EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6012
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Unraveling Symmetry Properties in a Three-Dimensional Nonlinear Evolution Model via the Lie Group Method

Akhtar Hussain[1,*], M. Usman[2], Ahmed M. Zidan[3], Jorge Herrera[4]

[1] *Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan*
[2] *College of Electrical and Mechanical Engineering (CEME), National University of Sciences and Technology (NUST), H-12 Islamabad 44000, Pakistan*
[3] *Department of Mathematics, College of Science, King Khalid University, Abha 61413, Saudi Arabia*
[4] *Facultad de Ciencias Naturales e Ingenieria, Universidad de Bogota Jorge Tadeo Lozano, Bogota 110311, Colombia*

Abstract. This research focuses on a (3+1)-dimensional nonlinear evolution model originating from the Jaulent-Miodek hierarchy. Several tools can be used to examine the symmetry of the model. However, our primary emphasis lies in harnessing one of the most significant and powerful analytical tools available for this purpose: the Lie group method. The Lie group method is an effective approach for uncovering the symmetry properties inherent in a model and exploring group-invariant solutions using symmetry algebra. In addition, we applied Ibragimov's method to study conservation laws relevant to the considered model. Our study is significant as it contributes to the investigation of this model and addresses a particular gap in the group-theoretic approach in this context. Our findings represent novel contributions to the study of the model under consideration.

2020 Mathematics Subject Classifications: 70G65, 35-XX, 70S10, 22E70

Key Words and Phrases: Lie group method, Nonlinear evolution model, Symmetry algebra, Jaulent-Miodek hierarchy, Conservation laws

1. Introduction

Many physical phenomena encountered in mechanics, physics, engineering, biology, and chemistry have been effectively described through nonlinear partial differential equations (PDEs). Nonlinear wave equations frequently encompass a multitude of factors, including dispersion, diffusion, dissipation, reaction, and convection. The pursuit of exact solutions to these equations plays a significant role in our understanding of the wave

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6012

Email addresses: akhtarhussain21@sms.edu.pk (A. Hussain)

propagation. Solitons [1, 2], for instance, emerge from a delicate balance between nonlinearity and linear dispersion, whereas a distinct class of solitons known as "compactons," characterized by the absence of exponential tails, arises from the interplay between nonlinearity and genuinely nonlinear dispersion. The solitary wave theory encompasses a wide array of phenomena that researchers are eager to investigate, with the aim of uncovering the underlying structures within the resulting wave solutions. In references [3–5], four (2+1)-dimensional nonlinear models, stemming from the Jaulent-Miodek hierarchy, were formulated in the following manner

$$\begin{aligned}
\psi_t &= -(\psi_{xx} - 2\psi^3)_x - \frac{3}{2}(\psi_x \partial_x^{-1}\psi_y + \psi\psi_y),\\
\psi_t &= \frac{1}{2}(\psi_{xx} - 2\psi^3)_x + \frac{3}{2}\left(- \frac{1}{4}\partial_x^{-1}\psi_{yy} + \psi\psi_y\right),\\
\psi_t &= -\frac{1}{4}(\psi_{xx} - 2\psi^3)_x - \frac{3}{4}\left(\frac{1}{4}\partial_x^{-1}\psi_{yy} + \psi_x\partial_x^{-1}\psi_y\right),\\
\psi_t &= 2(\psi_{xx} - 2\psi^3)_x - \frac{3}{4}\left(\partial_x^{-1}\psi_{yy} - 2\psi_x\partial_x^{-1}\psi_y - 6\psi\psi_y\right),
\end{aligned} \tag{1}$$

where ∂_x^{-1} is the inverse of ∂_x with $\partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = 1$, and

$$(\partial_x^{-1} g)(x) = \int_{-\infty}^{x} g(t)dt, \tag{2}$$

under decay conditions of infinity. The four models were examined in [3–5] utilizing robust methodologies such as the Wronskian form, simplified Hirota's method, and the perturbation technique introduced by Herman and Nuseir [6]. These investigations resulted in derivation of N soliton solutions for each model.

Wazwaz [7] extended the research presented in [3–5] by delving into the analysis of four (3+1)-dimensional nonlinear models derived from the Jaulent-Miodek hierarchy. He formulated these (3+1)-dimensional models in the following

$$\begin{aligned}
\psi_t &= -(\psi_{xx} - 2\psi^3)_x - \frac{3}{2}(\psi_x \partial_x^{-1}\psi_y + \psi\psi_y) + a\partial_x^{-1}\psi_{zz},\\
\psi_t &= \frac{1}{2}(\psi_{xx} - 2\psi^3)_x + \frac{3}{2}\left(- \frac{1}{4}\partial_x^{-1}\psi_{yy} + \psi\psi_y\right) + a\partial_x^{-1}\psi_{zz},\\
\psi_t &= -\frac{1}{4}(\psi_{xx} - 2\psi^3)_x - \frac{3}{4}\left(\frac{1}{4}\partial_x^{-1}\psi_{yy} + \psi_x\partial_x^{-1}\psi_y\right) + a\partial_x^{-1}\psi_{zz},\\
\psi_t &= 2(\psi_{xx} - 2\psi^3)_x - \frac{3}{4}\left(\partial_x^{-1}\psi_{yy} - 2\psi_x\partial_x^{-1}\psi_y - 6\psi\psi_y\right) - \frac{3}{4}\partial_x^{-1}\psi_{zz} - \frac{1}{4}\psi_z - \frac{1}{2}\psi_y,
\end{aligned} \tag{3}$$

where a is a constant. It is evident that these (3+1)-dimensional nonlinear models (3) are constructed by augmenting the first three models in (1) with the term $a\partial_x^{-1}\psi_{zz}$ and by introducing the terms $-\frac{3}{4}\partial_x^{-1}\psi_{zz} - \frac{1}{4}\psi_z - \frac{1}{2}\psi_y$ to the last model in (1). To eliminate the integral term in the first model of Eq (3), we introduce the potential function as follows

$$\psi(x, y, z, t) = u_x(x, y, z, t),$$

which allows us to transform the model into the equation

$$u_{xt} + u_{xxxx} - 6u_x^2 u_{xx} + \frac{3}{2} u_{xx} u_y + \frac{3}{2} u_x u_{xy} - a u_{zz} = 0, \tag{4}$$

referred to as (4):

The main objective of this study is to explore the integrability of nonlinear model (3), which is equivalent to (4). To achieve this, our approach begins with an exploration of the symmetry algebra of the model, which is then used to analyze the associated symmetry groups. We proceed by performing potential symmetry reductions using symmetry algebra to derive the invariant solutions. Furthermore, we aim to leverage the symmetry algebra to delve into the local conservation laws of the given model Eq. (4), following the principles outlined in Ibragimov's new conservation theorem. It is worth noting that our main focus is on the first model. However, it is crucial to emphasize that Wazwaz, as mentioned in [7], highlighted the significance of the solutions for the second and third models, perfectly aligned with those of the first model.

The Lie symmetry method [8–10], which is based on the principles of symmetry and invariance, offers a systematic approach to analytically solve differential equations. Its origins are traced back to the work of Sophus Lie (1842-1899), and it has since become a fundamental mathematical tool for researchers investigating mathematical models in fields ranging from physics and engineering to natural sciences. Comprehensive discussions of Lie group analysis methods [11] can be found in various books, including those authored by Ovsyannikov [12], Bluman and Kumei [13], Olver [14], and Ibragimov [15], among others.

It is well established that conservation laws play a pivotal role in solving the differential equations. The existence of a substantial number of conservation laws within a set of PDEs signifies their potential for integrability, as elucidated by Bluman and Kumei (1989). An insightful comparison of different approaches for deriving conservation laws for specific PDEs in the field of fluid mechanics can be found in the study conducted by Naz [16] and his team published in 2008.

The remainder of this paper is organized as follows. In Section 2, we explore the symmetry algebra of nonlinear model (4). Subsequently, in Section 3, we use it to perform several symmetry reductions on the same model. In section 4, we provide a recap of key definitions and theorems related to conservation laws. We then constructed conservation laws for model (4) using Ibragimov's new conservation theorem. Finally, Section 5 presents the concluding remarks.

2. Lie Group Method for Eq (4)

In this section, an analysis of the Lie symmetries and the optimal system pertaining to the equation denoted by (4) is conducted. Consider the one-parameter Lie group of transformation

$$\begin{aligned} \tilde{x} &\to x + \varepsilon\phi_1(x,y,z,t,u) + O(\varepsilon^2), \\ \tilde{y} &\to y + \varepsilon\phi_2(x,y,z,t,u) + O(\varepsilon^2), \\ \tilde{z} &\to z + \varepsilon\phi_3(x,y,z,t,u) + O(\varepsilon^2), \\ \tilde{t} &\to t + \varepsilon\phi_4(x,y,z,t,u) + O(\varepsilon^2), \\ \tilde{u} &\to u + \varepsilon\varsigma(x,y,z,t,u) + O(\varepsilon^2), \end{aligned} \tag{5}$$

where ε is the group parameter. For Eq (4), the vector field is defined as

$$V = \phi_1\frac{\partial}{\partial x} + \phi_2\frac{\partial}{\partial y} + \phi_3\frac{\partial}{\partial z} + \phi_4\frac{\partial}{\partial t} + \varsigma\frac{\partial}{\partial u}. \tag{6}$$

The task at hand involves determining the coefficient functions $\phi_1, \phi_2, \phi_3, \phi_4$, and ς, such that the operator V satisfies the Lie symmetry condition

$$V^{[4]}(u_{xt} + u_{xxxx} - 6u_x^2u_{xx} + \frac{3}{2}u_{xx}u_y + \frac{3}{2}u_xu_{xy} - au_{zz})|_{(4)} = 0, \tag{7}$$

where $V^{[4]}$ denotes the fourth prolongation of V.

Solving Eq (7) yields the set of determining equations, expressed as

$$\begin{aligned} &\varsigma_{zz} - \frac{2}{3a}\phi_{2tt} = 0, \phi_{3zt} = 0, \phi_{3zz} = 0, \varsigma_u = 0, \varsigma_x - \frac{2}{3}\phi_{2t} = 0, \varsigma_y - \frac{2}{3}\phi_{1t} = 0, \phi_{4y} = 0, \\ &\phi_{4t} - \frac{3}{2}\phi_{3z} = 0, \phi_{4u} = 0, \phi_{4x} = 0, \phi_{4z} = 0, \phi_{1u} = 0, \phi_{1x} - \frac{\phi_{3z}}{2} = 0, \phi_{1y} + \frac{8}{3}\phi_{2t} = 0, \\ &\phi_{1z} - \frac{\phi_{3t}}{2a} = 0, \phi_{2u} = 0, \phi_{2x} = 0, \phi_{2y} - \phi_{3z} = 0, \phi_{2z} = 0, \phi_{3u} = 0, \phi_{3x} = 0, \phi_{3y} = 0. \end{aligned} \tag{8}$$

Solution of system (8) gives infinitesimals given as

$$\begin{aligned} \phi_1 &= \frac{c_1}{3}x - \frac{8}{3}f_{1t}y + \frac{f_{2t}z}{2\alpha} + \frac{3}{2}f_5 + c_3, \\ \phi_2 &= \frac{2}{3}c_1y + f_1, \\ \phi_3 &= \frac{2}{3}c_1z + f_2, \\ \phi_4 &= c_1t + c_2, \\ \varsigma &= \frac{1}{9\alpha}\Big((-8\alpha y^2 + 3z^2)f_{1tt} + 3yzf_{2tt} + 9\alpha(\frac{2}{3}f_{1t}x + f_{5t}y + f_6z + f_7)\Big). \end{aligned} \tag{9}$$

By assuming $f_2 = c_4, f_6 = c_5, f_7 = c_6$ and $f_1 = f_5 = 0$, we obtain the following symmetry generators

$$V_1 = \frac{\partial}{\partial x},\ V_2 = \frac{\partial}{\partial z},\ V_3 = \frac{\partial}{\partial t},\ V_4 = \frac{\partial}{\partial u},\ V_5 = z\frac{\partial}{\partial u},\ V_6 = \frac{x}{3}\frac{\partial}{\partial x} + \frac{2y}{3}\frac{\partial}{\partial y} + \frac{2z}{3}\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}. \tag{10}$$

The commutator relation for the symmetry algebra (10) is defined as

$$[V_i, V_i] = V_iV_j - V_jV_i, \tag{11}$$

where $[V_i, V_j]$ denotes the Lie product. The commutator relation for symmetry algebra (10) is given in Table 1.

Table 1: Commutator Table.

$[V_i, V_j]$	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	0	0	0	0	$\frac{1}{3}V_1$
V_2	0	0	0	0	V_4	$\frac{2}{3}V_2$
V_3	0	0	0	0	0	V_3
V_4	0	0	0	0	0	0
V_5	0	$-V_4$	0	0	0	$-\frac{2}{3}V_5$
V_6	$-\frac{1}{3}V_1$	$-\frac{2}{3}V_2$	$-V_3$	0	$\frac{2}{3}V_5$	0

Theorem 1. *The vector fields V_i ($i = 1, 2, \cdots, 6$) for (4) constitute six-dimensional Lie algebra.*

Proof. As indicated in Table 1, an antisymmetrical pattern is noticeable, and zero diagonal elements are evident. The determination of the structure constants is easily accomplished by examining the commutator table, and Jacobi identity verification is straightforward.

Lie group transformations associated with the symmetry generators (10) are written as

$$\begin{aligned}
&\mathcal{S}_1 : (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}) \to (x + \varepsilon, y, z, t, u),\\
&\mathcal{S}_2 : (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}) \to (x, y, z + \varepsilon, t, u),\\
&\mathcal{S}_3 : (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}) \to (x, y, z, t + \varepsilon, u),\\
&\mathcal{S}_4 : (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}) \to (x, y, z, t, \varepsilon + u),\\
&\mathcal{S}_5 : (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}) \to (x, y, z, t, \varepsilon z + u),\\
&\mathcal{S}_6 : (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}) \to (e^{\frac{\varepsilon}{3}}x, e^{\frac{2\varepsilon}{3}}y, e^{\frac{2\varepsilon}{3}}z, e^{\epsilon}t, u).
\end{aligned}$$

3. Group-Invariant Solutions

Case 1: $V_3 + V_4 = \frac{\partial}{\partial t} + \frac{\partial}{\partial u}$.
In the case of the symmetry generator $V_3 + V_4$, the Lagrange equation can be expressed

as shown below,

$$\frac{du}{1} = \frac{dx}{0} = \frac{dy}{0} = \frac{dz}{0} = \frac{dt}{1},$$

and produces a similarity transformation, $u(x, y, z, t) = t + \vartheta(\alpha, \beta, \gamma)$, $\alpha = x, \beta = y, \gamma = z$. Through the application of this transformation, we continue with equation (4) in its reduced form as shown below,

$$2\vartheta_{\alpha\alpha\alpha\alpha} + 3\left(-4\vartheta_\alpha^2 + \vartheta_\beta\right)\vartheta_{\alpha\alpha} - 2a\vartheta_{\gamma\gamma} + 3\vartheta_\alpha\vartheta_{\alpha\beta} = 0, \tag{12}$$

The application of the Lie symmetry method to this (2+1) dimensional PDE unveils additional infinitesimals

$$\begin{aligned} \phi_\alpha &= \frac{c_3\alpha}{2} + c_6, \\ \phi_\beta &= c_3\beta + c_5, \\ \phi_\gamma &= c_3\gamma + c_4, \\ \varsigma_\vartheta &= c_1\gamma + c_2. \end{aligned} \tag{13}$$

Case 1.1: For $c_2 = c_5 = 1$, the symmetry generator $\frac{\partial}{\partial\beta} + \frac{\partial}{\partial\vartheta}$ leads to characteristic form

$$\frac{d\vartheta}{1} = \frac{d\alpha}{0} = \frac{d\beta}{1} = \frac{d\gamma}{0},$$

which gives rise to similarity variables $\vartheta(\alpha, \beta, \gamma) = \beta + p(k, \tau)$, where $k = \alpha, \tau = \gamma$. Applying these invariants, Eq (12) transforms into the following (1+1) PDE

$$2p_{kkkk} - 12p_{kk}p_k^2 + 3p_{kk} - 2ap_{\tau\tau} = 0. \tag{14}$$

The application of the Lie symmetry method to this (1+1) dimensional PDE unveils additional infinitesimals

$$\begin{aligned} \phi_k &= c_3, \\ \phi_\tau &= c_4, \\ \varsigma_p &= c_1\tau + c_2. \end{aligned} \tag{15}$$

Case 1.1.1: For $c_3 = 1$, the symmetry generator $\frac{\partial}{\partial k}$ leads to characteristic form

$$\frac{dp}{0} = \frac{dk}{1} = \frac{d\tau}{0},$$

which gives rise to invariant variables $p(k, \tau) = h(s)$, where $s = \tau$. Applying these invariants, Eq (14) transforms into the following ODE $-2ah'' = 0$ with solution

$$h(s) = c_1 s + c_2, \tag{16}$$

this gives,

$$p(k, \tau) = c_1\tau + c_2, \tag{17}$$

which yields,

$$\vartheta(\alpha,\beta,\gamma) = c_1\gamma + c_2 + \beta. \tag{18}$$

As a outcome, the invariant solution for the (3+1)-dimensional model (4) is formulated as follows

$$u(x,y,z,t) = c_1 z + c_2 + y + t, \tag{19}$$

where c_1, c_2 are constants.

Case 2: $V_2 + V_3 + V_4 = \frac{\partial}{\partial z} + \frac{\partial}{\partial t} + \frac{\partial}{\partial u}$.
In the case of the symmetry generator $V_2+V_3+V_4$, the Lagrange equation can be expressed as shown below,

$$\frac{du}{1} = \frac{dx}{0} = \frac{dy}{0} = \frac{dz}{1} = \frac{dt}{1},$$

and produces a similarity transformation, $u(x,y,z,t) = z + \vartheta(\alpha,\beta,\gamma)$, $\alpha = x, \beta = y, \gamma = -z + t$. Through the application of this transformation, we continue with equation (4) in its reduced form as shown below,

$$2\vartheta_{\alpha\alpha\alpha\alpha} + 3\left(-4\vartheta_\alpha^2 + \vartheta_\beta\right)\vartheta_{\alpha\alpha} - 2a\vartheta_{\gamma\gamma} + 3\vartheta_\alpha\vartheta_{\alpha\beta} + 2\vartheta_{\alpha\gamma} = 0. \tag{20}$$

The application of the Lie symmetry method to this (2+1) dimensional PDE unveils additional infinitesimals

$$\begin{aligned}
\phi_\alpha &= \frac{c_1\alpha}{2} - \frac{c_1\gamma}{4a} + c_6,\\
\phi_\beta &= c_1\beta + c_2,\\
\phi_\gamma &= c_1\gamma + c_5,\\
\varsigma_\vartheta &= -\frac{c_1\beta}{6a} + c_3\gamma + c_4.
\end{aligned} \tag{21}$$

Case 2.1: For $c_2 = c_3 = 1$, the symmetry generator $\frac{\partial}{\partial\beta} + \gamma\frac{\partial}{\partial\vartheta}$ leads to characteristic form

$$\frac{d\vartheta}{\gamma} = \frac{d\alpha}{0} = \frac{d\beta}{1} = \frac{d\gamma}{0},$$

This gives rise to variables $\vartheta(\alpha,\beta,\gamma) = \beta\gamma + p(k,\tau)$, where $k = \alpha, \tau = \gamma$. Applying these invariants, Eq (20) transforms into the following (1+1) PDE

$$p_{kkkk} + 3\left(-4p_k^2 + \tau\right)p_{kk} - 2ap_{\tau\tau} + 2p_{k\tau} = 0. \tag{22}$$

The application of the Lie symmetry method to this (1+1) dimensional PDE unveils additional infinitesimals

$$\begin{aligned}
\phi_k &= c_3,\\
\phi_\tau &= 0,\\
\varsigma_p &= c_1\tau + c_2.
\end{aligned} \tag{23}$$

Case 2.1.1: For $c_1 = c_2 = c_3 = 1$, the symmetry generator $(1+\tau)\frac{\partial}{\partial p} + \frac{\partial}{\partial k}$ leads to characteristic form

$$\frac{dp}{\tau+1} = \frac{dk}{1} = \frac{d\tau}{0},$$

which gives rise to invariant variables $p(k,\tau) = k\tau + k + h(s)$, where $s = \tau$. Applying these invariants, Eq (22) transforms into the following ODE $-2ah'' + 2 = 0$, with solution

$$h(s) = \frac{s^2}{2a} + c_1 s + c_2, \tag{24}$$

this gives,

$$p(k,\tau) = \frac{\tau^2}{2a} + c_1\tau + c_2 + k\tau + k, \tag{25}$$

which yields,

$$\vartheta(\alpha,\beta,\gamma) = \frac{\gamma^2}{2a} + c_1\gamma + c_2 + \alpha\gamma + \alpha + \beta\gamma. \tag{26}$$

As a outcome, the invariant solution for the (3+1)-dimensional model (4) is formulated as follows

$$u(x,y,z,t) = \frac{(t-z)^2}{2a} + c_1(t-z) + c_2 + x(t-z) + x + y(t-z) + z, \tag{27}$$

where c_1, c_2 are any constants.

Case 3: $V_2 + V_3 + V_5 = \frac{\partial}{\partial z} + z\frac{\partial}{\partial u} + \frac{\partial}{\partial t}$.
In the case of the symmetry generator $V_2+V_3+V_5$, the Lagrange equation can be expressed as shown below,

$$\frac{du}{z} = \frac{dx}{0} = \frac{dy}{0} = \frac{dz}{1} = \frac{dt}{1},$$

and produces a similarity transformation $u(x,y,z,t) = \frac{z^2}{2} + \vartheta(\alpha,\beta,\gamma)$, $\alpha = x, \beta = y, \gamma = -z + t$. Through the application of this transformation, we continue with equation (4) in its reduced form as shown below,

$$2\vartheta_{\alpha\alpha\alpha\alpha} + (-12\vartheta_\alpha^2 + 3\vartheta_\beta)\vartheta_{\alpha\alpha} - 2a\vartheta_{\gamma\gamma} + 3\vartheta_\alpha\vartheta_{\alpha\beta} - 2a + 2\vartheta_{\alpha\gamma} = 0, \tag{28}$$

The application of the Lie symmetry method to this (2+1) dimensional PDE unveils additional infinitesimals

$$\begin{aligned} \phi_\alpha &= c_3, \\ \phi_\beta &= c_4, \\ \phi_\gamma &= c_5, \\ \varsigma_\vartheta &= c_1 + c_2\gamma. \end{aligned} \tag{29}$$

Case 3.1: For $c_1 = c_3 = c_4 = 1$ and the symmetry generator $\frac{\partial}{\partial\beta} + \frac{\partial}{\partial\vartheta} + \frac{\partial}{\partial\alpha}$ leads to characteristic form

$$\frac{d\vartheta}{1} = \frac{d\alpha}{1} = \frac{d\beta}{1} = \frac{d\gamma}{0},$$

which yields variables $\vartheta(\alpha,\beta,\gamma)=\alpha+p(k,\tau)$, where $k=\gamma,\tau=-\alpha+\beta$. Applying these invariants, Eq (28) transforms into the following (1+1) PDE

$$2p_{\tau\tau\tau\tau}+(-12p_\tau^2+30p_\tau-15)p_{\tau\tau}-2ap_{kk}-2a-2p_{k\tau}=0. \tag{30}$$

The application of the Lie symmetry method to this (1+1) dimensional PDE unveils additional infinitesimals

$$\begin{aligned}\phi_k&=c_3,\\ \phi_\tau&=c_4,\\ \varsigma_p&=c_1+c_2\tau.\end{aligned} \tag{31}$$

Case 3.1.1: For $c_1=c_4=1$ and the symmetry generator $\frac{\partial}{\partial\tau}+\frac{\partial}{\partial p}$ leads to characteristic form

$$\frac{dp}{1}=\frac{dk}{0}=\frac{d\tau}{1},$$

with invariant variables $p(k,\tau)=\tau+h(s)$, where $k=s$, applying these invariants, (30) transforms into ODE $-2ah''-2a=0$. The solution for this ODE is

$$h(s)=-\frac{s^2}{2}+c_1s+c_2, \tag{32}$$

this gives,

$$p(k,\tau)=-\frac{k^2}{2}+c_1k+c_2+\tau, \tag{33}$$

which yields,

$$\vartheta(\alpha,\beta,\gamma)=-\frac{\gamma^2}{2}+c_1\gamma+c_2+(-\alpha+\beta)+\alpha. \tag{34}$$

As a outcome, the invariant solution for the (3+1)-dimensional model (4) is formulated as follows

$$u(x,y,z,t)=-\frac{(-z+t)^2}{2}+c_1(-z+t)+c_2+y+\frac{z^2}{2}, \tag{35}$$

where c_1,c_2 are any constants.

Case 4: $V_2+V_5=\frac{\partial}{\partial z}+z\frac{\partial}{\partial u}$.
In the case of the symmetry generator V_2+V_5, the Lagrange equation can be expressed as shown below,

$$\frac{du}{z}=\frac{dt}{0}=\frac{dz}{1}=\frac{dy}{0}=\frac{dx}{0},$$

and produces a similarity transformation, $u(x,y,z,t)=\frac{z^2}{2}+\vartheta(\alpha,\beta,\gamma)$, $\alpha=t,\beta=x,\gamma=y$. Through the application of this transformation, we continue with equation (4) in its reduced form as shown below,

$$2\vartheta_{\beta\beta\beta\beta}+3\left(-4\vartheta_\beta^2+\vartheta_\gamma\right)\vartheta_{\beta\beta}+3\vartheta_\beta\vartheta_{\beta\gamma}-2a+2\vartheta_{\alpha\beta}=0. \tag{36}$$

The application of the Lie symmetry method to this (2+1) dimensional PDE unveils additional infinitesimals

$$\begin{aligned}
\phi_\alpha &= c_1\alpha + c_2,\\
\phi_\beta &= \frac{3f_1(\alpha)}{2} + \frac{(-16a\alpha\gamma+\beta)\,c_1}{3} - \frac{8c_3\gamma}{3} + c_5,\\
\phi_\gamma &= \frac{\left(3a\,\alpha^2+2\gamma\right)c_1}{3} + c_3\alpha + c_4,\\
\varsigma_\vartheta &= f_{1\alpha}\gamma + f_2(\alpha) + \frac{2\,(2a\alpha c_1+c_3)\,\beta}{3} - \frac{16c_1 a\,\gamma^2}{9}.
\end{aligned} \tag{37}$$

Case 4.1: For $c_4 = 1$, the symmetry generator $\frac{\partial}{\partial\gamma}$ leads to characteristic form

$$\frac{d\vartheta}{0} = \frac{d\alpha}{0} = \frac{d\beta}{0} = \frac{d\gamma}{1},$$

which yields variables $\vartheta(\alpha,\beta,\gamma) = p(k,\tau)$, where $k = \alpha, \tau = \beta$. Applying these invariants, Eq (36) transforms into the following (1+1) PDE

$$2p_{\tau\tau\tau\tau} - 12p_{\tau\tau}p_\tau^2 - 2a + 2p_{k\tau} = 0. \tag{38}$$

The application of the Lie symmetry method to this (1+1) dimensional PDE unveils additional infinitesimals

$$\begin{aligned}
\phi_k &= c_1,\\
\phi_\tau &= c_2,\\
\varsigma_p &= f_1(k)\,.
\end{aligned} \tag{39}$$

Case 4.1.1: For $c_1 = 1$ and the symmetry generator $\frac{\partial}{\partial k}$ leads to characteristic form

$$\frac{dp}{0} = \frac{dk}{1} = \frac{d\tau}{0},$$

which gives rise to the invariant variables $p(k,\tau) = h(s)$ where, $s = \tau$. Applying these invariants, Eq (38) transforms into the ODE

$$h^{(iv)} - 6{h'}^2h'' - a = 0. \tag{40}$$

Eq (40) admits a Lagrangian $\mathcal{L} = \frac{{h''}^2}{2} + \frac{{h'}^4}{2} - ah$, which in turn exhibits variational symmetries given by

$$\frac{\partial}{\partial s},\ \frac{\partial}{\partial h}. \tag{41}$$

For $\frac{\partial}{\partial h}$, the corresponding Noether first integral is

$$I = 2{h'}^3 + as. \tag{42}$$

Now, $D_s I = 0$ implies $I = c$, that is

$$2h'^3 + as = c, \tag{43}$$

where c denotes a constant.
The solution of equation (43) is

$$h(s) = \frac{3\,\iota\,(as - c)\left(\iota \pm \sqrt{3}\right)(-4as + 4c)^{\frac{1}{3}}}{16a}, \tag{44}$$

this gives,

$$p(k,\tau) = \frac{3\,\iota\,(a\tau - c)\left(\iota \pm \sqrt{3}\right)(-4a\tau + 4c)^{\frac{1}{3}}}{16a}, \tag{45}$$

which yields,

$$\vartheta(\alpha,\beta,\gamma) = \frac{3\,\iota\,(a\beta - c)\left(\iota \pm \sqrt{3}\right)(-4a\beta + 4c)^{\frac{1}{3}}}{16a}. \tag{46}$$

As a outcome, the invariant solution for the (3+1)-dimensional model (4) is formulated as follows

$$u(x,y,z,t) = \frac{z^2}{2} + \frac{3\,\iota\,(ax - c)\left(\iota \pm \sqrt{3}\right)(-4ax + 4c)^{\frac{1}{3}}}{16a}, \tag{47}$$

where c denotes a constant.

Case 5: $V_3 = \frac{\partial}{\partial t}$.
In the case of the symmetry generator V_3, the Lagrange equation can be expressed as shown below,

$$\frac{dt}{1} = \frac{dz}{0} = \frac{dy}{0} = \frac{dx}{0} = \frac{du}{0},$$

and produces a similarity transformation, $u(x,y,z,t) = \vartheta(\alpha,\beta,\gamma)$, $\alpha = x, \beta = y, \gamma = z$. Through the application of this transformation, we continue with equation (4) in its reduced form as shown below,

$$2\vartheta_{\alpha\alpha\alpha\alpha} + (-12\vartheta_\alpha^2 + 3\vartheta_\beta)\vartheta_{\alpha\alpha} - 2a\vartheta_{\gamma\gamma} + 3\vartheta_\alpha\vartheta_{\alpha\beta} = 0, \tag{48}$$

The application of the Lie symmetry method to this (2+1) dimensional PDE unveils additional infinitesimals

$$\begin{aligned} \phi_\alpha &= c_4\alpha + c_3, \\ \phi_\beta &= 2c_4\beta + c_5, \\ \phi_\gamma &= 2c_4\gamma + c_6, \\ \varsigma_\vartheta &= c_1 + c_2\gamma. \end{aligned} \tag{49}$$

Case 5.1: For $c_6 = 1$ and the symmetry generator $\frac{\partial}{\partial\gamma}$ leads to characteristic form

$$\frac{d\vartheta}{0} = \frac{d\alpha}{0} = \frac{d\beta}{0} = \frac{d\gamma}{1},$$

which yields variables $\vartheta(\alpha,\beta,\gamma) = p(k,\tau)$, where $k = \alpha, \tau = \beta$. Applying these invariants, Eq (48) transforms into the following (1+1) PDE

$$2p_{kkkk} + (-12p_k^2 + 3p_\tau)p_{kk} + 3p_k p_{k\tau} = 0. \tag{50}$$

The application of the Lie symmetry method to this (1+1) dimensional PDE unveils additional infinitesimals

$$\begin{aligned} \phi_k &= c_2 + c_3 k, \\ \phi_\tau &= 2c_3\tau + c_4, \\ \varsigma_p &= c_1. \end{aligned} \tag{51}$$

Case 5.1.1: For $c_4 = 1$ and the symmetry generator $\frac{\partial}{\partial \tau}$ leads to characteristic form

$$\frac{dp}{0} = \frac{dk}{0} = \frac{d\tau}{1},$$

which gives rise to invariant variables $p(k,\tau) = h(s)$, where $k = s$. Applying these invariants, Eq (50) transforms into the following ODE

$$h^{(iv)} - 6h'^2 h'' = 0. \tag{52}$$

Eq (52) admits a Lagrangian $\mathcal{L} = \frac{h''^2}{2} + \frac{h'^4}{2}$, which in turn exhibits variational symmetries given by

$$\frac{\partial}{\partial s}, \ \frac{\partial}{\partial h}. \tag{53}$$

For $\frac{\partial}{\partial h}$, the corresponding Noether first integral is

$$I = h''' - 2h'^3. \tag{54}$$

Now, $D_s I = 0$ implies $I = c$, that is

$$h''' - 2h'^3 = c, \tag{55}$$

where c is any constant.
The polynomial solution of equation (55) is

$$h(s) = \Big(- \frac{(-4c)^{\frac{1}{3}}}{4} \pm \frac{i\sqrt{3}}{4}(-4c)^{\frac{1}{3}} \Big) s + c_1. \tag{56}$$

By back substitution of variables, we get

$$p(k,\tau) = \Big(- \frac{(-4c)^{\frac{1}{3}}}{4} \pm \frac{i\sqrt{3}}{4}(-4c)^{\frac{1}{3}} \Big) k + c_1, \tag{57}$$

this implies,

$$\vartheta(\alpha,\beta,\gamma) = \Big(- \frac{(-4c)^{\frac{1}{3}}}{4} \pm \frac{i\sqrt{3}}{4}(-4c)^{\frac{1}{3}} \Big) \alpha + c_1. \tag{58}$$

As a outcome, the invariant solution for the (3+1)-dimensional model (4) is formulated as follows

$$u(x,y,z,t) = \Big(-\frac{(-4c)^{\frac{1}{3}}}{4} \pm \frac{i\sqrt{3}}{4}(-4c)^{\frac{1}{3}} \Big) x + c_1, \tag{59}$$

where c_1 denotes a constant.

Case 6: $V_2 + V_4 = \frac{\partial}{\partial u} + \frac{\partial}{\partial z}$.
In the case of the symmetry generator $V_2 + V_4$, the Lagrange equation can be expressed as shown below,

$$\frac{du}{1} = \frac{dx}{0} = \frac{dy}{0} = \frac{dz}{1} = \frac{dt}{0},$$

and produces a similarity transformation, $u(x,y,z,t) = z + \vartheta(\alpha,\beta,\gamma)$, $\alpha = t, \beta = x, \gamma = y$. Through the application of this transformation, we continue with equation (4) in its reduced form as shown below,

$$2\vartheta_{\beta\beta\beta\beta} + (-12\vartheta_\beta^2 + 3\vartheta_\gamma)\vartheta_{\beta\beta} + 3\vartheta_\beta\vartheta_{\beta\gamma} + 2\vartheta_{\alpha\beta} = 0, \tag{60}$$

The application of the Lie symmetry method to this (2+1) dimensional PDE unveils additional infinitesimals

$$\begin{aligned} \phi_\alpha &= 3c_7\alpha + c_5, \\ \phi_\beta &= \frac{3}{2}\alpha c_2 + c_7\beta - 4\gamma c_3 + c_6, \\ \phi_\gamma &= \frac{3}{2}\alpha c_3 + 2\gamma c_7 + c_8, \\ \varsigma_\vartheta &= c_2\gamma + c_3\beta + c_4\alpha + c_1. \end{aligned} \tag{61}$$

Case 6.1: For $c_8 = 1$ and the symmetry generator $\frac{\partial}{\partial\gamma}$ leads to characteristic form

$$\frac{d\vartheta}{0} = \frac{d\alpha}{0} = \frac{d\beta}{0} = \frac{d\gamma}{1},$$

with the invariant variables $\vartheta(\alpha,\beta,\gamma) = p(k,\tau)$, where $k = \alpha, \tau = \beta$. Applying these invariants, Eq (60) transforms into the following (1+1) PDE

$$2p_{\tau\tau\tau\tau} - 12p_\tau^2 p_{\tau\tau} + 2p_{k\tau} = 0, \tag{62}$$

The application of the Lie symmetry method to this (1+1) dimensional PDE unveils additional infinitesimals

$$\begin{aligned} \phi_k &= 3c_5k + c_3, \\ \phi_\tau &= c_5\tau + c_4, \\ \varsigma_p &= c_2k + c_1. \end{aligned} \tag{63}$$

Case 6.1.1: For $c_3 = c_4 = 1$ and the symmetry generator $\frac{\partial}{\partial k} + \frac{\partial}{\partial\tau}$ leads to characteristic form

$$\frac{dp}{0} = \frac{dk}{1} = \frac{d\tau}{1},$$

which gives rise to invariant variables $p(k,\tau) = h(s)$, where $-k+\tau = s$. Applying these invariants, Eq (62) transforms into the following ODE

$$h^{(iv)} - 6h'^2 h'' - h'' = 0. \tag{64}$$

Eq (64) admits a Lagrangian $\mathcal{L} = \frac{h''^2}{2} + \frac{h'^4}{2} + \frac{h'^2}{2}$, which in turn exhibits variational symmetries given by

$$\frac{\partial}{\partial s}, \ \frac{\partial}{\partial h}. \tag{65}$$

For $\frac{\partial}{\partial h}$, the corresponding Noether first integral is

$$I = h''' - 2h'^3 - h'. \tag{66}$$

Now, $D_s I = 0$ implies $I = c$, that is

$$h''' - 2h'^3 - h' = c, \tag{67}$$

where c is any constant.
The polynomial solution of equation (67) is

$$h(s) = \left(-\frac{(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}}{12} + \frac{1}{2(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}} - \frac{i\sqrt{3}}{2}\Big(\frac{(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}}{6} + \frac{1}{(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}}\Big)\right) s + c_1. \tag{68}$$

By back substitution of variables, we get

$$p(k,\tau) = \left(-\frac{(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}}{12} + \frac{1}{2(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}} - \frac{i\sqrt{3}}{2}\Big(\frac{(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}}{6} + \frac{1}{(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}}\Big)\right) (-k+\tau) + c_1, \tag{69}$$

this implies,

$$\vartheta(\alpha,\beta,\gamma) = \left(-\frac{(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}}{12} + \frac{1}{2(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}} - \frac{i\sqrt{3}}{2}\Big(\frac{(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}}{6} + \frac{1}{(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}}\Big)\right) (-\alpha+\beta) + c_1. \tag{70}$$

As a outcome, the invariant solution for the (3+1)-dimensional model (4) is formulated as follows

$$u(x,y,z,t) = \Bigg(-\frac{(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}}{12} + \frac{1}{2(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}} - \frac{i\sqrt{3}}{2}\Big(\frac{(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}}{6} + \frac{1}{(-54c+6\sqrt{81c^2+6})^{\frac{1}{3}}}\Big)\Bigg)(-t+x) + c_1 + z, \tag{71}$$

where c_1 is any constant.

Comment 1. 4. Conservation Laws

Let Λ denote the multiplier or characteristic function [16] depending on x, y, z, t, u and the first- and second-order derivatives of u. Then for (4), we have the following relation

$$D_tT^t + D_xT^x + D_yT^y + D_zT^z = \Lambda(u_{xt} + u_{xxxx} - 6u_x^2u_{xx} + \frac{3}{2}u_{xx}u_y + \frac{3}{2}u_xu_{xy} - au_{zz}). \tag{72}$$

By applying the Euler operator $\frac{\delta}{\delta u}$ on (72), we acquire

$$\frac{\delta}{\delta u}(u_{xt} + u_{xxxx} - 6u_x^2u_{xx} + \frac{3}{2}u_{xx}u_y + \frac{3}{2}u_xu_{xy} - au_{zz}) = 0. \tag{73}$$

The determining equations obtained using (73) are described as

$$\begin{gathered} \Lambda_{zz} = 0, \quad \Lambda_x = 0, \quad \Lambda_u = 0, \quad \Lambda_{u_{xt}} = 0, \\ \Lambda_{u_{xx}} = 0, \quad \Lambda_{u_{xy}} = 0, \quad \Lambda_{u_{zz}} = 0. \end{gathered} \tag{74}$$

The solution of (3) yields the multipliers for (4), which are stated as follows

$$\Lambda = f_1(y,t)z + f_2(y,t). \tag{75}$$

To obtain the local conservation laws for (4), the results of (75) indicate further classification of the functions $f_1(y,t)$ and $f_2(y,t)$. In addition, the multiplier can be used to obtain the potential symmetries.

5. Non-Local Conservation Laws for the Eq (4)

In the referenced work [17], Ibragimov's pioneering theorem centers on the conserved flow of differential equations. This theorem demonstrates significant applicability to systems of differential equations, where the number of equations aligns with the number of dependent variables.
We consider a kth order PDE given as

$$\mathcal{G} = \mathcal{G}(\mathbf{x}, u, u_1, u_2, ..., u_p). \tag{76}$$

In this context, $u = u(\mathbf{x})$ and $\mathbf{x} = \mathbf{x}(x_1, x_2, ..., x_m)$. Given the formal Lagrangian for Eq (76), an adjoint equation can be followed by

$$\mathcal{G}^\star \equiv \frac{\delta}{\delta u}(v\mathcal{G}), \tag{77}$$

where the Euler-Lagrange operator $\frac{\delta}{\delta u}$ in symmetric form is given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{i=1}^{\infty}(-1)^s \mathcal{D}_{i_1}...\mathcal{D}_{i_s}\frac{\partial}{\partial u_{i_1...i_s}}, \tag{78}$$

and the differential operator $\mathcal{D}_i$ is defined by

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \dots\ . \tag{79}$$

Theorem 2. *For any symmetry Lie point, Lie-Bäcklund, or nonlocal of Eq (76) given by*

$$V = \phi^i\frac{\partial}{\partial x^i} + \varsigma\frac{\partial}{\partial u}, \tag{80}$$

where $\mathcal{L}$ serves as a formal Lagrangian, the conserved vectors for Eq (77) can be formally defined as

$$\begin{aligned}\mathcal{W}^i =& \phi^i\mathcal{L} + N\left[\frac{\partial\mathcal{L}}{\partial u_i} - \mathcal{D}_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}}\right) + \mathcal{D}_j\mathcal{D}_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right) + ...\right] \\ &+ \mathcal{D}_j(N)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}} - \mathcal{D}_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right) + ...\right] + \mathcal{D}_j\mathcal{D}_k(N)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}} + ...\right]\cdots\end{aligned} \tag{81}$$

In this context, N is defined as

$$N = \varsigma - \phi^i u_i, \tag{82}$$

where $\mathcal{D}_i\ (\mathcal{W}^i) = 0$.

Theorem 3. *[17] The adjoint equation for the nonlinear model described in Eq (4) is represented as follows*

$$\mathcal{G}^\star = 12u_xu_{xx}v_x + 3u_{xy}v_x + \frac{3}{2}u_yv_{xx} - 6v_{xx}u_x^2 + \frac{3}{2}u_xv_{xy} + v_{xt} - \alpha v_{zz} + v_{xxxx} = 0, \tag{83}$$

where

$$\mathcal{L} = v(x,y,z,t)\big(u_{xt} + u_{xxxx} - 6u_x^2u_{xx} + \frac{3}{2}u_{xx}u_y + \frac{3}{2}u_xu_{xy} - au_{zz}\big). \tag{84}$$

According to the Ibragimov theorem, every symmetry generator corresponds to a conserved vector. Consequently, we calculated the conserved vectors using Theorem 2.

(I) When considering the vector field $V_1 = \frac{\partial}{\partial x}$ and $N = -u_x$, the resulting conserved vectors are as follows

$$\begin{aligned}
\mathcal{W}_1^t =& v_x u_x - u_{xx} v, \\
\mathcal{W}_1^x =& \mathcal{L} - u_x\big(\frac{3}{2}u_{xy}v + 6u_x^2 v_x + \frac{3}{2}u_y v_x - \frac{3}{2}u_x v_y - v_t - v_{xxxx}\big) - \\
& u_{xx}\big(v_{xx} - 6u_x^2 v + \frac{3}{2}u_y v\big) - \frac{3}{2}u_x u_{xy} v - v u_{xt} + v_x u_{xxx} \\
& - v u_{xxxx}, \\
\mathcal{W}_1^y =& \frac{3}{2}u_x\big(u_x v_x - u_{xx} v\big), \\
\mathcal{W}_1^z =& - a v_z u_x + a v u_{xz}.
\end{aligned} \tag{85}$$

(II) When considering the vector field $V_2 = \frac{\partial}{\partial z}$ and $N = -u_z$, the resulting conserved vectors are as follows

$$\begin{aligned}
\mathcal{W}_2^t =& u_z v_x - v u_{xz}, \\
\mathcal{W}_2^x =& - u_z\big(\frac{3}{2}u_{xy}v + 6u_x^2 v_x + \frac{3}{2}u_y v_x - \frac{3}{2}u_x v_y - v_t - v_{xxxx}\big) - \\
& u_{xz}\big(v_{xx} - 6u_x^2 v + \frac{3}{2}u_y v\big) - \frac{3}{2}u_x u_{yz} v - v u_{zt} + v_x u_{xxz} \\
& - v u_{xxxz}, \\
\mathcal{W}_2^y =& \frac{3}{2}u_x(u_z v_x - u_{xz} v), \\
\mathcal{W}_2^z =& v\big(u_{xt} + u_{xxxx} - 6u_x^2 u_{xx} + \frac{3}{2}u_{xx}u_y + \frac{3}{2}u_x u_{xy} - a u_{zz}\big)
\end{aligned} \tag{86}$$

(III) When considering the vector field $V_3 = \frac{\partial}{\partial t}$ and $N = -u_t$, the resulting conserved vectors are as follows

$$\begin{aligned}
\mathcal{W}_3^t =& v\big(u_{xxxx} - 6u_x^2 u_{xx} + \frac{3}{2}u_{xx}u_y + \frac{3}{2}u_x u_{xy} - a u_{zz}\big) + u_t v_x, \\
\mathcal{W}_3^x =& - u_t\big(\frac{3}{2}u_{xy}v + 6u_x^2 v_x + \frac{3}{2}u_y v_x - \frac{3}{2}u_x v_y - v_t - v_{xxxx}\big) - \\
& u_{xt}\big(v_{xx} - 6u_x^2 v + \frac{3}{2}u_y v\big) - \frac{3}{2}u_x u_{yt} v - v u_{tt} + v_x u_{xxt} \\
& - v u_{xxxt}, \\
\mathcal{W}_3^y =& \frac{3}{2}u_x\big(u_t v_x - u_{xt} v\big), \\
\mathcal{W}_3^z =& - a v_z u_t + a v u_{zt}.
\end{aligned} \tag{87}$$

(IV) When considering the vector field $V_4 = \frac{\partial}{\partial u}$ and $N = 1$, the resulting conserved vectors are as follows

$$\begin{aligned}\mathcal{W}_4^t =& -v_x,\\ \mathcal{W}_4^x =& \frac{3}{2}u_{xy} + 6v_x u_x^2 + \frac{3}{2}u_y v_x + v_{xt} + \frac{3}{2}u_x v_y - v_t - v_{xxxx},\\ \mathcal{W}_4^y =& -\frac{3}{2}u_x v_x,\\ \mathcal{W}_4^z =& a v_z.\end{aligned} \tag{88}$$

(V) When considering the vector field $V_5 = z\frac{\partial}{\partial u}$ and $N = z$, the resulting conserved vectors are as follows

$$\begin{aligned}\mathcal{W}_5^t =& -zv_x,\\ \mathcal{W}_5^x =& z\big(\frac{3}{2}u_{xy}v + 6u_x^2 v_x + \frac{3}{2}u_y v_x - \frac{3}{2}u_x v_y - v_t - v_{xxxx}\big),\\ \mathcal{W}_5^y =& -\frac{3}{2}zu_x v_x,\\ \mathcal{W}_5^z =& -av.\end{aligned} \tag{89}$$

(VI) When considering the vector field $V_6 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} + 3t\frac{\partial}{\partial t}$ and $N = -(xu_x + 2yu_y + 2zu_z + 3tu_t)$, the resulting conserved vectors are as follows

$$\begin{aligned}\mathcal{W}_6^t =& (3t)\mathcal{L} + u_x(xu_x + 2yu_y + 2zu_z + 3tu_t) - v\mathcal{D}_x(xu_x + 2yu_y + 2zu_z + 3tu_t),\\ \mathcal{W}_6^x =& x\mathcal{L} + (xu_x + 2yu_y + 2zu_z + 3tu_t)\big(\frac{3}{2}u_{xy}v + 6u_x^2 v_x + \frac{3}{2}u_y v_x - \frac{3}{2}u_x v_y - v_t - v_{xxxx}\big) +\\ & \mathcal{D}_x(xu_x + 2yu_y + 2zu_z + 3tu_t)\big(v_{xx} - 6u_x^2 v + \frac{3}{2}u_y v\big) + \frac{3}{2}u_x v\mathcal{D}_y(xu_x + 2yu_y\\ & + 2zu_z + 3tu_t) + v\mathcal{D}_t(xu_x + 2yu_y + 2zu_z + 3tu_t) - v_x\mathcal{D}_x^2(xu_x + 2yu_y + 2zu_z + 3tu_t)\\ & + v\mathcal{D}_x^3(xu_x + 2yu_y + 2zu_z + 3tu_t),\\ \mathcal{W}_6^y =& (2y)\mathcal{L} + \frac{3}{2}u_x v_x(xu_x + 2yu_y + 2zu_z + 3tu_t) - \frac{3}{2}u_x v\mathcal{D}_x(xu_x + 2yu_y + 2zu_z + 3tu_t),\\ \mathcal{W}_6^z =& (2z)\mathcal{L} - av_z(xu_x + 2yu_y + 2zu_z + 3tu_t) + av\mathcal{D}_z(xu_x + 2yu_y + 2zu_z + 3tu_t).\end{aligned} \tag{90}$$

Note that in all the cases discussed above, $\mathcal{L}$ is the formal Lagrangian deduced from Eq (84), and the divergence relation

$$\mathcal{D}_t\mathcal{W}^t + \mathcal{D}_x\mathcal{W}^x + \mathcal{D}_y\mathcal{W}^y + \mathcal{D}_z\mathcal{W}^z = 0, \tag{91}$$

holds for all the conserved vectors.

6. Conclusions

In this research, our primary focus was on examining a (3+1)-dimensional nonlinear model (4) derived from the Jaulent-Miodek hierarchy. We utilized the Lie group method to analyze the integrability features of this model. This method led to the identification of six-dimensional symmetry algebra, which allowed us to determine the symmetry groups associated with the nonlinear model (4). Using this symmetry algebra, we successfully derived polynomial group-invariant solutions. In the literature, various approaches have been used to obtain solitary wave, soliton, and other types of solutions for the desired model [3–6]. Notably, in a prior study conducted by Wazwaz [7], rational function solutions were obtained including single, two, and three soliton solutions. However, our results differ significantly from those of previous studies. Our research underscored the effectiveness of the Lie group method in unveiling not only the inherent symmetry properties of the model, but also in facilitating the exploration of group-invariant solutions through symmetry algebra. Furthermore, we apply Anco's method to investigate the conservation laws pertinent to model (4). Our study assumed considerable significance as it contributed to the understanding of this model and addressed a specific gap in the group theoretic approach within this context. Consequently, our findings represent pioneering contributions to the study of the examined model. Motivated by these outcomes, we hope to apply the same technique to other nonlinear models in the future.

Acknowledgements

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through large group Research Project under grant number RGP.2/51/46.

Ethics approval and consent to participate: All the authors approve their consent.
Consent for publication: The authors approve this version for publication.
Availability of data and materials: All data generated or analyzed during this study are included in this published article.
Competing interests: The authors profess no conflicts of interest.

References

[1] Z Li, S Tian, and J Yang. On the soliton resolution and the asymptotic stability of N-soliton solution for the Wadati-Konno-Ichikawa equation with finite density initial data in space-time solitonic regions. *Advances in Mathematics*, (409):108639, 2022.

[2] Z Li, S Tian, and J Yang. On the asymptotic stability of N-soliton solution for the short pulse equation with weighted Sobolev initial data. *Journal of Differential Equations*, (377):121–87, 2023.

[3] A Wazwaz. Multiple kink solutions and multiple singular kink solutions for (2+1)-dimensional nonlinear models generated by the Jaulent–Miodek hierarchy. *Physics Letters A*, 21(373):1844–6, 2009.

[4] X Geng, C Cao, and H Dai. Quasi-periodic solutions for some (2+1)-dimensional

integrable models generated by the Jaulent-Miodek hierarchy. *Journal of Physics A: Mathematical and General*, 5(34):989, 2001.

[5] X Geng and Y Ma. N-soliton solution and its Wronskian form of a (3+1)-dimensional nonlinear evolution equation. *Physics Letters A*, 4(369):285–9, 2007.

[6] W Hereman and A Nuseir. Symbolic methods to construct exact solutions of nonlinear partial differential equations. *Mathematics and Computers in Simulation*, 1(43):13–27, 1997.

[7] A Wazwaz. Multiple soliton solutions for some (3+1)-dimensional nonlinear models generated by the Jaulent-Miodek hierarchy. *Applied Mathematics Letters*, 11(25):1936–40, 2012.

[8] N Zinat, A Hussain, A Kara, and F Zaman. Lie group analysis and conservation laws for the time-fractional 3D Bateman-Burgers equation. *Afrika Matematika*, 2(36):1–6, 2025.

[9] A Hussain. Invariant analysis and equivalence transformations for the non-linear wave equation in elasticity. *Partial Differential Equations in Applied Mathematics*, (13):101123, 2025.

[10] A Hussain, M Usman, F Zaman, A Zidan, and J Herrera. Noether and partial Noether approach for the nonlinear (3+1)-dimensional elastic wave equations. *PloS one*, 1(20):e0315505, 2025.

[11] S Tian, M Xu, and T Zhang. A symmetry-preserving difference scheme and analytical solutions of a generalized higher-order beam equation. *Proceedings of the Royal Society A*, 2255(477):20210455, 2021.

[12] L Ovsyannikov. *Lectures on the Theory of Group Properties of Differential Equations.* World Scientific Publishing Company, 2013.

[13] G Bluman and S Anco. *Symmetry and Integration Methods for Differential Equations.* Springer Science & Business Media, 2008.

[14] P Olver. *Applications of Lie Groups to Differential Equations.* . Springer Science & Business Media, 1993.

[15] N Ibragimov. *CRC Handbook of Lie Group Analysis of Differential Equations.* . CRC Press, 1995.

[16] R Naz, F Mahomed, and D Mason. Comparison of different approaches to conservation laws for some partial differential equations in fluid mechanics. *Applied Mathematics and Computation*, 1(205):212–30, 2008.

[17] N Ibragimov. A new conservation theorem. *Journal of Mathematical Analysis and Applications*, 1(333):311–28, 2007.