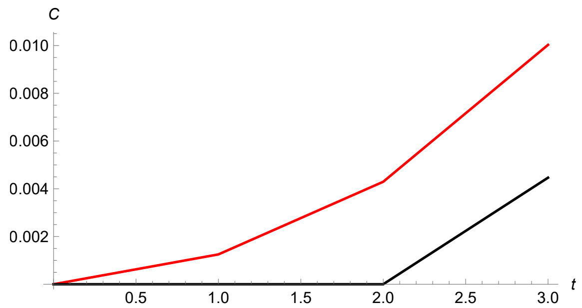

Figure 3: The advertising cost functions of Firm 1 and Firm 2 at the time t

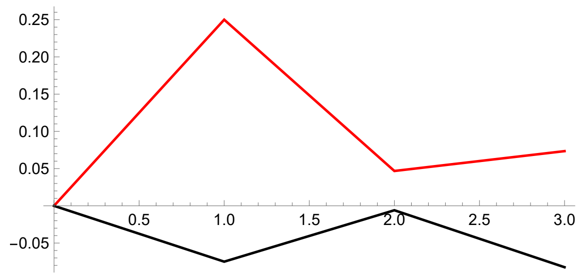

Figure 4: The costate variables of Firm 1 and Firm 2 at time t.

4. Stability Analysis of the Fourth-Order Runge-Kutta Method Using the Jacobian Matrix

In a system of differential equations, stability analysis is performed to determine whether small perturbations around equilibrium points cause the system to return to equilibrium (stable) or to diverge away (unstable). For a nonlinear system:

$$\begin{aligned} \frac{dx}{dt} &= \frac{\lambda_1}{P_1}(1-x)^2 + \frac{\lambda_2}{P_2}x^2, \\ \frac{d\lambda_1(t)}{dt} &= \frac{\lambda_1^2}{P_1}(1-x) - \frac{\lambda_1\lambda_2(t)}{P_2}x - \phi_1, \\ \frac{d\lambda_2(t)}{dt} &= \frac{\lambda_1\lambda_2(t)}{P_1}(1-x) - \frac{\lambda_2^2}{P_2}x + \phi_2. \end{aligned} \tag{55}$$

Equilibrium points (or steady states) of the system are obtained by solving:

$$\begin{aligned} f_1 = \frac{\lambda_1}{P_1}(1-x)^2 + \frac{\lambda_2}{P_2}x^2 = 0, \\ f_2 = \frac{\lambda_1^2}{P_1}(1-x) - \frac{\lambda_1\lambda_2(t)}{P_2}x - \phi_1 = 0, \\ f_3 = \frac{\lambda_1\lambda_2(t)}{P_1}(1-x) - \frac{\lambda_2^2}{P_2}x + \phi_2 = 0. \end{aligned} \tag{56}$$

After solving the system (56) simultaneously, we obtain the equilibrium point $(x^*, \lambda_1^*, \lambda_2^*)$. Next, we construct the Jacobian matrix, which is a square matrix of the first-order partial derivatives of the system's functions with respect to the state variables. The Jacobian matrix J is defined as:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial \lambda_1} & \frac{\partial f_1}{\partial \lambda_2} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial \lambda_1} & \frac{\partial f_2}{\partial \lambda_2} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial \lambda_1} & \frac{\partial f_3}{\partial \lambda_2} \end{bmatrix},$$

Then the jacobian matrix evaluation,

$$J = \begin{bmatrix} \frac{-2\lambda_1^*}{p_1}(1-x^*) + \frac{2\lambda_2^*}{p_2}x & \frac{(1-x^*)^2}{p_1} & \frac{(x^*)^2}{p_2} \\ \frac{-(\lambda_1^*)^2}{p_1} - \frac{\lambda_1\lambda_2^*}{p_2} & \frac{2\lambda_1^*(1-x^*)}{p_1} - \frac{\lambda_2(t)}{P_2}x & -\frac{\lambda_1^* x^*}{p_2} \\ -\frac{\lambda_1^*\lambda_2^*}{p_1} - \frac{(\lambda_2^*)^2}{p_2} & \frac{\lambda_2^*(1-x^*)}{p_1} & \frac{\lambda_1^*(1-x^*)}{p_1} - \frac{2\lambda_2^* x^*}{p_2} \end{bmatrix}.$$

To analyze the stability at an equilibrium point $(x^*, \lambda_1^*, \lambda_2^*)$, the Jacobian matrix $J(x^*, \lambda_1^*, \lambda_2^*)$ is evaluated at the equilibrium. The stability of the system depends on the eigenvalues μ_i of $J(x^*, \lambda_1^*, \lambda_2^*)$. **Stable Equilibrium:** All eigenvalues satisfy $\text{Re}(\mu_i) < 0$, implying local asymptotic stability.

Unstable Equilibrium: At least one eigenvalue satisfies $\text{Re}(\mu_i) > 0$, implying instability.

Marginally Stable Equilibrium: All eigenvalues satisfy $\mathrm{Re}(\mu_i) = 0$; stability cannot be fully determined and the system may be marginally stable or exhibit oscillatory behavior.

To study the stability of the system (55), we first determine the equilibrium points by solving the system (56). We obtain the following equilibrium points:

$$\begin{aligned}
(x^*, \lambda_1^*, \lambda_2^*)_1 &= (1.85786, 0 - 0.617178i, 0 + 0.263177i),\\
(x^*, \lambda_1^*, \lambda_2^*)_2 &= (1.85786, 0 + 0.617178i, 0 - 0.263177i),\\
(x^*, \lambda_1^*, \lambda_2^*)_3 &= (-0.916188, 0 + 0.0640229i, 0 - 0.560109i),\\
(x^*, \lambda_1^*, \lambda_2^*)_4 &= (-0.916188, 0 - 0.0640229i, 0 + 0.560109i),\\
(x^*, \lambda_1^*, \lambda_2^*)_5 &= (0.391662, 0.100038, -0.482684),\\
(x^*, \lambda_1^*, \lambda_2^*)_6 &= (0.391662, -0.100038, 0.482684),
\end{aligned}$$

Then the Jacobian matrix evaluation,

$$J = \begin{bmatrix} \frac{-2\lambda_1^*}{0.25}(1 - x^*) + \frac{2\lambda_2^*}{0.5}x & -\frac{(1-x^*)^2}{0.25} & \frac{(x^*)^2}{0.5} \\ \frac{-(\lambda_1^*)^2}{0.25} - \frac{\lambda_1^*\lambda_2^*}{0.5} & \frac{2\lambda_1^*(1-x^*)}{0.25} & -\frac{\lambda_1^* x^*}{0.5} \\ -\frac{\lambda_1^*\lambda_2^*}{0.25} - \frac{(\lambda_2^*)^2}{0.5} & \frac{\lambda_2^*(1-x^*)}{0.25} & \frac{\lambda_1^*(1-x^*)}{0.25} - \frac{2\lambda_2^* x^*}{0.5} \end{bmatrix}.$$

The eigenvalues μ_i of the Jacobian matrix are computed,

$$\begin{aligned}
J(x^*, \lambda_1^*, \lambda_2^*)_1 &= (-2.23227 - 4.23449i, -2.23227 - 0.97901i, -6.85728 \times 10^{-15} - 2.279837i),\\
J(x^*, \lambda_1^*, \lambda_2^*)_2 &= (-2.23227 + 4.23449i, -2.23227 + 0.97901i, -6.85728 \times 10^{-15} + 2.279837i),\\
J(x^*, \lambda_1^*, \lambda_2^*)_3 &= (-1.04710 - 2.79990i, -1.04710 - 0.79213i, 8.88260 \times 10^{-16} - 1.07122i),\\
J(x^*, \lambda_1^*, \lambda_2^*)_4 &= (-1.04710 + 2.79990i, -1.04710 + 0.79213i, 8.88260 \times 10^{-16} + 1.07122i),\\
J(x^*, \lambda_1^*, \lambda_2^*)_5 &= (-1.24304, -0.62152, -0.51276),\\
J(x^*, \lambda_1^*, \lambda_2^*)_6 &= (-1.24304, -1.24304, -0.62152).
\end{aligned}$$

From the computed eigenvalues of the Jacobian matrix, all eigenvalues have negative real parts, indicating that the equilibrium point of the system is locally asymptotically stable. This means that small perturbations in the market shares or advertising efforts of either firm will decay over time, and the system will return to the equilibrium state. Economically, this implies that both Firm 1 and Firm 2 can maintain stable market shares under the given dynamics, and neither firm will experience unbounded fluctuations in influence or costs.

5. Conclusion

In this paper, we discussed the numerical solution of an open-loop Nash differential game using the fourth-order Runge-Kutta method. This approach allowed us to analyze competition between firms in a dynamic market environment. We studied the convergence of the Runge-Kutta method to ensure the reliability of the numerical solution, and we examined the stability of the solution to confirm that numerical errors do not amplify over time. We also demonstrated that increasing advertising campaigns leads to higher market participation for the firms. Finally, illustrative figures were presented to demonstrate the effect of advertising campaigns on each firm's market share. In future work, we plan to expand this study in several ways. First, we aim to incorporate stochastic elements into the model to analyze how uncertainty influences market behavior and firm strategies. Second, we plan to include more firms, allowing multiple companies to compete simultaneously, which increases the complexity of interactions and may reveal new patterns. Third, we aim to allow strategies to adapt based on the current state of the system, using feedback Nash equilibria, making the model more flexible and closer to real-world competition.

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