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On Multivalued Contractions via θ-Hyperbolic Sine Distance Functions

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Abstract. Very recently, the concept of θ-hyperbolic sine distance functions has been introduced by Jleli and Samet in [1]. In this work, we prove some related fixed points results for several classes of multivalued mappings including manageable functions on metric spaces.

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1. Introduction

In 1906, Fréchet [2] defined the concept of a metric space (MS).

Definition 1. *[2] Let X be any nonempty set. A function $d : X \times X \to [0, +\infty)$ is said to be a distance function or metric on X if for any $\varpi, \varsigma, s \in X$,*

(i) $d(\varpi, \varsigma) = 0$ *iff* $\varsigma = \varpi$,

(ii) $d(\varpi, \varsigma) = d(\varsigma, \varpi)$,

(iii) $d(\varpi, \varsigma) \leq d(\varsigma, s) + d(s, \varpi)$ *(triangle inequality).*

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Banach fixed point (FP) theorem [3] is an fundamental tool in the theory of MSs. It affirms the existence of a unique FP of contraction maps on complete MSs. Later, it has been generalized and extended in several directions, either by weakening the topology of the metric, or by generalizing the contraction itself. Several works arise in this sense, like [4–9].

On a MS (X, d), $CB(X)$ is the set of nonempty bounded and closed subsets of X. For $\Pi, \Xi \in CB(X)$, the Hausdorff-Pompieu metric induced by d is

$$H(\Pi, \Xi) = \max\left\{\sup_{a\in\Pi} \delta(a, \Xi), \sup_{b\in\Xi} \delta(b, \Pi)\right\},$$

where $\delta(\varsigma, \Pi) = \inf\{d(\varsigma, a) \mid a \in \Pi\}$ is the distance from ς to the set Π.

Definition 2. *Let X be any nonempty set. An element $\varsigma \in X$ is said to be a FP of a multivalued mapping $T : X \to 2^X$ if $\varsigma \in T(\varsigma)$, where 2^X denotes the collection of all nonempty subsets of X.*

Nadler [10] studied the existence of FPs for multivalued contractions.

Theorem 1. *[10] Let (X, d) be a complete MS and $T : X \to CB(X)$ be a contraction, i.e.,*

$$H(T\varsigma, T\varpi) \leq kd(\varpi, \varsigma),$$

for all $\varpi, \varsigma \in X$, where $k \in [0, 1)$. Then, there is a FP of T.

After the work of Nadler [10], many FP results for multivalued mappings appeared in literature. For more details, see [11–13].

Motivated by the fact that hyperbolic functions have variant applications in many fields, like physics, mathematics, engineering, etc, recently, Jleli and Samet [1] introduced the notion of θ-hyperbolic sine distance functions associated to a certain metric and obtained some nice FPs results. Following this direction, we aim to establish some FPs results for some classes of contractive multivalued mappings on MSs involving the θ-hyperbolic sine distance function.

For $\tau > 0$, let Θ_τ be the collection of functions $\theta : [0, +\infty) \to [0, +\infty)$ so that

$$\theta(t) \geq ct^\tau, \tag{1}$$

for all $t \geq 0$, where $c > 0$ is a constant.

Definition 3. Let (X, d) be a MS. For all $\tau > 0$ and $\theta \in \Theta_\tau$, consider $d_\theta : X^2 \to [0, +\infty)$ as

$$d_\theta(\varpi, \varsigma) = \theta(\sinh(d(\varpi, \varsigma))), \quad \forall \varpi, \varsigma \in X,$$

where sinh the hyperbolic sine function is given as

$$\sinh t = \frac{e^t - e^{-t}}{2}, \quad t \in \mathbb{R}.$$

The mapping d_θ is called the θ-hyperbolic sine distance function associated to the metric d.

Some properties of the θ-hyperbolic sine distance function are provided below.

Proposition 1. *[1] Let (X,d) be a MS and $\theta \in \Theta_\tau$ for some $\tau > 0$. Then, for all $\varpi, \varsigma \in X$, we have*

(i) $d_\theta(\varpi,\varsigma) = 0 \implies \varsigma = \varpi$.

(ii) *If $\theta(0) = 0$, then $d_\theta(\varsigma,\varsigma) = 0$.*

(iii) $d_\theta(\varpi,\varsigma) = d_\theta(\varsigma,\varpi)$.

Notice that a θ-hyperbolic sine distance function is not necessarily a metric, even if $\theta(0) = 0$. The next example shows this fact.

Example 1. Let $X = \mathbb{R}$ and $d(\varpi,\varsigma) = |\varsigma - \varpi|$ for all $\varpi,\varsigma \in X$. Let $\theta(t) = \sqrt{t}$ for all $t \geq 0$. Then, $\theta \in \Theta_{\frac{1}{2}}$. The θ-hyperbolic sine distance function associated to the metric d is defined by

$$d_\theta(\varpi,\varsigma) = \theta(\sinh(|\varsigma - \varpi|))$$

for all $\varpi,\varsigma \in X$. On the other hand, we have

$$\begin{aligned}
\frac{d_\theta(1,5)}{d_\theta(1,3) + d_\theta(3,5)} &= \frac{\theta(\sinh(4))}{\theta(\sinh(2)) + \theta(\sinh(2))} \\
&= \frac{\sqrt{\sinh(4)}}{2\sqrt{\sinh 2}} \\
&= \frac{1}{2}\sqrt{e^2 + e^{-2}} > 1,
\end{aligned}$$

which shows that d_θ does not verify the triangle inequality. Consequently, d_θ is not a metric on X.

Proposition 2. *[1] Let (X,d) be a MS.*

(i) *Let d_θ be the θ-hyperbolic sine distance function associated to d, where $\theta \in \Theta_\tau$ for some $\tau > 0$. Then, for all $\iota > 0$, we have*

$$\iota d_\theta = d_{\theta_\iota},$$

where $\theta_\iota = \iota\theta$.

(ii) *Let $\theta_1, \theta_2 \in \Theta_\tau$ for some $\tau > 0$. Then,*

$$d_{\theta_1} + d_{\theta_2} = d_\theta,$$

where $\theta = \theta_1 + \theta_2$.

Proposition 3. *[1] Let $\theta \in \Theta_\tau$ for some $\tau > 0$. Assume that:*

(i) $\theta(0) = 0$;

(ii) *There exists $r^* > 0$ such that*

$$\theta(\sinh r^*) = r^*.$$

Then, for every nonempty set X, there exists a metric d on X such that the θ-hyperbolic sine distance function associated to d coincides with d, i.e., $d_\theta = d$.

2. The Hausdorff θ-hyperbolic sine distance function

Our work is concerned with multivalued mappings $T : X \to 2^X$. For this, let (X, d) be a MS. Let $\theta \in \Theta_\tau$ for some $\tau > 0$. For two bounded and closed subsets Π, Ξ in X, consider

$$H_\theta(\Pi, \Xi) = \max\{\Delta_\theta(\Pi, \Xi), \Delta_\theta(\Xi, \Pi)\},$$

where

$$\Delta_\theta(\Pi, \Xi) = \sup\{\theta(\sinh(\delta(a, \Xi))) : a \in \Pi\} = \sup_{a\in\Pi} \inf_{b\in\Xi}\{\theta(\sinh(d(a, b)))\}.$$

The mapping H_θ is called the Hausdorff θ-hyperbolic sine distance function associated to the metric d. Some properties of H_θ are provided below.

Proposition 4. *Let (X, d) be a MS and $\theta \in \Theta_\tau$ for some $\tau > 0$. Then, for every $\Pi, \Xi \in CB(X)$,*

(i) $H_\theta(\Pi, \Xi) = 0 \implies \Pi = \Xi$.

(ii) *If $\theta(0) = 0$, then $H_\theta(\Pi, \Pi) = 0$.*

(iii) $H_\theta(\Pi, \Xi) = H_\theta(\Xi, \Pi)$.

Proof. (i)$H_\theta(\Pi, \Xi) = 0 \implies \Delta_\theta(\Pi, \Xi) = \Delta_\theta(\Xi, \Pi) = 0$. In the case $\Delta_\theta(\Pi, \Xi) = 0$, we get

$$\sup_{a\in\Pi}\{\theta(\sinh(\delta(a, \Xi)))\} = 0, \text{ i.e } \theta(\sinh(\delta(a, \Xi))) = 0\,\forall a \in \Pi.$$

Then for all $a \in \Pi$,

$$\exists (b_n) \subset \Xi \text{ so that } \lim_{n\to+\infty} \theta(\sinh(d(a, b_n))) = 0.$$

By (1), we obtain

$$\lim_{n\to+\infty}(\sinh(d(a, b_n)))^\tau = 0, \text{ for some } \tau > 0,$$

which implies

$$\lim_{n\to+\infty} \sinh(d(a, b_n)) = 0.$$

Thus, for all $a \in \Pi$,

$$\lim_{n\to+\infty} \delta(a, b_n) = 0, \text{ i.e } a \in \overline{\Xi} = \Xi.$$

So, $\Pi \subset \Xi$. Similarly, as $\Delta_\theta(\Xi, \Pi) = 0$, we have $\Xi \subset \Pi$. Finally, we obtain $\Pi = \Xi$.
(ii) If $\theta(0) = 0$, then

$$\begin{aligned}
H_\theta(\Pi, \Pi) &= \Delta_\theta(\Pi, \Pi) \\
&= \sup_{a\in\Pi} \inf_{b\in\Xi} \theta(\sinh(d(a, b))) \\
&\le \sup_{a\in\Pi} \theta(\sinh(d(a, a))) \\
&= \sup_{a\in\Pi} \theta(\sinh(0)) = \sup_{a\in\Pi} \theta(0) = \sup_{a\in\Pi}(0) = 0.
\end{aligned}$$

Thus, $H_\theta(\Pi,\Pi)=0$.
(iii) It is obvious.

Notice that a Hausdorff θ-hyperbolic sine distance function is not necessarily a Hausdorff metric, even $\theta(0)=0$. The following example shows this fact.

Example 2. Let $X=\mathbb{R}$ and $d(\varpi,\varsigma)=|\varsigma-\varpi|$ for all $\varpi,\varsigma\in X$. Let $\theta(t)=t$ for any $t\geq 0$. Take $A=\{0\}$, $B=\{2n\}$ and $C=\{n\}$, with $n\geq 1$. We write

$$\begin{aligned}\frac{H_\theta(\Pi,\Xi)}{H_\theta(\Pi,C)+H_\theta(C,\Xi)}&=\frac{\theta(\sinh(2n))}{2\theta(\sinh(n))}\\&=\frac{1}{2}(e^n+e^{-n})\to++\infty\,\text{as}\,n\to+\infty,\end{aligned}$$

which shows that H_θ does not satisfy the triangle inequality, and so H_θ is not a Hausdorff metric on $CB(X)$.

Definition 4. Let (X,d) a MS. A function $f:X\to[0,+\infty)$ is termed as lower semi-continuous if for $\{\varsigma_n\}\subset X$ and $\varsigma\in X$, we have

$$\lim_{n\to+\infty}d(\varsigma_n,\varsigma)=0\Rightarrow f(\varsigma)\leq\liminf_{n\to+\infty}f(\varsigma_n).$$

For $T:X\to CB(X)$, define $f_T:X\to[0,+\infty)$ by

$$f_T(\varsigma)=d(\varsigma,T\varsigma)\;\text{for all}\,\varsigma\in X.$$

3. FP results

In this part, we present FP results for some multivalued contractions via θ-hyperbolic sine functions.

3.1. Multivalued θ-hyperbolic contractions

Jleli and Samet [1] introduced the following class of single-valued mappings.

Definition 5. Let (X,d) be a MS and $\theta\in\Theta_\tau$ for some $\tau>0$. A mapping $T:X\to X$ is called a θ-hyperbolic contraction on X, if there is $k\in(0,1)$ so that

$$d_\theta(T\varsigma,T\varpi)\leq kd_\theta(\varpi,\varsigma)\tag{2}$$

for all $\varpi,\varsigma\in X$.

Also, they established the following FP theorem.

Theorem 2. *[1] Let (X,d) be a complete MS and $\theta\in\Theta_\tau$ for some $\tau>0$. Given $T:X\to X$ so that:*

(I) *T is a θ-hyperbolic contraction on X;*

(II) *For all* $\varpi, \varsigma \in X$*, if* $\lim_{n\to+\infty} d(T^n\varpi, \varsigma) = 0$*, then there exists a subsequence* $\{T^{n_k}\varsigma\}$ *of* $\{T^n\varsigma\}$ *such that* $\lim_{k\to+\infty} d(T(T^{n_k}\varsigma), T\varpi) = 0$.

Then, T *possesses one and only one FP. Moreover, for all* $w_0 \in X$*,* $\{T^n w_0\}$ *is convergent to this unique FP.*

We need the next lemma for the rest.

Lemma 1. *Let* $\Pi, \Xi \in CB(X)$*,* $a \in \Pi$ *and* $\theta \in \Theta_\tau$ *for some* $\tau > 0$*. Thus, for each* $\varepsilon > 0$*, there is* $b \in \Xi$ *so that*

$$d_\theta(a, b) \le H_\theta(\Pi, \Xi) + \varepsilon.$$

Proof. Let $\Pi, \Xi \in CB(X)$ and $a \in \Pi$. We have $d_\theta(a, \Xi) \le \Delta_\theta(\Pi, \Xi) \le H_\theta(\Pi, \Xi)$. Then, for every $\varepsilon > 0$, there is $b \in B$ so that

$$d_\theta(a, b) \le d_\theta(a, \Xi) + \varepsilon.$$

Consequently,

$$d_\theta(a, b) \le H_\theta(\Pi, \Xi) + \varepsilon.$$

The next result corresponds to the extension of Theorem 2 to multivalued mappings. It is stated as follows:

Theorem 3. *Let* (X, d) *be a complete MS and* $\theta \in \Theta_\tau$ *for* $\tau > 0$*. Let* $T : X \to CB(X)$ *be a mapping such that*

$$H_\theta(T\varsigma, T\varpi) \le k d_\theta(\varpi, \varsigma), \tag{3}$$

for all $\varpi, \varsigma \in X$*, where* $k \in [0, 1)$*. Assume that,* f_T *is lower semi-continuous, then* T *has a FP in* X*.*

Proof.
Let $\varsigma_0 \in X$ and $\varsigma_1 \in T\varsigma_0$. When $d_\theta(\varsigma_0, \varsigma_1) = 0$, so by Proposition 1 (i), one gets $\varsigma_0 = \varsigma_1$ and so ς_0 is a FP of T. Suppose that $d_\theta(\varsigma_0, \varsigma_1) > 0$. Since $T\varsigma_0, T\varsigma_1 \in CB(X)$ and $\varsigma_1 \in T\varsigma_0$, using Lemma 1, there is $\varsigma_2 \in T\varsigma_1$, so that

$$d_\theta(\varsigma_1, \varsigma_2) \le H_\theta(T\varsigma_0, T\varsigma_1) + \frac{1-k}{2} d_\theta(\varsigma_0, \varsigma_1). \tag{4}$$

If $d_\theta(\varsigma_1, \varsigma_2) = 0$, then by Proposition 1 (i), we have $\varsigma_1 = \varsigma_2$ and so ς_1 is a FP of T. When $d_\theta(\varsigma_1, \varsigma_2) > 0$, then by Lemma 1, there is $\varsigma_3 \in T\varsigma_2$, so that

$$d_\theta(\varsigma_2, \varsigma_3) \le H_\theta(T\varsigma_1, T\varsigma_2) + \frac{1-k}{2} d_\theta(\varsigma_1, \varsigma_2). \tag{5}$$

Continuing as above, we construct $\{\varsigma_n\} \subset X$ so that $\varsigma_{n+1} \in T(\varsigma_n)$ with $d_\theta(\varsigma_n, \varsigma_{n+1}) > 0$ and

$$d_\theta(\varsigma_{n+1}, \varsigma_n) \le H_\theta(T\varsigma_n, T\varsigma_{n-1}) + \frac{1-k}{2} d_\theta(\varsigma_n, \varsigma_{n-1}).$$

Then, using (3), we get

$$d_\theta(\varsigma_{n+1},\varsigma_n) \le k d_\theta(\varsigma_n,\varsigma_{n-1}) + \frac{1-k}{2} d_\theta(\varsigma_n,\varsigma_{n-1}) = \frac{1+k}{2} d_\theta(\varsigma_n,\varsigma_{n-1}).$$

By induction,

$$d_\theta(\varsigma_n,\varsigma_{n+1}) \le (\frac{1+k}{2})^n d_\theta(\varsigma_0,\varsigma_1), \quad \forall n \ge 1.$$

On the other hand, by (1), and since $\sinh t \ge t \, \forall t \ge 0$, we get

$$\theta(\sinh(d(\varsigma_n,\varsigma_{n+1}))) \ge c(\sinh(d(\varsigma_n,\varsigma_{n+1})))^\tau \ge c(d(\varsigma_n,\varsigma_{n+1}))^\tau, \quad \forall n \ge 1.$$

This yields to

$$d(\varsigma_n,\varsigma_{n+1}) \le ((\frac{1+k}{2})^{\frac{1}{\tau}})^n (\frac{d_\theta(\varsigma_0,\varsigma_1)}{c})^{\frac{1}{\tau}}, \quad \forall n \ge 1.$$

Since $k \in [0,1)$ and $\tau > 0$, one has

$$\sum_{n=0}^{+} \infty((\frac{1+k}{2})^{\frac{1}{\tau}})^n < +\infty.$$

So, for all $p \ge 0$, we have

$$d(\varsigma_n,\varsigma_{n+p}) \le d(\varsigma_n,\varsigma_{n+1}) + d(\varsigma_{n+1},\varsigma_{n+2}) + \cdots + d(\varsigma_{n+p-1},\varsigma_{n+p}).$$

That is,

$$d(\varsigma_n,\varsigma_{n+p}) \le (\frac{d_\theta(\varsigma_0,\varsigma_1)}{c})^{\frac{1}{\tau}} \sum_{i=n}^{n+p-1} ((\frac{1+k}{2})^{\frac{1}{\tau}})^n.$$

By summing the geometric series, we find

$$\sum_{i=n}^{+\infty}((\frac{1+k}{2})^{\frac{1}{\tau}})^i \to 0 \quad \text{as } n \to +\infty. \tag{3.4}$$

The symmetry of d leads to

$$\lim_{n,m\to+\infty} d(\varsigma_n,\varsigma_m) = 0. \tag{3.5}$$

This implies that $\{\varsigma_n\}$ is Cauchy in the complete MS (X,d). so $\{\varsigma_n\}$ converges to some $\varsigma^* \in X$. Next, the lower semi-continuity of f_T yields that

$$d(x^*,T\varsigma^*) = f_T(\varsigma^*) \le \liminf_{n\to+\infty} d(\varsigma_n,T\varsigma_n) \le \liminf_{n\to+\infty} d(\varsigma_n,\varsigma_{n+1}) = 0.$$

Finally, we get $d(\varsigma^*,T\varsigma^*) = 0$, that is, $\varsigma^* \in \overline{T\varsigma^*} = T\varsigma^*$. Then, ς^* is a FP of T.

3.2. $\theta-$hyperbolic contractions via manageable functions

In 2014, a new class of mappings called manageable functions was explored by Du and Khojasteh [14]. They used this class to obtain some FP theorems. In 2017, Hussain et al. [15] established some FP theorems in the setting of MSs for contraction mappings via manageable functions.

Definition 6. *[14] A manageable function $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function so that:*

($\eta 1$) $\eta(\hbar, \ell) < \ell - \hbar$ *for all* $\hbar, \ell > 0$*;*

($\eta 2$) *For each bounded* $\{\hbar_n\}$ *in* $(0, +\infty)$ *and each non-increasing* $\{\ell_n\}$ *in* $(0, +\infty)$*,*

$$\limsup_{n\to+\infty} \frac{\hbar_n + \eta(\hbar_n, \ell_n)}{\ell_n} < 1.$$

Let $\widehat{\mathrm{Man}(\mathbb{R})}$ be the set of manageable functions. We give the next two examples.

Example 3. *[14] Let $\wp \in [0, 1)$. Then $\eta_k : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$\eta_k(\hbar, \ell) = \wp\ell - \hbar$$

is a manageable function.

Example 4. *[16] Let $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$\eta(\hbar, \ell) = \begin{cases} \psi(\ell) - \phi(\hbar) & \text{if } (\hbar, \ell) \in [0, +\infty) \times [0, +\infty), \\ f(\ell, \hbar) & \text{otherwise,} \end{cases}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a given function and $\psi, \phi : [0, +\infty) \to \mathbb{R}$ are two functions sp that

- $\psi(\hbar) < \hbar \leq \phi(t)$ *for all* $\hbar > 0$*, and*
- $\limsup_{r\to\hbar^+} \frac{\psi(r)}{r} < 1$ *for any* $t \geq 0$*.*

Then, $\eta \in \widehat{Man(\mathbb{R})}$. Indeed, for any $s, t > 0$,

$$\eta(t, s) = \psi(s) - \phi(t) < s - t,$$

so, ($\eta 1$) holds. Let $\{t_n\}$ be a bounded and $\{s_n\}$ be a non-increasing in $(0, +\infty)$. Then $\lim_{n\to+\infty} s_n$ exists in $[0, +\infty)$. Hence,

$$\limsup_{n\to+\infty} \frac{\psi(s_n)}{s_n} = \limsup_{r\to t^+} \frac{\psi(r)}{r} < 1.$$

Thus, we get

$$\limsup_{n\to+\infty} \frac{t_n + \eta(t_n, s_n)}{s_n} = \limsup_{n\to+\infty} \frac{\psi(s_n) + t_n - \phi(t_n)}{s_n} \leq \limsup_{n\to+\infty} \frac{\psi(s_n)}{s_n} < 1.$$

It follows that ($\eta 2$) holds.

The next lemma is needful.

Lemma 2. *Let (X,d) be a MS, $\Xi \in CB(X)$ and $k>0$. Let $\theta \in \Theta_\tau$ for some $\tau > 0$. When $a \in X$ and $d_\theta(a,\Xi) < k$, then there is $b \in \Xi$ so that*

$$d_\theta(a,b) < k.$$

Proof. Let $a \in X$ and suppose that $d_\theta(a,B) < k$. Recall that

$$d_\theta(a,B) := \inf_{b\in B} d_\theta(a,b).$$

Suppose in the contrary, for all $b \in B$ we have $d_\theta(a,b) \geq k$. Thus, $\inf_{b\in B} d_\theta(a,b) = d_\theta(a,B) \geq k$, which is impossible. So there exists a point $b \in B$ such that $d_\theta(a,b) < k$.

Our second result for multivalued mappings involves manageable functions.

Theorem 4. *Let (X,d) be a complete MS and $T : X \to CB(X)$. Let $\theta \in \Theta_\tau$ for some $\tau > 0$. Suppose there is $\eta \in \widehat{Man}(\mathbb{R})$ so that*

$$\eta(H_\theta(Tx,Ty), d_\theta(\varpi,\varsigma)) \geq 0 \, \forall \, \varpi, \varsigma \in X. \tag{6}$$

If f_T is lower semi-continuous, then T admits a FP.

Proof. Let $\varsigma_0 \in X$ and $\varsigma_1 \in T\varsigma_0$. When $\varsigma_1 = \varsigma_0$ or $\varsigma_1 \in T\varsigma_1$, one has ς_1 is a FP of T. Otherwise, suppose $\varsigma_1 \neq \varsigma_0$ and $\varsigma_1 \notin T\varsigma_1$. So, $d_\theta(\varsigma_0,\varsigma_1) > 0$ and $d_\theta(\varsigma_1, T\varsigma_1) > 0$. By (6), we have

$$\eta(H_\theta(T\varsigma_0, T\varsigma_1), d_\theta(\varsigma_0,\varsigma_1)) \geq 0. \tag{7}$$

Define the function $\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\lambda(t,s) = \begin{cases} \frac{t+\eta(t,s)}{s} & \text{if } t,s>0, \\ 0 & \text{otherwise.} \end{cases}$$

By (η1), we have

$$0 < \lambda(t,s) < 1 \text{ for every } t,s > 0. \tag{8}$$

Also, if $\eta(t,s) \geq 0$, then

$$0 < t \leq s\lambda(t,s) \text{ for all } t,s>0. \tag{9}$$

From (7) and (8), we get

$$0 < \lambda(H_\theta(T\varsigma_0, T\varsigma_1), d_\theta(\varsigma_0,\varsigma_1)) < 1. \tag{10}$$

Since $d_\theta(\varsigma_1, T\varsigma_1) > 0$, by using (10), we have

$$d_\theta(\varsigma_1, T\varsigma_1) < \frac{1}{\sqrt{\lambda(H_\theta(T\varsigma_0, T\varsigma_1), d_\theta(\varsigma_0,\varsigma_1))}} d_\theta(\varsigma_1, T\varsigma_1).$$

Using Lemma 2, there is $\varsigma_2 \in T\varsigma_1$ so that

$$d_\theta(\varsigma_1, \varsigma_2) < \frac{1}{\sqrt{\lambda(H_\theta(T\varsigma_0, T\varsigma_1), d_\theta(\varsigma_0, \varsigma_1))}} d_\theta(\varsigma_1, T\varsigma_1). \tag{11}$$

It follows that

$$d_\theta(\varsigma_1, T\varsigma_1) \leq d_\theta(\varsigma_0, \varsigma_1)\lambda(H_\theta(T\varsigma_0, T\varsigma_1), d_\theta(\varsigma_0, \varsigma_1)). \tag{12}$$

Combining (11) and (12), we get

$$d_\theta(\varsigma_1, \varsigma_2) \leq \sqrt{\lambda(H_\theta(T\varsigma_0, T\varsigma_1), d_\theta(\varsigma_0, \varsigma_1))} d_\theta(\varsigma_0, \varsigma_1).$$

Note that $\varsigma_2 \neq \varsigma_1$ because $\varsigma_1 \notin T\varsigma_1$. When $\varsigma_2 \in T\varsigma_2$, ς_2 is a FP of T. Suppose $\varsigma_2 \notin T\varsigma_2$. Hence, by (6),

$$\eta(H_\theta(T\varsigma_1, T\varsigma_2), d_\theta(\varsigma_1, \varsigma_2)) \geq 0.$$

Since $d_\theta(\varsigma_2, T\varsigma_2) > 0$, by using (10), we have

$$d_\theta(\varsigma_2, T\varsigma_2) < \frac{1}{\sqrt{\lambda(H_\theta(T\varsigma_1, T\varsigma_2), d_\theta(\varsigma_1, \varsigma_2))}} d_\theta(\varsigma_1, T\varsigma_2).$$

Lemma 2 implies the existence of a point $\varsigma_3 \in T\varsigma_2$ such that

$$d_\theta(\varsigma_2, \varsigma_3) < \frac{1}{\sqrt{\lambda(H_\theta(T\varsigma_1, T\varsigma_2), d_\theta(\varsigma_1, \varsigma_2))}} d_\theta(\varsigma_2, T\varsigma_2).$$

Similarly, we get

$$d_\theta(\varsigma_2, \varsigma_3) \leq \sqrt{\lambda(H_\theta(T\varsigma_1, T\varsigma_2), d_\theta(\varsigma_1, \varsigma_2))} d_\theta(\varsigma_1, \varsigma_2).$$

Continuing the same work, we build $\{\varsigma_n\}$ in X so that for any $n \geq 1$,

$$d_\theta(\varsigma_n, \varsigma_{n+1}) \leq \sqrt{\lambda(H_\theta(T\varsigma_{n-1}, T\varsigma_n), d_\theta(\varsigma_{n-1}, \varsigma_n))} d_\theta(\varsigma_{n-1}, \varsigma_n). \tag{13}$$

From (8) and (13), we get $0 < d_\theta(\varsigma_n, \varsigma_{n+1}) < d_\theta(\varsigma_{n-1}, \varsigma_n)$ for all n, which yields that $\{d_\theta(\varsigma_{n-1}, \varsigma_n)\}$ is non-increasing and positive, so it is convergent. Also,

$$0 < H_\theta(T\varsigma_{n-1}, T\varsigma_n) < d_\theta(\varsigma_{n-1}, \varsigma_n),$$

for all n, which yields that $\{H_\theta(T\varsigma_{n-1}, T\varsigma_n)\}$ is bounded. From (η_2),

$$\limsup_{n\to+\infty} \lambda(H_\theta(T\varsigma_{n-1}, T\varsigma_n), d_\theta(\varsigma_{n-1}, \varsigma_n)) < 1. \tag{14}$$

Let

$$\lambda_n = \sqrt{\lambda(H_\theta(T\varsigma_{n-1}, T\varsigma_n), d_\theta(\varsigma_{n-1}, \varsigma_n))}, \quad \forall n \geq 1. \tag{15}$$

From (13), we get

$$d_\theta(\varsigma_n, \varsigma_{n+1}) \leq \lambda_n d_\theta(\varsigma_{n-1}, \varsigma_n), \quad \forall n \geq 1. \tag{16}$$

By (14), there are $\alpha \in (0,1)$ and $n_0 \in \mathbb{N}$ so that

$$\lambda_n \leq \alpha, \quad \forall n \geq n_0.$$

Hence, by (16), we get

$$d_\theta(\varsigma_n, \varsigma_{n+1}) \leq \alpha d_\theta(\varsigma_{n-1}, \varsigma_n), \quad \forall n \geq n_0.$$

Thus,

$$d_\theta(\varsigma_n, \varsigma_{n+1}) \leq \alpha^{n-n_0+1} d_\theta(\varsigma_{n_0-1}, \varsigma_{n_0}), \quad \forall n \geq n_0.$$

Moreover, by (1), and $\sinh(t) \geq t$ for all $t \geq 0$, one gets

$$d(\varsigma_n, \varsigma_{n+1}) \leq (\alpha^{\frac{1}{\tau}})^{n-n_0+1} (\frac{d_\theta(\varsigma_{n_0-1}, \varsigma_{n_0})}{c})^{\frac{1}{\tau}}, \quad \forall n \geq n_0.$$

Now, for $m > n \geq n_0$, we have

$$d(\varsigma_n, \varsigma_m) \leq \sum_{i=n}^{m-1} d(\varsigma_i, \varsigma_{i+1}) \leq (\frac{d_\theta(\varsigma_{n_0-1}, \varsigma_{n_0})}{c})^{\frac{1}{\tau}} \sum_{i=n}^{+\infty} (\alpha^{\frac{1}{\tau}})^{i-n_0+1} \to 0 \quad \text{as } n \to +\infty.$$

Thus,

$$\lim_{n,m \to +\infty} d(\varsigma_n, \varsigma_m) = 0.$$

So $\{\varsigma_n\}$ is a Cauchy sequence in the complete MS (X, d). Then, there is $u \in X$ so that

$$\lim_{n \to +\infty} d(\varsigma_n, u) = 0.$$

By the lower semi-continuity of T, we obtain that $u \in Tu$.

Remark 1. *From Theorem 4, several corollaries could be derived following particular cases of manageable functions.*

The next example inspired from Example 3.3 in [1] makes effective Theorem 4. Here, the theorem of Nadler [10] is not applicable..

Example 5. Let $X = \{1, 2, 3\}$. Take d the metric on X given as

$$d(\varpi, \varsigma) = d(\varsigma, \varpi),\ d(\varsigma, \varsigma) = 0\, \forall \varpi, \varsigma \in X$$

and

$$d(1,2) = 1, \ d(1,3) = 4, \ d(2,3) = 5.$$

Notice that (X, d) is a complete MS. Choose $T : X \to CB(X)$ as

$$T1 = T3 = \{1\} \text{ and } T2 = \{1, 3\}.$$

We point out that T is not a contraction in the sense of Nadler [10]. Indeed,

$$H(T1, T2) = \max\{d(1,1), d(1,3)\} = 4 > 1 = d(1,2).$$

We now introduce the mapping $\theta : [0, +\infty) \to [0, +\infty)$ defined by

$$\theta(t) = \begin{cases} \dfrac{7t}{\sinh 1} & \text{if} \quad 0 \le t \le \sinh 1, \\ \dfrac{2t}{\sinh 4} & \text{if} \quad \sinh 1 < t < \sinh 4, \\ \dfrac{5t}{4\sinh 5} & \text{if} \quad t \ge \sinh 4. \end{cases}$$

Clearly,

$$\theta(t) \ge \frac{2}{\sinh 5} t, \quad \forall t \ge 0,$$

which shows that $\theta \in \Theta_1$ and $\theta(0) = 0$. Furthermore, take $\eta(t, s) = ks - t$ for all $s, t \in \mathbb{R}$ with $k \in [\frac{2}{5}, 1)$.
We have

$$\begin{aligned} H_\theta(T1, T2) &= \max\{d_\theta(1,3), d_\theta(1,1)\} = d_\theta(1,3) = \theta(\sinh 4) = 2 \\ &= \frac{2}{7} \times 7 = \frac{2}{7} \times \theta(\sinh 1) = \frac{2}{7} d_\theta(1,2) \le \frac{2}{5} d_\theta(1,2). \end{aligned}$$

Also,

$$H_\theta(T2, T3) = 2 = \frac{2}{5} \times 5 = \frac{2}{5}\theta(\sinh 5) = \frac{2}{5} d_\theta(2,3).$$

Furthermore,

$$H_\theta(T1, T3) = 0 \le \frac{2}{5} d_\theta(1,3),$$

which implies

$$H_\theta(T\varsigma, T\varpi) \le \frac{2}{5} d_\theta(\varpi, \varsigma) \, \forall \, \varpi, \varsigma \in X.$$

We also have

$$\eta(H_\theta(T\varsigma, T\varpi), d_\theta(\varpi, \varsigma)) = kH_\theta(T\varsigma, T\varpi) - d_\theta(\varpi, \varsigma) \ge (k - \frac{2}{5}) d_\theta(\varpi, \varsigma) \ge 0 \, \forall \, \varpi, \varsigma \in X.$$

Let $\varsigma \in X$ and $\{\varsigma_n\} \subset X$ so that $\lim_{n \to +\infty} d(\varsigma_n, \varsigma) = 0$. Then, there is $n_0 \ge 0$ so that $\varsigma_n = \varsigma$ for all $n_0 \ge 0$. Then $T\varsigma_n = T\varsigma$ for all $n_0 \ge 0$. Henc,e $d(\varsigma, T\varsigma) = d(\varsigma_n, T\varsigma_n)$ for all $n_0 \ge 0$. Finally, we get

$$d(\varsigma, T\varsigma) = \liminf_{n \to +\infty} d(\varsigma_n, T\varsigma_n).$$

Consequently, all required hypothesises of Theorem 4 hold. Here, 1 is a FP of T.

4. Conclusion

In this work, we initiated the concept of a Hausdorff θ-hyperbolic sine distance function. We proved two FP results for multivalued contraction mappings, one of Nadler type , and the second using manageable functions. As open problems, we suggest to prove further FP results for multivalued mappings, using either different types of control function, like:
(i) implict functions;
(ii) α-admissibility,
or, via generalized metrics.

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