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Notes on Finite Groups with Nearly Sylow-permutable and Nearly Sylow-permutable-Transitive Subgroups

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Abstract. Let G be a finite group and let H be a subgroup of G. We called H is nearly S-permutable in G if for every prime p such that (p: $|H|= 1$) p-subgroup of K. We shall denote this by (H is *NSP* in G). We introduce the class of nearly S-permutable transitive -groups as those groups in which nearly S-permutability is transitive among subgroups. That is, if A is *NSP* in B; and B is *NSP* in G, then A is *NSP* in G: In this paper we study some characterize finite groups using *NSP* and *NSPT* and we compare some subgroups with groups under study, supported by theorems and examples.

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1. Introduction

In this paper, all groups under discussion are finite. A subgroup H of a group G is said to commute with another subgroup K if the product HK is also a subgroup of G. If H commutes with every subgroup (or every Sylow subgroup) of G, it is called a permutable (or S-permutable) subgroup [1]. A well-known fact in group theory is that normal p-Sylow subgroups imply nilpotency, as normality plays a central role in subgroup structure. One fundamental property of normal subgroups is that if N is a normal subgroup of G and H is any subgroup of G, then $NH = HN$.

However, it has been observed that some subgroups, although not normal, still commute with every subgroup in the group [2]. According to [1], an S-permutable subgroup of a group is subnormal. In contrast, nearly S-permutability does not necessarily imply subnormality [3]. A notable example is the dihedral group $D_{18} = \langle r, s : r^9 = s^2 = e, rs = sr^8 \rangle$, which contains nearly S-permutable subgroups that are not subnormal [4, 5]. This

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highlights the significance of studying the structural differences between these subgroup properties.

Motivated by this, our research focuses on nearly S-permutable subgroups and introduces a new class of groups called $NSPT$-groups, in which the property of nearly S-permutability is transitive among subgroups. That is, if A is nearly S-permutable in B, and B is nearly S-permutable in G, then A is nearly S-permutable in G [6].

Several generalizations of normality and subnormality have been investigated in the literature, including c-normality and S-permutability. For example, soluble T-groups (where every subnormal subgroup is normal) were studied in [7], and similar results for S-permutability were developed in [8, 9]. Analogous results for nearly S-permutability were given in [10], where the structure of soluble groups with nearly S-permutable subnormal subgroups was described.

2. Preliminaries

This section will introduce fundamental theories and concepts related to groups and subgroups, which serve as the foundation for abstract algebra and will be applied later.

Definition 1. *[10] A subgroup H of G is termed nearly S-permutable in G if, for every prime p that does not divide the order of H, and for every subgroup K of G containing H, the normalizer $N_K(H)$ includes at least one Sylow p-subgroup of K. We use the notation $H\,nsp\,G$ to signify that H is nearly S-permutable in G.*

Definition 2. *[11] A group G is referred to as an $NSPT$-group if the property of nearly S-permutability is transitive within G. Specifically, G is an $NSPT$-group if, for any subgroups H and K of G such that H is nearly S-permutable in K and K is nearly S-permutable in G, it follows that H is nearly S-permutable in G.*

Lemma 1. *[12] Let $N \trianglelefteq G$ and suppose that $P \in Sylp(N)$, then $G = N_G(P)N$.*

Definition 3. *[8] A subgroup H of G is said to be S-permutable in G if $HP = PH$ holds for every Sylow p-subgroup of G and for every prime p in the set of prime divisors of the order of G, denoted by $\sigma(G)$.*

Proposition 1. *[13] Let G be a group. Then the following properties hold:*

(i) If H is normal in G, then H is c-normal in G.

(ii) The group G is c-simple if and only if G is simple.

(iii) If H is c-normal in G and $H \leq K \leq G$, then H is c-normal in K.

(iv) Let K be a normal subgroup of G such that $K \leq H$. Then H is c-normal in G if and only if H/K is c-normal in G/K.

Definition 4. *[9] A T-group is a group where normality is a transitive property, meaning that every subnormal subgroup is normal. Specifically, if $H \trianglelefteq K$ and $K \trianglelefteq G$, then $H \trianglelefteq G$.*

Lemma 2. *Every normal subgroup is nearly S-permutability subgroup.*

Proof. See [11]

Examples of T-groups include abelian groups, Dedekind groups, and simple groups.

Definition 5. *[14] Let G be a group we called a CT-group if the property of c-normality is transitive in G. Specifically, G is a CT-group if for all subgroups H and K of G, whenever H is c-normal in K and K is c-normal in G, it follows that H is c-normal in G.*

Corollary 1. *[15] All maximal subgroups of a solvable CT-group is a CT-group.*

3. Main Results

Nilpotent groups can be viewed as an extension of the concept of P-groups. This section explores finite groups where the property of nearly S-permutability is transitive, some results and theorems of finite solvable $NSPT$-groups were proven. It is clear that all abelian groups and all nilpotent groups are examples of groups with nearly S-permutable groups. But not all groups satisfies this property as the following example shows:

Example 1. *The alternating group on 4-letters A_4 does not satisfiy the nearly S-permutable. Specifically, any of the Sylow 3-subgroups in A_4 will not be a nearly S-permutable in A_4.*

Remark 1. *Transitive realation of nearly S-permutable is not true for all groups.*

Lemma 3. *Every normal subgroup is nearly S-permutable.*

Proof. Let G be agroup and H is normal subgroup of G and let $H \leq K \leq G$ for every prime number $p \in P$ with $(P, |H|)$, since $H \trianglelefteq G$ implies $H \trianglelefteq K$. Then $N_K(H) = K$.If $P \in Syl_p(K)$ that implies $P \leq K = N_K(H)$, that mean H is nearly S-permutable.The proof is complete.

Proposition 2. *The intersection of two nearly S-permutable subgroups does not necessarily imply that the intersection is nearly S-permutable, and a subgroup being nearly S-permutable does not imply that all of its subgroups are nearly S-permutable.*

Proof. Let G be a group of order 18 defined as the direct product of the symmetric group S_3 and the cyclic group Z_3, i.e.,

$G = S_3 \times Z_3$. $G = \{(e, 0), ((12), 0), ((13), 0), ((23), 0),$
$((132), 0)((123), 0), (e, 1), ((12), 1), ((13), 1), ((23), 1),$
$((132), 1), ((123), 1), (e, 2), ((12), 2), ((13), 2), ((23), 2), ((132), 2), ((123), 2)\}$.

The normal subgroups in G are:

order $9 : \langle((132), 0), (e, 1)\rangle$.

order $6 : \langle((12), 0), ((13), 0)\rangle$.

order $3 : \langle(e, 1)\rangle$ *and order* 1

In this case, every normal subgroup is nearly S-permutable.

However, the intersection of two nearly S-permutable subgroups does not necessarily maintain the nearly S-permutable property. Moreover, a subgroup being nearly S-permutable does not guarantee that all its subgroups will also be nearly S-permutable.

For the subgroups of order 6, (non-normal subgroup) there are 3 conjugacy classes are nearly S-permutable:

$H_1 = \langle (13), 1) \rangle$.

$H_2 = \langle (23), 1) \rangle$.

$H_3 = \langle (12, 1) \rangle$.

Now in this group G we have 9 subgroups that are nearly S-permutable.

Take H_1 with N a normal subgroup of order 6, we have $H_1 \cap N = \langle (12), 0) \rangle$,

now H_1 is NSP and N is NSP but the intersection doesnt nearly S-permutable.

Now takeH_2,the subgroup of H_2 is$\langle (23, 0) \rangle$,its clear that H_2 is NSP.

But the subgroup from H_2 is not NSP. The proof is complete.

Lemma 4. *If H is a p-subgroup of G, then H is contained in some Sylow a p-subgroup of G.*

Proof. Let $L = Sylp(G)$, and H be p-subgroup of G. Consider the action $H \times L \to L$, $h(p) = h^{-1}ph$, then L is a G-set so $|L| \cong |L_H| (modp)$.

$But |L| = |Sylp(G)| = \eta p \equiv 1 (mod\ p) \ldots (1)$

Let us examine L_H. $P \in L_H$ if and only if $hp = p\ \forall h \in H$ and $h^{-1}ph \forall h \in H. If H \leq N_G(P)$, from (1) there exist at least one $P \in L$, such that $H \leq N_G(P)$, then $HP = PH$, it can be seen that HP is a subgroup of $N_G(P)$. Note HP is Sylow p-subgroup $N_G(P)$. Hence,$HP = P$; thus, $H \leq P$.

Remark 2. *There exists a CT-group which is not an $NSPT$-group.*

Proof. Let G be a finite group of order 18, given by $G = S_3 \times Z_3$. The elements of G are:

$G = \{(e, 0), ((12), 0), ((13), 0), ((23), 0), ((13), 2), ((23), 2), ((132), 2), ((123), 2)\}$.

Normal subgroups in G are:

order $9 : \langle ((132), 0), (e, 1) \rangle$.

order $6 : \langle ((12), 0), ((13), 0) \rangle$.

order $3 : \langle (e, 1) \rangle$ *and order*1.

All normal subgroups of G satisfy the c-normality condition. Additionally, subgroups with order 6 are distributed across 3 conjugacy classes:

$H_1 = \{(e, 0), (e, 1), (e, 2), ((23), 0), ((23), 1), ((23), 2)\}$.

$H_2 = \{(e, 0), (e, 1), (e, 2), ((13), 0), ((13), 1), ((13), 2)$.

$H_3 = \{(e, 0), (e, 1), (e, 2), ((12), 0), ((12), 1), ((12), 2)\}$.

For each H_i, we find that:

If H_1 is paired with N, a normal subgroup of order 9, then $G = H_1N$ and $H_1 \cap N = \langle (e, 1) \rangle$. This ensures $H_1 \cap N \subseteq (H_1)$, making subgroups of order 6 are c-normal.

For subgroups of order 3, there are two distinct conjugacy classes:

$H_1 = \{(e, 0), ((123), 1), ((132), 2\}$.

$H_2 = \{(e,0),((132),1),((123),2\}$.

Similarly, when H_1 *is paired with* N*, a normal subgroup of order 6, we have* $G = H_1N$ *and* $H_1 \cap N = \langle(e,0)\rangle H_1 \cap N = \langle(e,0)\rangle$*. This guarantees that subgroups of order 3 are c-normal.*

For subgroups of order 2, there are 3 conjugacy classes:

$H_1 = \{(e,0),((23),0)\}$.

$H_2 = \{(e,0),((13),0)\}$.

$H_3 = \{(e,0),((12),0)\}$.

Subgroups of order 2 are c-normal, as demonstrated by similar reasoning.

Now the sylow 3-subgroup of G *is* $\langle((132),(e,1)\rangle \simeq Z_3 \times \ A_3 = P$*, the indexed* $[G : P] = 2$.

Now p *is normal in* G*, such that* $N_G(H) = G$*; take* H *of order 3. Now* H *is nearly S-permutable, since* $N_G(H) = P$ *and* P *is nearly S-permutable in* G*, but* H *is not nearly S-permutable in* G*, Hence,* G *is not* $NSPT$ *-group.*

Thus, G *is not an* $NSPT$*-group, demonstrating that a* CT*-group need not be an* $NSPT$*-group.*

Remark 3. *There exists a finite group which is an* $NSPT$*-group but not a* CT*-group.*

Proof. *Let* $E_8 = Z_2 \times Z_2 \times Z_2$*, (E: elementary* 2*-group of order* 8*). Now Aut* $(E_8) \simeq PSL(3,2)$*-Simple group of order* 168 *and it has* 179 *subgroups,* 35*-subgroups of which are of order* 4.

If $H \simeq Z_4 \leq Aut(E_8) = PSL(3,2)$ *and* $G = E_8 \rtimes H(Z_2 \times Z_2 \times Z_2) \rtimes Z_4$*, then* G *is example of an* $NSPT$*-group that is not a* CT*-group.*

G *is not* CT*-group,* $G = E_8 \times Z_4$*. More specifically,* $G = (Z_2 \times Z_2 \times Z_2) \rtimes H \cdot H = \langle d\rangle$*, note* H *is a c-normal in* G*, since there exist* $K \leq G; K \trianglelefteq G$ *and* $k \cap h \leq HG = 1$ *and* $HK = G$*; since,* $|H| = 4$*every subgroup of* H *is normal. But* $H_1 = \langle d^2\rangle \trianglelefteq H$*, c-normal in* G*,*

$H_1\{1,d^2\} \simeq Z_2$ *and note* $H_1 \leq M$ *for any subgroup* M *of* G *with* $|M| = 16$ *implies* $M \trianglelefteq G$*, there are* 3 *such* $M.MG = 1$*-identity.*

If $KH_1 = G$ *with* $K \trianglelefteq G$ *implies* $|K| = 16$*, but* $H \leq K$ *for any* K*.* $NSPT$*-group follows from* $|G| = 25$ *and* G *is* p*-group that implies, every subgroup is nearly S-permutable.*

From Remark 2 and Remark 3, we conclude that the classes of CT*-groups and* $NSPT$*-groups are incomparable. That is, a* CT*-group need not be*$NSPT$*, and an* $NSPT$*-group need not be* CT*-groups.*

Lemma 5. *Let* G *be a finite group and let* P *be a Sylow* p*-subgroup of* G*, it follows that* $N_G(N_G(P) = N_G(P)$.

Proof. *Since* $P \subseteq N_G(P) \subseteq G$ *and* P *is a Sylow* p*-subgroup of* G *therefore,* P *is a Sylow* p*-subgroup of* $N_G(P)$.

Furthermore, P *is normal in* $N_G(P)$*, making* P *is unique Sylow* p*-subgroup of* G*. Now let* $a \in N_G(N_G(P))$.

We will aim show that $a \in N_G(P)$. *Observe that:*
$aPa^{-1} \subseteq aN_G\ (P)a^{-1} = N_G(P)$.
This implies that, aPa^{-1} *is a Sylow p-subgroup of* $N_G(P)$, *we have* $aPa^{-1} = P$.
This means $a \in N_G(P)$.

Theorem 1. *Let* G *is a nilpotent group and let* H *subgroup of* G. *Then* $H \leq N_G(H)$.

Proof. *Since* G *is nilpotent, it has a central series*$\{N_i \mid 0 < i < r\}$, *and we have* $N_0 = 1 \subseteq H$ *and* $N_r = G \nsubseteq H$.

Consequently, there exists an index k *with* $0 < k < r$, *such that* $N_k \subseteq H$, *but*$N_{k+1} \subsetneq H$. *We will show that in fact,*$N_{k+1} \subseteq N_G(H)$, *and it will follow that* $H \leq N_G(H)$, *as required.*

Theorem 2. *Let* G *be a finite group. Then the following are equivalent.*

(i) G *is nilpotent.*
(ii) $H \leq N_G(H)$ *for every subgroup* $H \leq G$.
(iii) All maximal subgroups of G *are nearly S-permutable.*

Proof. We saw that (i) implies (ii) in Theorem 1 That (ii) implies (iii) is clear, since every maximal subgroup M *in* G *satisfies* $N_G(M) \geq M$. *it follows that* $N_G(M) = G$.

Now assume condition (iii) holds and let $P \in Syl_p(G)$ *for some prime* p. *If* $N_G(P)$ *is proper in* G, *it must be contained in some maximal subgroup* M, *and we have* $M \leq G$. *Since* $P \in Syl_p(M)$, *it follows by Lemma 1 and Lemma 5, that* $G = N_G(P)M \subseteq M$, *and this is a contradiction. Thus* P *is normal in* G *and by Lemma 3,* P *is nearly S-permutable.*

Example 2. *Let* $G = Z_2 \times Z_2 \times Z_2$, *an abelian group of order 8. Then:* G *is nilpotent (since all abelian groups are nilpotent). Every subgroup* $H \leq G$ *satisfies* $H \leq N_G(H) = G$. *Every maximal subgroup is of order 4 and is normal in* G, *and hence nearly S-permutable. This example confirms that all three conditions hold simultaneously, illustrating the equivalence.*

4. Conclusion

New facts on Sylow p-subgroups and nearly S-permutable subgroups were discovered and established. The relationship between CT*-groups and* $NSPT$*-groups was clarified. It was demonstrated that every normal subgroup is nearly S-permutable, whereas a subnormal subgroup is not necessarily nearly S-permutable. Moreover, a new class of groups, termed* $NSPT$*-groups, was introduced, characterized by the transitivity of nearly S-permutability. Additionally, it was proven that c-normal subgroups do not necessarily exhibit nearly S-permutability, and conversely, nearly S-permutability does not imply c-normality.*

Future research may focus on the following directions:

1. Investigating relationships between $NSPT$*-groups and other group classes with similar subgroup properties.*

2. Constructing new group classes inspired by $NSPT$*-groups and comparing them with nilpotent and solvable groups.*

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