EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6030
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Symmetries and Novel Exact Solutions for (2+1)-D QZK equation via Lie-Symmetry and Kudryashov-Auxaliry Method

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Abstract. In this paper, the (2+1)-D quantum Zakharov-Kuznetsov (QZK) equation that describes how nonlinear ion-acoustic waves diffuse in magnetized plasma is studied. Firstly, The governing equation was transformed into a number of ordinary differential equations using symmetry analysis. After that, We used Kudryashov-Auxaliry Method (KAM) to develop a new kind of accurate answers for the QZK equation. The discovered solutions included a number of arbitrary constants that improved their dynamic characteristics. The resulting solutions represent solitary wave, single wave, and multisolitons solutions and include hyperbolic and trigonometric functions.

2020 Mathematics Subject Classifications: 35C05, 35Q60, 35A30, 35B06

Key Words and Phrases: Lie-Symmetry method, Quantum Zakharov-Kuznetsov, Kudryashov-Auxaliry Method, Wave solutions

1. Introduction

Over the past centuries, scientists have sought to understand and explain many natural and physical phenomena. The discovery of partial differential equations (PDEs) has led to the explanation of many of these events. Many scientists have turned to discovering the types of PDEs and their approximate and accurate solutions. This has helped in the ease of understanding and developing many models and how to use them in scientific development. Through that, mathematicians have made remarkable progress in inventing

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6030

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many methods to solve nonlinear PDEs for explaining many physical phenomena. Many nonlinear PDEs have been used in the fields of engineering, physics, and other scientific and social sciences. Modern science has produced many equations, such as Korteweg-de-Vries [1, 2] , Burgers [2] , Boltzmann [4] and other famous equations.

On the other hand, scientists have discovered and developed many methods to solve nonlinear partial differential equations and find accurate or approximate solutions. These methods include Lie-Symmetry method [5 − 9], Kudryashov method [7, 10], Exp-function method [11], $\tan(\varphi/2)$-expansion method [12], Jacobi elliptic function [13] , (G′/G)-expansion method [14, 15] and others [16 − 19].

The main challenge of this work are studying symmetries and obtaining a novel exact solutions for the (2+1)-dimensional QZK equation [15, 20 − 22], which can be written in the form

$$v_t + pv\ v_x + q(v_{x,x,x} + v_{y,y,y}) + r(v_{x,y,y} + v_{x,x,y}) = 0, \tag{1}$$

where $v = v(x, y, t)$. Many researchers have been interested in this equation [20 − 22]. Nuruddeen et al. [21] reduced the governing equation to a single form and then used tanh method to obtain solutions. On the other hand, Vinita and Ray [20] reduced the governing equation and obtained some solutions

2. Symmetries

Proposition 1: The QZK equation (1) has the following five Lie point symmetries:

$$\begin{aligned} W_1 &= t\frac{\partial}{\partial t} + \frac{1}{3}x\frac{\partial}{\partial x} + \frac{1}{3}y\frac{\partial}{\partial y} - \frac{2}{3}v\frac{\partial}{\partial v}, \quad W_2 = \frac{\partial}{\partial t}, \\ W_3 &= t\frac{\partial}{\partial x} + \frac{1}{a}\frac{\partial}{\partial v},\ W_4 = \frac{\partial}{\partial x},\ W_5 = \frac{\partial}{\partial y}, \end{aligned} \tag{2}$$

Proof: A Lie group with infinitesimal on the space of independent and dependent variables with one parameter ε is examined as follows:

$$\begin{aligned} t^* &= t + \varepsilon A(x, y, t, v) + \vartheta(\varepsilon^2), x^* = x + \varepsilon B(x, y, t, v) + \vartheta(\varepsilon^2), \\ y^* &= y + \varepsilon C(x, y, t, v) + \vartheta(\varepsilon^2), v^* = v + \varepsilon \Phi(x, y, t, v) + \vartheta(\varepsilon^2). \end{aligned} \tag{3}$$

The third vector field which can generated the Lie algebra of the (2+1)-dimensional QZK equation can be expressed as follows:

$$\Gamma^{(3)} = \chi + \Phi_{[x]}\frac{\partial}{\partial v_x} + \Phi_{[xxx]}\frac{\partial}{\partial v_{x,x.x}} + \Phi_{[yyy]}\frac{\partial}{\partial v_{y,y,y}} + \Phi_{[xxy]}\frac{\partial}{\partial v_{x,x,y}} + \Phi_{[xyy]}\frac{\partial}{\partial v_{x,y,y}}. \tag{4}$$

A description of the infinitesimal vector χ associated with the aforementioned transformations is given.

$$\chi = T\frac{\partial}{\partial t} + A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial v}. \tag{5}$$

where the components $\Phi_{[t]}, \Phi_{[x]}$, $\Phi_{[xxx]}$, $\Phi_{[xxy]}$, $\Phi_{[yxx]}$.... establish as the expressions:

$$\Phi_{[x]} = D_x\Phi - v_t D_x A - v_x D_x B - v_y D_x C,$$
$$\Phi_{[xy]} = D_y\Phi - v_{tx} D_y A - v_{xx} D_y B - v_{xx} D_y C. \tag{6}$$

The invariance condition is satisfied [5, 6]

$$\Gamma^{(3)}(\Delta) = 0, \tag{7}$$

where

$$\Delta = v_t + pv\ v_x + q(v_{x,x,x} + v_{y,y,y}) + r(v_{x,y,y} + v_{x,x,y}) = 0, \tag{8}$$

This invariance condition leads to a specific system of PDEs. When we solve this condition, we get

$$A = c_1 t + c_2, \qquad B = \frac{1}{3}x + c_3 t + c_4,$$
$$C = \frac{1}{3}c_1 y + c_5,\ \Phi = -\frac{2}{3}c_1 v - \frac{1}{a}c_3. \tag{9}$$

where Ω is sum of $W_1, ...W_5$. The commutator relations are given by Table 1.

Table 1: The commutator table of Ω

	W_1	W_2	W_3	W_4	W_5
W_1	0	$\frac{-1}{3}W_2$	0	$\frac{-1}{3}W_4$	$\frac{-1}{3}W_5$
W_2	$\frac{1}{3}W_2$	0	$-W_3$	W_4	0
W_3	0	W_3	0	0	0
W_4	$\frac{1}{3}W_4$	$-W_4$	0	0	0
W_5	$\frac{1}{3}W_5$	0	0	0	0

Table 2: The adjoint table of Ω

	W_1	W_2	W_3	W_4	W_5
W_1	W_1	$e_2^{\frac{\epsilon}{3}}W_2$	W_3	$e_4^{\frac{\epsilon}{3}}W_4$	$e_5^{\frac{\epsilon}{3}}W_5$
W_2	$W_1 - \frac{1}{3}W_2$	W_2	$e_3^{\epsilon}W_3$	$e_4^{-\epsilon}W_4$	W_5
W_3	W_1	$W_2 - W_3$	W_3	W_4	W_5
W_4	$W_1 - \frac{1}{3}W_4$	$W_2 - W_4$	W_3	W_4	W_5
W_5	$W_1 - \frac{1}{3}W_5$	W_2	W_3	W_4	W_5

where $Adj(exp(\varepsilon W_i))W_j = W_j - \epsilon[W_i, W_j] + \frac{\epsilon^2}{2}[W_i, [W_i, W_j]],\ \ i = 1, 2, 3$ and $[W_i, W_j] = W_{iWj} - W_{jWi}$.

The asymmetric probability are derived from the preceding table under the following scenarios:

(I) $W_1 + m\ W_2$
(II) $W_1 + W_3$
(III) $W_2 +\ m_1 W_4 + m_2 W_5$
(IV) $W_4 + mW_5$

3. The reductions and exact solutions

The constant transformation can be obtained by applying the characteristic equation that follows:

$$\frac{dt}{c_1 t + c_2} = \frac{dx}{\frac{1}{3}x + c_3 t + c_4} = \frac{dy}{\frac{1}{3}c_1 y + c_5} = \frac{du}{-\frac{2}{3}c_1 v - \frac{1}{a}c_3}. \tag{10}$$

Case I: Substituting W_1,and W_2 into Eq.(10) correspondingly, the invariant variables are

$$\zeta_1 = \frac{x}{(t+m)^{\frac{1}{3}}},\ \zeta_2 = \frac{y}{(t+m)^{\frac{1}{3}}},\ v = \frac{F(\zeta_1, \zeta_2)}{(t+m)^{\frac{2}{3}}}, \tag{11}$$

where $m = c_3$.

Utilizing Eq.(11) into Eq.(1), then Eq.(1) is reduced to the posterior equation

$$-2F - \zeta_1 F_{\zeta_1} - \zeta_2 F_{\zeta_2} + 3pFF_{\zeta_1} + 3q(F_{\zeta_1\zeta_1\zeta_1} + F_{\zeta_2\zeta_2\zeta_2}) + 3r(F_{\zeta_1\zeta_2\zeta_2} + F_{\zeta_2\zeta_1\zeta_1}) = 0. \tag{12}$$

In this case, by putting $\theta = h\zeta_1 + k\zeta_2$, Eq.(12) can be written in the following form:

$$-2F - \theta F' + 3pFF' + 3[q(k^3 + h^3) + r(k^2 h + kh^2)]F''' = 0. \tag{13}$$

Taking the solution to the previous equation in the following form:

$$F = a_0 + \sum_{i=1}^{m} (a_i \theta^i + b_i \theta^{-i}) \tag{14}$$

Substituting Eq.(14) into Eq.(13) and obtainig the arbitrary constant a_0, a_i and b_i .The closed form solution of Eq.(13), takes the following expression:

$$F = \frac{-1}{pk}(\zeta_1 + \zeta_2). \tag{15}$$

The general solution of Eq. (1) writtes in the form:

$$v(x,y,t) = \frac{-1}{pk(t+m)^{\frac{2}{3}}}\left(\frac{k\ x}{(t+m)^{\frac{1}{3}}} + \frac{h\ y}{(t+m)^{\frac{1}{3}}}\right) \tag{16}$$

Case II: Similarly, in the earlier case, substituting W_1,and W_3 in Eq.(2). The form of the invariant variables as follows

$$\zeta_1 = \frac{x - \frac{2}{3}t}{t^{\frac{1}{3}}},\ \zeta_2 = \frac{y}{t^{\frac{1}{3}}},\ v = \frac{1}{t^{\frac{2}{3}}}F(\zeta_1, \zeta_2). \tag{17}$$

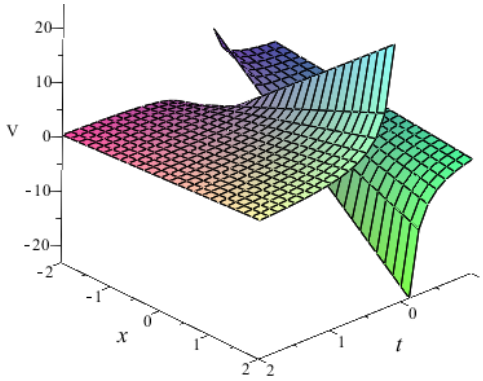

Figure 1: The solitary wave solution of (16).

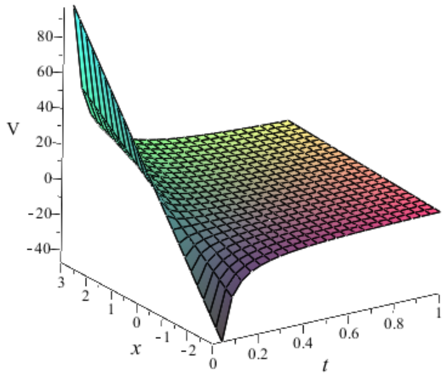

Figure 2: The Double wave solution of (21)

We obtained the following equation by substituting Eq. (17) into Eq. (1).

$$-2F - \zeta_1 F_{\zeta_1} - \zeta_2 F_{\zeta_2} + 3pFF_{\zeta_1} + 3q(F_{\zeta_1\zeta_1\zeta_1} + F_{\zeta_2\zeta_2\zeta_2}) + 3r(F_{\zeta_1\zeta_2\zeta_2} + F_{\zeta_2\zeta_1\zeta_1}) = 0. \quad (18)$$

We take $\theta = k\zeta_1 + h\zeta_2$, then Eq.(18) takes the form

$$-2F - \theta F' + 3pFF' + 3[q(k^3 + h^3) + r(k^2h + kh^2)]F''' = 0. \quad (19)$$

The following formula represents the closed form solution of Eq. (13):

$$F = \frac{-1}{pk}(\zeta_1 + \zeta_2). \quad (20)$$

The exact solution of Eq. (1) writtes in the form:

$$v(x,y,t) = \frac{-1}{pkt^{\frac{2}{3}}}\left(\frac{k\ (x - \frac{3}{2}t)}{t^{\frac{1}{3}}} + \frac{h\ y}{t^{\frac{1}{3}}}\right) \quad (21)$$

Case III: The vectors of W_2, W_4 and W_5 are substituted in the characteristic equation, then the following is the form of the invariant variables.

$$\zeta_1 = x - m_1 t,\ \zeta_2 = y - m_2 t,\ v = F(\zeta_1, \zeta_2), \quad (22)$$

Utilizing the relations of Eq.(22) to reduce Eq.(1) in the form

$$-m_1F_{\zeta_1} - m_2F_{\zeta_2} + pFF_{\zeta_1} + q(F_{\zeta_1\zeta_1\zeta_1} + F_{\zeta_2\zeta_2\zeta_2}) + r(F_{\zeta_1\zeta_2\zeta_2} + F_{\zeta_2\zeta_1\zeta_1}) = 0. \tag{23}$$

We take $\theta = k\zeta_1 + h\zeta_2$, then Eq.(23) takes the form

$$(-m_1k - m_2h)F' + pkFF' + [q(k^3 + h^3) + r(k^2h + kh^2)]F''' = 0. \tag{24}$$

We employ Kudryashov-Auxalіry Method [23], which is expressed as follows:

$$F(\theta) = A_0 + \sum_{i=1}^{m} \frac{A_i}{[1 + \Psi(\theta)]^i}, \tag{25}$$

where $\Psi(\theta)$ satisfies the following auxaliry equation [23]

$$\Psi'(\theta) = R + Q\ \Psi^2(\theta) + P\ \Psi^4(\theta), \tag{26}$$

Balncing between linear term F''' and non linear term FF', we get

$$F(\theta) = A_0 + \frac{A_1}{1 + \Psi(\theta)} + \frac{A_2}{[1 + \Psi(\theta)]^2}. \tag{27}$$

Substituting (27) into (24) and equating the coefficients of all powers of $\Psi(\theta)$ to zero. Utilizing Maple for solving the algebraic equations of A_i, we get:

$$A_1 = \frac{6}{pm_1k}(2qk^3P + 2rk^2hP + 2rkh^2P + rkh^2Q + qh^3Q + 2qh^3P + qk^3Q + rk^2hQ), R = -(P + Q)$$
$$m_2 = \frac{1}{h}(6qk^3P + 6rk^2Ph(1 + h) + rkhQ(k + h) + qh^3Q + 6qh^3P + pm_1kA_0 - m_1k + qk^3Q). \tag{28}$$

The appropriate solitary wave solutions of Eq.(2) yield:

Case 1: $P = 1, Q = 1, R = 0$,

$$v(x, y, t) = A_0 + \frac{6h^2(qm^3 + 2qm^3 - q - 2q - m^2r - 2m^2r + 2rm + rm)}{pm[1 + \csc(k(x - m_1) + h(y - m_2))]}. \tag{29}$$

where $A_0 = \frac{-1}{pmh}(-6h^3(rm^2 + h^3q) + h^3qm^3 + 6h^3qm(m^2 + 1) + hQrm(1 - m) - h^3q - k)$.

Case 2: $P = 1, Q = -1, R = 1$,

$$v(x, y, t) = A_0 + \frac{h^2(qm^3 - 2qmP - q + 2q - m^2r + 2m^2r - 2rm + rm)}{pm[1 + \tan(k(x - m_1) + h(y - m_2))]}. \tag{30}$$

where $A_0 = \frac{-1}{pmh}(6h^3(rm^2 + h^3q) + h^3qm^3 - 6h^3qm(m^2 + 1) + h^3rm(1 - m) - h^3q - k)$.

Case 3: $P = 0, Q = -1, R = 1$,

$$v(x, y, t) = A_0 + \frac{6h^2(qm^3 - q - m^2r + rm)6h^2(qm^3 - q - m^2r + rm)}{pm[1 + \sin(k(x - m_1) + h(y - m_2))]}. \tag{31}$$

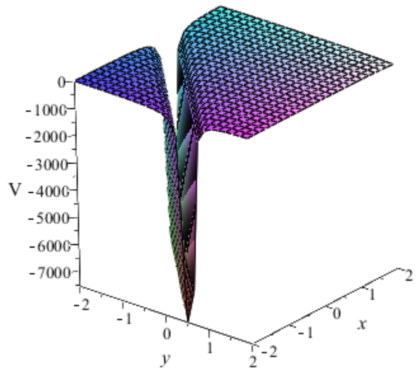

Figure 3: The single wave solution of (29).

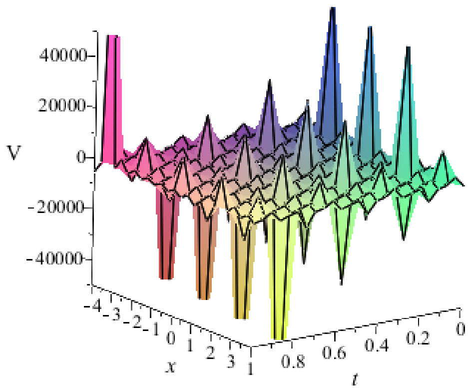

Figure 4: The multi-soliton solution of (30).

where $A_0 = \frac{-1}{pmh}(h^3qm^3 + h^3rm(1-m) - h^3q - k)$.

Case IV: Utilizing vectors W_4 and W_5 for obtainig the invariant variables as follow

$$\zeta_1 = t,\ \zeta_2 = y - mx,\ v = F(\zeta_1, \zeta_2), \tag{32}$$

Using the invariant variables in Eq.(32). Eq.(1) is converted to the format

$$F_{\zeta_1} - pmFF_{\zeta_1} + q(-m^3F_{\zeta_1\zeta_1\zeta_1} + F_{\zeta_2\zeta_2\zeta_2}) + r(m^2F_{\zeta_1\zeta_2\zeta_2} + F_{\zeta_2\zeta_1\zeta_1}) = 0. \tag{33}$$

Taking $\theta = k\zeta_1 + h\zeta_2$, then Eq. (33) can be written as following

$$kF' - pmhFF' + [qh^3(-m^3+1) + rh^3(m^2-1)]F''' = 0, \tag{34}$$

Substituting Eq.(27) into Eq.(34), equating to zero the coefficients of all powers of $\Psi(\theta)$ yields a set of algebraic equations for A_i. By applying Maple we solve this algebraic equation, to yield

$$A_0 = \frac{-1}{pmh}(-6h^3P(rm^2 + h^3q) + h^3qm^3Q + 6h^3qPm(m^2+1) + h^3Qrm(1-m) - h^3qQ - k),$$

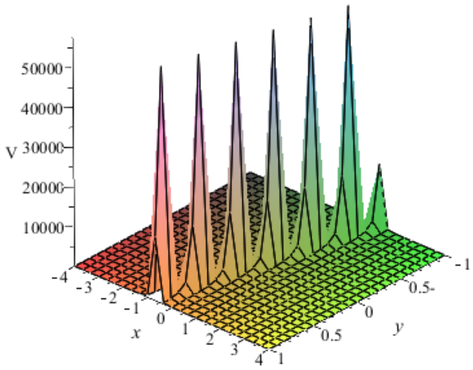

Figure 5: The oscillating-solitons solution of (31).

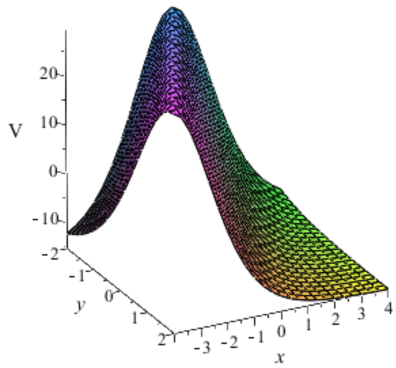

Figure 6: The wave solution solution of (36).

$$A_1 = \frac{6h^2}{pm}(qm^3Q + 2qm^3P - qQ - 2qP - m^2rQ - 2m^2rP + 2rmP + rmQ) \tag{35}$$

The general exact solutions of Eq.(2) taking form

Case 1: $P = -1, Q = 1, R = 0,$

$$v(x, y, t) = A_0 + \frac{6h^2(qm^3 - 2qmP - q + 2q - m^2r + 2m^2r - 2rm + rm)}{pm[1 + \operatorname{sech}(kt + h(y - m_2x))]}. \tag{36}$$

where $A_0 = \frac{-1}{pmh}(6h^3(rm^2 + h^3q) + h^3qm^3 - 6h^3qm(m^2 + 1) + h^3rm(1 - m) - h^3q - k)$.

Case 2: $P = 1, Q = 1, R = 0,$

$$v(x, y, t) = A_0 + \frac{6h^2(qm^3 + 2qm^3 - q - 2q - m^2r - 2m^2r + 2rm + rm)}{pm[1 + \operatorname{csch}(kt + h(y - m_2x))]}. \tag{37}$$

where $A_0 = \frac{-1}{pmh}(-6h^3(rm^2 + h^3q) + h^3qm^3 + 6h^3qm(m^2 + 1) + hQrm(1 - m) - h^3q - k)$.

Case 3: $P = 0, Q = 1, R = -1,$

$$v(x, y, t) = A_0 + \frac{6h^2(qm^3 - q - m^2r + rm)6h^2(qm^3 - q - m^2r + rm)}{pm[1 + \cosh(kt + h(y - m_2x))]}. \tag{38}$$

where $A_0 = \frac{-1}{pmh}(h^3qm^3 + h^3rm(1-m) - h^3q - k)$.

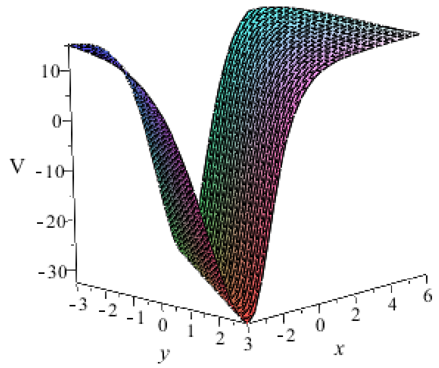

Figure 7: The wave solution of (37).

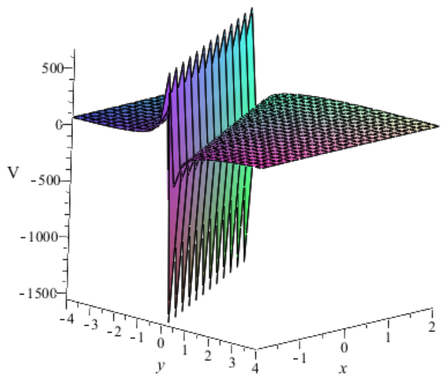

Figure 8: The single wave solution of (38).

4. Discussion and Results

We contrast our exact answers and similarity reduction findings with those of earlier research:

1) Nuruddeen et al. [20] reduced the govern equation the one case of ordinary differential equations. They used the tanh technique to get precise answers for these cases. In our study, we employed a novel traveling wave method and achieved numerous accurate answers for four common examples.

2) Vinita and Ray[21] investigated the similarity reductions for the gonvering equation. Their reductions, however, are subcases of ours. We obtained new accurate solutions for four general examples in our study.

5. Conclusion

In this work, we reduced the governing equation to four distinct ordinary differential equations using symmetry analysis. In order to generate new types of solutions, we finally used traveling wave approach called KAM. We graphed the solutions to display their attributes. The solutions that were discovered are entirely different from those that were obtained in earlier research.

Acknowledgements

The author Ahmed A. Gaber would like to thank the Deanship of Scientific Research, Majmaah University, Saudi Arabia, for supporting this work under project No. R-2025-1786

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