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Applications of Weighted Tangent Similarity Measure of Picture Hesitant Fuzzy Sets

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Abstract. Picture hesitant fuzzy sets (PHFSs) provide a powerful framework for modeling uncertainty by incorporating the degrees of membership, non-membership, neutrality, and refusal. This paper introduces a novel similarity measure, the Weighted Tangent Similarity Measure (ωTSM), designed to improve sensitivity to subtle variations and to capture nonlinear interactions among PHFS components. The proposed measure is theoretically established and empirically validated through two real-world decision-making problems: medical diagnosis and building material classification. In both cases, ωTSM effectively distinguishes between closely related alternatives and consistently outperforms existing methods. The comparative analysis highlights its robustness, flexibility, and practical value in environments characterized by ambiguity and hesitation. Overall, this study advances fuzzy decision-making models and provides a foundation for future applications in knowledge representation and intelligent systems.

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Key Words and Phrases: Picture hesitant fuzzy sets, Tangent similarity measure, Weighted tangent similarity measure, Medical diagnosis, Building material recognition

1. Introduction

In 1965, Zadeh [1] introduced the concept of fuzzy sets (FS), marking a pivotal advancement in the mathematical modeling of uncertainty. In this framework, the membership degree of an element in a fuzzy set is defined by a characteristic function mapping to the unit interval $[0, 1]$, enabling a flexible representation of vagueness. Later, Atanassov [2] extended this foundation by proposing intuitionistic fuzzy sets (IFS), which incorporate both membership and non-membership degrees, thereby enhancing the capacity to represent and analyze imprecise information.

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Building on these developments, Cuong and Kreinovich [3, 4] introduced picture fuzzy sets (PFS), which extend FS and IFS by incorporating additional degrees of neutrality and refusal alongside membership and non-membership. This richer representation allows for more nuanced modeling of human evaluations, particularly in contexts where responses may fall into categories such as "yes," "no," "neutral," or "refusal." Concurrently, Torra [5] proposed hesitant fuzzy sets (HFS), in which the membership degree of an element is characterized by a set of possible values within $[0, 1]$. Combining these two concepts, Wang et al. [6] developed picture hesitant fuzzy sets (PHFS), providing a flexible and comprehensive framework for handling uncertainty in complex decision-making scenarios.

Similarity measures play a crucial role in numerous fuzzy set applications, including decision-making, pattern recognition, and medical diagnosis. Several similarity measures for FS have been proposed by researchers such as Pappis and Karacapilidis [7], Chen [8], and Lohrmann [9]. Within the IFS framework, Dengfeng and Chuntian [10] introduced similarity measures that have been successfully applied to pattern recognition. Nevertheless, conventional approaches often encounter limitations when faced with increasingly complex and uncertain data. To address these challenges, recent studies by Wei [11] and Zhang et al. [12] proposed similarity measures for PFS and HFS based on cosine and grey similarity functions, achieving promising results in applications such as material classification and medical diagnosis.

More recently, Alshehri et al. [13] developed new PHFS distance measures with applications in medical diagnosis. Li [14] introduced distance and similarity measures for PFS. Mostafa [15] proposed a novel PFS distance measure ρ^* together with a proof of the triangular inequality. Cao et al. [16] presented advanced PFS distance measures for multi-criteria group decision-making in healthcare. In addition, Palanikumar et al. [17] designed complex Pythagorean normal interval-valued fuzzy aggregation operators for solving medical diagnosis problems, while Palanikumar et al. [18] explored various distance measures between generalized Diophantine fuzzy sets with applications in multi-criteria decision making. Collectively, these studies highlight the continuing evolution of distance and similarity measures in fuzzy set theory and their growing practical significance.

Despite these advances, existing similarity measures such as the picture hesitant fuzzy weighted hybrid vector similarity measure (PHFWHVSM) [19] and other weighted similarity methods [20] continue to face challenges in capturing the complex relationships among membership, non-membership, and neutrality degrees. This difficulty becomes more pronounced in environments characterized by nonlinear data. A key limitation of such approaches is their reliance on linear assumptions, which restricts their ability to represent the true behavior of PHFS. In addition, these measures often exhibit low sensitivity to subtle variations in fuzzy parameters, making it difficult to distinguish between closely related PHFS instances. Another drawback is computational inefficiency, as many of these methods require significant processing resources, limiting their scalability in large-scale decision-making problems.

To address these shortcomings, this study introduces a novel similarity measure called the Weighted Tangent Similarity Measure (ωTSM). This measure leverages the nonlinear properties of the tangent function to enhance sensitivity to small variations and more accurately model the intricate relationships among fuzzy components. By exploiting these characteristics, the proposed measure provides a robust analytical framework for processing complex and ambiguous information. Its effectiveness is demonstrated through rigorous theoretical validation and comprehensive comparative experiments, demonstrating improvements in both computational efficiency and classification accuracy.

The practicality of ωTSM is further illustrated through real-world case studies involving medical diagnosis and building material classification. These applications confirm its ability to accurately associate symptoms with diseases and to improve material identification. By overcoming the limitations of conventional methods, the proposed approach contributes to advances in diverse domains, including clustering, image segmentation, and decision-making under uncertainty.

The main contributions of this study in the context of PHFS are summarized as follows:

(i) Introduction and validation of the Weighted Tangent Similarity Measure (ωTSM), which effectively evaluates the membership, neutrality, and non-membership degrees in PHFS while improving sensitivity to subtle variations.

(ii) Resolution of key limitations in existing similarity measures, particularly in medical diagnosis, by demonstrating the method's ability to accurately associate symptoms with diseases under uncertain conditions.

(iii) Development of detailed case studies using PHFS-based data in both medical and material classification tasks, highlighting the robustness, adaptability, and practical significance of the proposed measure.

(iv) Execution of a sensitivity analysis to evaluate the stability and consistency of ωTSM across various datasets and decision-making scenarios.

The remainder of this paper is organized as follows. Section 2 presents the necessary definitions and background on PHFS. Section 3 introduces the proposed similarity measure and discusses its theoretical properties. Section 4 provides real-world case studies in medical diagnosis and material classification. Finally, Section 5 concludes the paper and outlines directions for future research.

2. Preliminaries

Definition 1 [1] Let X be a non-empty set. A fuzzy set E on X can be represented as

$$E = \{\langle x, \mu_E(x)\rangle \mid x \in X\}, \quad \text{where } \mu_E : X \to [0, 1]. \tag{1}$$

Definition 2 [2] An intuitionistic fuzzy set (IFS) E on a universal set X is defined as

$$E = \{\langle x, \mu_E(x), \nu_E(x)\rangle \mid x \in X\}, \tag{2}$$

where $\mu_E : X \to [0,1]$ and $\nu_E : X \to [0,1]$ represent the *degree of membership* and the *degree of non-membership* of the element $x \in X$, respectively. These functions satisfy the condition

$$0 \leq \mu_E(x) + \nu_E(x) \leq 1, \quad \forall x \in X.$$

Definition 3 [3] A picture fuzzy set (PFS) E on a universal set X is defined as

$$E = \{\langle x, \mu_E(x), \eta_E(x), \nu_E(x)\rangle \mid x \in X\}, \tag{3}$$

where $\mu_E(x) \in [0,1]$ denotes the membership degree of x in E, $\eta_E(x) \in [0,1]$ denotes the neutrality degree, and $\nu_E(x) \in [0,1]$ denotes the non-membership degree. These values satisfy

$$0 \leq \mu_E(x) + \eta_E(x) + \nu_E(x) \leq 1, \quad \forall x \in X. \tag{4}$$

The refusal degree is given by

$$\pi_E(x) = 1 - (\mu_E(x) + \eta_E(x) + \nu_E(x)), \quad \forall x \in X. \tag{5}$$

Definition 4 [5] A hesitant fuzzy set (HFS) E on a universal set X is defined as

$$E = \{\langle x, E_h(x)\rangle \mid x \in X\}, \tag{6}$$

where $E_h(x) \subseteq [0,1]$ represents a set of possible membership degrees of the element x, reflecting hesitation in assigning a precise value.

Given hesitant fuzzy elements (HFEs) $E_h(x)$, $E_{h_1}(x)$, and $E_{h_2}(x)$, several operations have been introduced by Xia and Xu [21], and Liao et al. [22], including:

$$E_h^-(x) = \min E_h(x), \quad E_h^+(x) = \max E_h(x), \tag{7}$$

$$E_h^c(x) = \bigcup_{\gamma \in E_h(x)} \{1 - \gamma\}, \tag{8}$$

$$E_h^\lambda(x) = \bigcup_{\gamma \in E_h(x)} \{\gamma^\lambda\}, \quad \lambda > 0, \tag{9}$$

$$\lambda E_h(x) = \bigcup_{\gamma \in E_h(x)} \{1 - (1-\gamma)^\lambda\}, \quad \lambda > 0, \tag{10}$$

$$E_{h_1}(x) \cup E_{h_2}(x) = \bigcup_{\gamma_1 \in E_{h_1}(x), \gamma_2 \in E_{h_2}(x)} \{\max(\gamma_1, \gamma_2)\}, \tag{11}$$

$$E_{h_1}(x) \cap E_{h_2}(x) = \bigcup_{\gamma_1 \in E_{h_1}(x), \gamma_2 \in E_{h_2}(x)} \{\min(\gamma_1, \gamma_2)\}, \tag{12}$$

$$E_{h_1}(x) \oplus E_{h_2}(x) = \bigcup_{\gamma_1 \in E_{h_1}(x), \gamma_2 \in E_{h_2}(x)} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}, \tag{13}$$

$$E_{h_1}(x) \otimes E_{h_2}(x) = \bigcup_{\gamma_1 \in E_{h_1}(x), \gamma_2 \in E_{h_2}(x)} \{\gamma_1 \cdot \gamma_2\}, \tag{14}$$

Definition 5 [19] A picture hesitant fuzzy set (PHFS) E on a universal set X is defined as

$$E = \{\langle x, \mu_{iE}(x), \eta_{iE}(x), \nu_{iE}(x)\rangle \mid x \in X\}, \quad i = 1, 2, 3, \dots, z, \tag{15}$$

where $\mu_{iE}(x)$, $\eta_{iE}(x)$, and $\nu_{iE}(x)$ are finite subsets of $[0, 1]$. The corresponding picture hesitant fuzzy number(PHFN) is denoted as $\left(\mu_{iE_p}(x), \eta_{iE_p}(x), \nu_{iE_p}(x)\right)$, and the refusal degree is expressed as

$$\rho_{iE_p}(x) = 1 - \left(\mu_{iE_p}(x) + \eta_{iE_p}(x) + \nu_{iE_p}(x)\right). \tag{16}$$

The following operations on PHFNs are defined:

1. **Union:**

$$E_P \cup F_P = \left\{\left\langle x, \max\left(\mu_{iE}(x), \mu_{iF}(x)\right), \min\left(\eta_{iE}(x), \eta_{iF}(x)\right), \min\left(\nu_{iE}(x), \nu_{iF}(x)\right)\right\rangle \middle| x \in X\right\}. \tag{17}$$

2. **Intersection:**

$$E_P \cap F_P = \left\{\left\langle x, \min\left(\mu_{iE}(x), \mu_{iF}(x)\right), \max\left(\eta_{iE}(x), \eta_{iF}(x)\right), \max\left(\nu_{iE}(x), \nu_{iF}(x)\right)\right\rangle \middle| x \in X\right\}. \tag{18}$$

3. **Complement:**

$$E_P^c = \{\langle x, \nu_{iE}(x), \eta_{iE}(x), \mu_{iE}(x)\rangle \mid x \in X\}. \tag{19}$$

3. Weighted Tangent Similarity Measures for Picture Hesitant Fuzzy Sets

Definition 6 Let

$$E = \{\langle x, \mu_{iE}(x), \eta_{iE}(x), \nu_{iE}(x)\rangle \mid x \in X\}, \quad F = \{\langle x, \mu_{iF}(x), \eta_{iF}(x), \nu_{iF}(x)\rangle \mid x \in X\}$$

be two picture hesitant fuzzy sets defined on the universal set X. Then, the tangent similarity measure between E and F is defined as:

$$T_{S_M}(E, F) = 1 - \frac{1}{m}\Bigg(\tan\left(\frac{\pi}{12}\right) \sum_{j=1}^{m} \Bigg[\sum_{i=1}^{n} \Bigg(|\mu_{iE}(x_j) - \mu_{iF}(x_j)|$$

$$+ |\eta_{iE}(x_j) - \eta_{iF}(x_j)| + |\nu_{iE}(x_j) - \nu_{iF}(x_j)| \Bigg)\Bigg]\Bigg) \tag{20}$$

Theorem 1. Let E, F, and G be three picture hesitant fuzzy sets (PHFS). The tangent similarity measure T_{S_M} satisfies the following properties:

1. $0 \leq T_{S_M}(E,F) \leq 1$
2. $T_{S_M}(E,F) = T_{S_M}(F,E)$
3. $T_{S_M}(E,F) = 1$ if and only if $E = F$
4. If $E \subseteq F \subseteq G$, then $T_{S_M}(E,F) \geq T_{S_M}(E,G)$ and $T_{S_M}(F,G) \geq T_{S_M}(E,G)$

Proof.

1. As the membership,non-membership,and neutral degrees of picture hesitant fuzzy sets are all defined within the interval $[0,1]$, and the value of the tangent function used in the similarity measure is normalized to lie within $[0,1]$, the similarity measure based on the tangent function is also bounded in $[0,1]$. Hence,

$$0 \leq T_{S_M}(E,F) \leq 1.$$

2. From the definition 6, we have

$$\begin{aligned} T_{S_M}(E,F) &= 1 - \frac{1}{m}\Bigg(\tan\left(\frac{\pi}{12}\right) \sum_{j=1}^{m} \Bigg[\sum_{i=1}^{n} \Bigg(|\mu_{iE}(x_j) - \mu_{iF}(x_j)| \\ &\quad + |\eta_{iE}(x_j) - \eta_{iF}(x_j)| + |\nu_{iE}(x_j) - \nu_{iF}(x_j)| \Bigg)\Bigg]\Bigg) \\ &= 1 - \frac{1}{m}\Bigg(\tan\left(\frac{\pi}{12}\right) \sum_{j=1}^{m} \Bigg[\sum_{i=1}^{n} \Bigg(|\mu_{iF}(x_j) - \mu_{iE}(x_j)| \\ &\quad + |\eta_{iF}(x_j) - \eta_{iE}(x_j)| + |\nu_{iF}(x_j) - \nu_{iE}(x_j)| \Bigg)\Bigg]\Bigg) \\ &= T_{S_M}(F,E) \end{aligned}$$

Therefore, $T_{S_M}(E,F) = T_{S_M}(F,E)$

3.Let $E = F$, then we have

$$\mu_{iE}(x_j) = \mu_{iF}(x_j), \quad \eta_{iE}(x_j) = \eta_{iF}(x_j), \quad \nu_{iE}(x_j) = \nu_{iF}(x_j),$$

this implies that

$$|\mu_{iE}(x_j) - \mu_{iF}(x_j)| = 0, \quad |\eta_{iE}(x_j) - \eta_{iF}(x_j)| = 0, \quad |\nu_{iE}(x_j) - \nu_{iF}(x_j)| = 0.$$

Therefore,

$$T_{S_M}(E,F) = 1 - \frac{1}{m}\left(\sum_{j=1}^{m} \tan(0)\right) = 1.$$

Conversely, suppose that $T_{S_M}(E,F) = 1$.

This implies that

$$|\mu_{iE}(x_j) - \mu_{iF}(x_j)| = 0, \quad |\eta_{iE}(x_j) - \eta_{iF}(x_j)| = 0, \quad |\nu_{iE}(x_j) - \nu_{iF}(x_j)| = 0.$$

Hence,

$$\mu_{iE}(x_j) = \mu_{iF}(x_j), \quad \eta_{iE}(x_j) = \eta_{iF}(x_j), \quad \nu_{iE}(x_j) = \nu_{iF}(x_j).$$

Therefore, $E = F$.

4. Since $E \subseteq F \subseteq G$, for every $x_j \in X$ we have

$$\mu_{iE}(x_j) \le \mu_{iF}(x_j) \le \mu_{iG}(x_j), \quad \eta_{iE}(x_j) \le \eta_{iF}(x_j) \le \eta_{iG}(x_j), \quad \nu_{iE}(x_j) \ge \nu_{iF}(x_j) \ge \nu_{iG}(x_j).$$

Hence,

$$\begin{aligned} &|\mu_{iE}(x_j) - \mu_{iF}(x_j)| \le |\mu_{iE}(x_j) - \mu_{iG}(x_j)|, \\ &|\eta_{iE}(x_j) - \eta_{iF}(x_j)| \le |\eta_{iE}(x_j) - \eta_{iG}(x_j)|, \\ &|\nu_{iE}(x_j) - \nu_{iF}(x_j)| \le |\nu_{iE}(x_j) - \nu_{iG}(x_j)|. \end{aligned}$$

$$\begin{aligned} T_{S_M}(E,F) &= 1 - \frac{1}{m}\left(\sum_{j=1}^{m} \tan\Big(\frac{\pi}{12}\Big)\left[\sum_{i=1}^{n}\Big(|\mu_{iE}(x_j) - \mu_{iF}(x_j)| + |\eta_{iE}(x_j) - \eta_{iF}(x_j)| \right.\right. \\ &\qquad \left.\left. + |\nu_{iE}(x_j) - \nu_{iF}(x_j)|\Big)\right]\right) \\ &\ge 1 - \frac{1}{m}\left(\sum_{j=1}^{m} \tan\Big(\frac{\pi}{12}\Big)\left[\sum_{i=1}^{n}\Big(|\mu_{iE}(x_j) - \mu_{iG}(x_j)| + |\eta_{iE}(x_j) - \eta_{iG}(x_j)| \right.\right. \\ &\qquad \left.\left. + |\nu_{iE}(x_j) - \nu_{iG}(x_j)|\Big)\right]\right) \\ &= T_{S_M}(E,G). \end{aligned}$$

This implies that $T_{S_M}(E,F) \ge T_{S_M}(E,G)$.

Similarly, we can prove that $T_{S_M}(F,G) \ge T_{S_M}(E,G)$.

Therefore, the proof is complete.

Definition 7 Let

$$E = \{\langle x, \mu_{iE}(x), \eta_{iE}(x), \nu_{iE}(x)\rangle \mid x \in X\}, \quad F = \{\langle x, \mu_{iF}(x), \eta_{iF}(x), \nu_{iF}(x)\rangle \mid x \in X\}$$

be two picture hesitant fuzzy sets defined on the universal set X. Then, the weighted tangent similarity measure between E and F is defined as:

$$\omega T_{S_M}(E,F) = 1 - \frac{1}{m}\Bigg(\tan\left(\frac{\pi}{12}\right)\sum_{j=1}^{m}\omega_j\Bigg[\sum_{i=1}^{n}\Bigg(|\mu_{iE}(x_j) - \mu_{iF}(x_j)| + |\eta_{iE}(x_j) - \eta_{iF}(x_j)| + |\nu_{iE}(x_j) - \nu_{iF}(x_j)|\Bigg)\Bigg]\Bigg), \quad \text{where } \sum_{j=1}^{m}\omega_j = 1 \tag{21}$$

Theorem 2 Let E , F and G be three picture hesitant fuzzy set PHFS on X then:

1. $0 \leqslant \omega T_{S_M}(E,F) \leqslant 1$

2. $\omega T_{S_M}(E,F) = \omega T_{S_M}(F,E)$

3. $\omega T_{S_M}(E,F) = 1$ iff $E = F$

4. If $E \subseteq F \subseteq G$, then $\omega T_{S_M}(E,F) \geq \omega T_{S_M}(E,G)$ and $\omega T_{S_M}(F,G) \geq \omega T_{S_M}(E,G)$.

Proof :
The proof is similar to Theorem 1

The rationale for adopting the tangent function in both applications presented in this study is grounded in its mathematical characteristics and practical utility. Its ability to model nonlinear relationships and amplify subtle differences, particularly near the origin, enhances the discriminative capacity of the proposed measure, which is essential in distinguishing closely related PHFS elements. This sensitivity is particularly valuable in real-world scenarios such as medical diagnosis and building material classification, where minor variations can lead to significantly different outcomes. The integration of expert-defined weights further strengthens the context-awareness and accuracy of the decision-making process.

4. Applications of the Weighted Tangent Similarity Measure

Medical Diagnosis

Medical diagnosis plays a crucial role in healthcare, where physicians must analyze a wide variety of clinical symptoms and complex medical data to identify underlying diseases accurately. However, the inherent uncertainty and imprecision in medical information often limit the reliability of traditional diagnostic techniques. To address this issue, the proposed Weighted Tangent Similarity Measure (ωTSM) under the Picture Hesitant Fuzzy Set

(PHFS) environment provides a robust mathematical framework for quantifying the similarity between a patient's symptoms and the known symptom patterns of specific diseases. This approach incorporates three essential components membership degree, neutrality degree, and non-membership degree which collectively enhance the accuracy, consistency, and structure of the diagnostic process.

Example 1: Consider a physician evaluating a group of patients represented as:

$$P = \{\text{James}, \text{Harry}, \text{Jak}, \text{Oliver}\}.$$

The potential diseases under consideration are given by:

$$D = \{\text{Viral Fever}, \text{Malaria}, \text{Typhoid}, \text{Stomach Problem}, \text{Chest Problem}\}.$$

These diseases are diagnosed based on a set of clinical symptoms defined as:

$$V = \{\text{Fever}, \text{Cough}, \text{Shortness of Breath}, \text{Chest Pain}, \text{Headache}\}.$$

In the proposed method, the diagnostic process involves determining the degrees of membership, neutrality, and non-membership for each patient according to their exhibited symptoms.

The similarity between each patient's symptom profile and the characteristic symptom patterns of the diseases is computed using Definition (7). Based on the resulting similarity scores, a ranking of potential diagnoses is generated. This ranking enables informed clinical decision-making by identifying the disease with the highest similarity score as the most probable diagnosis. The method effectively distinguishes among competing diagnoses while accounting for uncertainty in patient data.

Table 1: Patients Data and Symptoms

	Fever	Cough	Shortness of breath	Chest pain	Headache
James	$\begin{pmatrix} \{0.25,0.24\} \\ \{0.30,0.28\} \\ \{0.40,0.37\} \end{pmatrix}$	$\begin{pmatrix} \{0.30,0.28\} \\ \{0.15,0.12\} \\ \{0.45,0.50\} \end{pmatrix}$	$\begin{pmatrix} \{0.32,0.28\} \\ \{0.40,0.35\} \\ \{0.20,0.15\} \end{pmatrix}$	$\begin{pmatrix} \{0.25,0.20\} \\ \{0.35,0.33\} \\ \{0.35,0.30\} \end{pmatrix}$	$\begin{pmatrix} \{0.55,0.50\} \\ \{0.05,0.0\} \\ \{0.35,0.40\} \end{pmatrix}$
Harry	$\begin{pmatrix} \{0.40,0.38\} \\ \{0.30,0.28\} \\ \{0.25,0.22\} \end{pmatrix}$	$\begin{pmatrix} \{0.38,0.35\} \\ \{0.08,0.0\} \\ \{0.45,0.40\} \end{pmatrix}$	$\begin{pmatrix} \{0.15,0.12\} \\ \{0.55,0.50\} \\ \{0.25,0.20\} \end{pmatrix}$	$\begin{pmatrix} \{0.30,0.35\} \\ \{0.20,0.20\} \\ \{0.45,0.40\} \end{pmatrix}$	$\begin{pmatrix} \{0.20,0.18\} \\ \{0.25,0.22\} \\ \{0.45,0.45\} \end{pmatrix}$
Jak	$\begin{pmatrix} \{0.40,0.35\} \\ \{0.35,0.28\} \\ \{0.20,0.20\} \end{pmatrix}$	$\begin{pmatrix} \{0.45,0.40\} \\ \{0.25,0.20\} \\ \{0.25,0.25\} \end{pmatrix}$	$\begin{pmatrix} \{0.33,0.30\} \\ \{0.25,0.20\} \\ \{0.35,0.30\} \end{pmatrix}$	$\begin{pmatrix} \{0.45,0.50\} \\ \{0.15,0.10\} \\ \{0.30,0.30\} \end{pmatrix}$	$\begin{pmatrix} \{0.48,0.50\} \\ \{0.05,0.0\} \\ \{0.40,0.45\} \end{pmatrix}$
Oliver	$\begin{pmatrix} \{0.18,0.15\} \\ \{0.25,0.20\} \\ \{0.50,0.45\} \end{pmatrix}$	$\begin{pmatrix} \{0.35,0.30\} \\ \{0.20,0.18\} \\ \{0.40,0.45\} \end{pmatrix}$	$\begin{pmatrix} \{0.38,0.30\} \\ \{0.30,0.25\} \\ \{0.30,0.35\} \end{pmatrix}$	$\begin{pmatrix} \{0.45,0.40\} \\ \{0.30,0.20\} \\ \{0.25,0.30\} \end{pmatrix}$	$\begin{pmatrix} \{0.38,0.35\} \\ \{0.10,0.08\} \\ \{0.45,0.50\} \end{pmatrix}$

Table 2: Diseases Data and Symptoms

	Fever	Cough	Shortness of breath	Chest pain	Headache
Viral fever	$\begin{pmatrix}\{0.36,0.36\}\\\{0.35,0.28\}\\\{0.28,0.35\}\end{pmatrix}$	$\begin{pmatrix}\{0.31,0.31\}\\\{0.31,0.38\}\\\{0.38,0.31\}\end{pmatrix}$	$\begin{pmatrix}\{0.40,0.41\}\\\{0.41,0.17\}\\\{0.17,0.40\}\end{pmatrix}$	$\begin{pmatrix}\{0.47,0.20\}\\\{0.20,0.33\}\\\{0.33,0.47\}\end{pmatrix}$	$\begin{pmatrix}\{0.59,0.19\}\\\{0.19,0.20\}\\\{0.20,0.59\}\end{pmatrix}$
Malaria	$\begin{pmatrix}\{0.29,0.43\}\\\{0.41,0.29\}\\\{0.29,0.28\}\end{pmatrix}$	$\begin{pmatrix}\{0.37,0.31\}\\\{0.31,0.31\}\\\{0.31,0.37\}\end{pmatrix}$	$\begin{pmatrix}\{0.14,0.50\}\\\{0.50,0.36\}\\\{0.35,0.13\}\end{pmatrix}$	$\begin{pmatrix}\{0.28,0.22\}\\\{0.22,0.49\}\\\{0.49,0.27\}\end{pmatrix}$	$\begin{pmatrix}\{0.41,0.32\}\\\{0.32,0.27\}\\\{0.27,0.40\}\end{pmatrix}$
Typhoid	$\begin{pmatrix}\{0.40,0.27\}\\\{0.27,0.33\}\\\{0.33,0.40\}\end{pmatrix}$	$\begin{pmatrix}\{0.26,0.26\}\\\{0.26,0.47\}\\\{0.47,0.26\}\end{pmatrix}$	$\begin{pmatrix}\{0.38,0.31\}\\\{0.31,0.31\}\\\{0.31,0.38\}\end{pmatrix}$	$\begin{pmatrix}\{0.63,0.07\}\\\{0.07,0.29\}\\\{0.29,0.63\}\end{pmatrix}$	$\begin{pmatrix}\{0.43,0.21\}\\\{0.21,0.36\}\\\{0.36,0.43\}\end{pmatrix}$
Stomach problem	$\begin{pmatrix}\{0.24,0.29\}\\\{0.29,0.47\}\\\{0.47,0.23\}\end{pmatrix}$	$\begin{pmatrix}\{0.53,0.06\}\\\{0.06,0.40\}\\\{0.40,0.52\}\end{pmatrix}$	$\begin{pmatrix}\{0.31,0.46\}\\\{0.46,0.23\}\\\{0.23,0.30\}\end{pmatrix}$	$\begin{pmatrix}\{0.29,0.29\}\\\{0.29,0.41\}\\\{0.41,0.29\}\end{pmatrix}$	$\begin{pmatrix}\{0.38,0.38\}\\\{0.38,0.23\}\\\{0.23,0.38\}\end{pmatrix}$
Chest problem	$\begin{pmatrix}\{0.57,0.14\}\\\{0.14,0.29\}\\\{0.29,0.57\}\end{pmatrix}$	$\begin{pmatrix}\{0.39,0.11\}\\\{0.11,0.50\}\\\{0.50,0.39\}\end{pmatrix}$	$\begin{pmatrix}\{0.23,0.46\}\\\{0.46,0.31\}\\\{0.31,0.23\}\end{pmatrix}$	$\begin{pmatrix}\{0.50,0.20\}\\\{0.20,0.10\}\\\{0.10,0.50\}\end{pmatrix}$	$\begin{pmatrix}\{0.20,0.47\}\\\{0.47,0.33\}\\\{0.33,0.20\}\end{pmatrix}$

The weight distribution for each criterion in the proposed tangent similarity measure was determined by a domain expert (e.g., a doctor or medical practitioner) as follows: $\omega_1 = 0.225$, $\omega_2 = 0.195$, $\omega_3 = 0.200$, $\omega_4 = 0.190$, and $\omega_5 = 0.190$. These weights reflect the relative importance of each clinical symptom in the similarity assessment process. Table 3 displays the computed weighted tangent similarity scores, with the highest score in each row indicating the most likely diagnosis for the corresponding patient. Based on these results, **James** is most likely diagnosed with *Stomach Problem*, while **Harry** is predicted to have *Malaria*. Similarly, **Jak** is most closely associated with *Viral Fever*, and **Oliver** with *Typhoid*.

Table 3: Computed Weighted Tangent Similarity Scores Between Patients and Diseases

Weighted Tangent Similarity Measure	Viral Fever	Malaria	Typhoid	Stomach Problem	Chest Problem
James	0.96393	0.96379	0.95921	**0.9656**	0.94963
Harry	0.95835	**0.9659**	0.95421	0.95699	0.95181
Jak	**0.96341**	0.9576	0.95917	0.95212	0.9453
Oliver	0.96198	0.95445	**0.96267**	0.96141	0.95156

Table 4: Comparative Analysis

Methods	**James**	**Harry**	**Jak**	**Oliver**
H_{WP} [19]	Stomach Problem	Malaria	Viral Fever	Viral Fever
W^1_{PHFS} [20]	Stomach Problem	Malaria	Viral Fever	Viral Fever
W^2_{PHFS} [20]	Viral Fever	Malaria	Viral Fever	Viral Fever
W^3_{PHFS} [20]	Chest Problem	Viral Fever	Stomach Problem	Chest Problem
ωT_{S_M}	Stomach Problem	Malaria	Viral Fever	Typhoid

To assess the effectiveness of the proposed weighted tangent similarity measure ωT_{S_M}, we conducted a comparative analysis against several established methods, including PH-FWHVSM [19], and the weighted cosine, set-theoretic, and grey similarity measures [20]. As summarized in Table 4, the results demonstrate both consistency and contrast across cases.

For *James* and *Harry*, the proposed measure yields diagnoses that align with those from other leading methods, supporting its reliability in typical scenarios. Notably, for *Jak*, our method correctly identifies *Viral Fever*, in agreement with most techniques, while avoiding less accurate alternatives such as *Stomach Problem*. For *Oliver*, the proposed measure uniquely predicts *Typhoid*, capturing distinctions that other methods favoring *Viral Fever* or *Chest Problem* appear to miss.
These findings highlight the superior discriminative capability of ωT_{S_M}, particularly in cases with overlapping symptoms and hesitant information. Its sensitivity to subtle variations ensures more nuanced and accurate classifications under the PHFS environment.

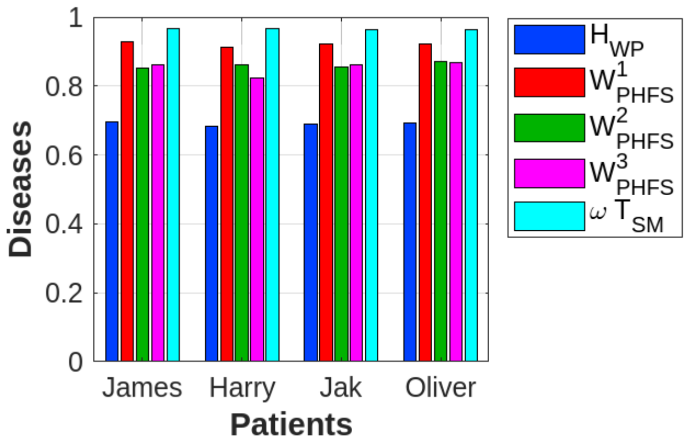

Figure 1: Comparison of similarity measures in disease diagnosis

Building Material Recognition

In this study, we classify an unknown building material using the weighted tangent similarity measure under the picture hesitant fuzzy set (PHFS) environment. This approach leverages expert knowledge of known building materials to construct a reference database. The similarity between the unknown material and each known material is computed using the weighted tangent similarity measure. The unknown material is then assigned to the category of the known material with the highest similarity score. The complete procedure is summarized below:

(i) Collect data for both known and unknown building materials, represented as picture hesitant fuzzy numbers (PHFNs).

(ii) Compute the weighted tangent similarity measure between the unknown material and each known material.

(iii) Rank the similarity scores and classify the unknown material based on the highest score.

Example 2: [20] Let $E_i (1 \leq i \leq 4)$ represent four building materials: brick, stone, muddy, and steel. Let $X = \{x_1, x_2, x_3, x_4\}$ be the space of attributes with the weights $\omega_1 = 0.27$,

$\omega_2 = 0.33$, $\omega_3 = 0.11$, and $\omega_4 = 0.29$. The details of the unknown and known building materials are shown in Table 4. We classified the unknown material as follows:

Table 5: Information of unknown and known building materials

	x_1	x_2	x_3	x_4
E_1	$(\{0.0, 0.1\}, \{0.0, 0.3\}, \{0.0, 0.5\})$	$(\{0.1\}, \{0.2, 0.4\}, \{0.3\})$	$(\{0.2, 0.4\}, \{0.0, 0.5\}, \{0.0, 0.01\})$	$(\{0.2, 0.3\}, \{0.0, 0.2\}, \{0.0, 0.3\})$
E_2	$(\{0.0, 0.2\}, \{0.1\}, \{0.0, 0.4\})$	$(\{0.0, 0.1\}, \{0.3\}, \{0.0\})$	$(\{0.0, 0.1\}, \{0.2, 0.4\}, \{0.0, 0.2\})$	$(\{0.2, 0.5\}, \{0.0, 0.2\}, \{0.0, 0.1\})$
E_3	$(\{0.22\}, \{0.23, 0.27\}, \{0.0\})$	$(\{0.1\}, \{0.2, 0.11\}, \{0.3\})$	$(\{0.0, 0.5\}, \{0.1, 0.3\}, \{0.0, 0.2\})$	$(\{0.0, 0.17\}, \{0.54, 0.63\}, \{0.1, 0.2\})$
E_4	$(\{0.42, 0.47\}, \{0.0, 0.53\}, \{0.0\})$	$(\{0.0, 0.15\}, \{0.0, 0.71\}, \{0.14\})$	$(\{0.1\}, \{0.0, 0.3\}, \{0.4, 0.5\})$	$(\{0.0, 0.1\}, \{0.02, 0.6\}, \{0.3, 0.35\})$
E	$(\{0.1, 0.2\}, \{0.2, 0.3\}, \{0.3, 0.4\})$	$(\{0.1, 0.2\}, \{0.0, 0.1\}, \{0.0, 0.4\})$	$(\{0.0, 0.2\}, \{0.4\}, \{0.3\})$	$(\{1.00\}, \{0.00\}, \{0.00\})$

By applying the definition (7), we obtain:

$$\omega T_{S_M}(E, E_1) = 0.910382, \quad \omega T_{S_M}(E, E_2) = \mathbf{0.925676},$$
$$\omega T_{S_M}(E, E_3) = 0.875248, \quad \omega T_{S_M}(E, E_4) = 0.863091$$

Table 6: Comparative Analysis of Similarity Measures

Similarity Measure	**(E, E$_1$)**	**(E, E$_2$)**	**(E, E$_3$)**	**(E, E$_4$)**	**Ranking (E, E$_i$)**
H_{WP}[19]	**0.5898**	0.5186	0.4423	0.3774	$(E_1) \succ (E_2) \succ (E_3) \succ (E_4)$
W^1_{PHFS}[20]	**0.8423**	0.8147	0.6418	0.3978	$(E_1) \succ (E_2) \succ (E_3) \succ (E_4)$
W^2_{PHFS}[20]	**0.6178**	0.5563	0.5022	0.2880	$(E_1) \succ (E_2) \succ (E_3) \succ (E_4)$
W^3_{PHFS}[20]	0.8739	**0.9101**	0.8728	0.7489	$(E_2) \succ (E_1) \succ (E_3) \succ (E_4)$
ωT_{S_M}	0.9104	**0.9257**	0.8752	0.8631	$(E_2) \succ (E_1) \succ (E_3) \succ (E_4)$

Table 6 presents a comparative analysis of various similarity measures applied to the classification task. Conventional measures such as H_{WP}, W^1_{PHFS}, and W^2_{PHFS} consistently identify E_1 as the material most similar to the unknown sample E. In contrast, both W^3_{PHFS} and the proposed measure ωT_{S_M} yield a different outcome, ranking E_2 as the most similar.

This divergence highlights the underlying methodological differences between the measures. Traditional approaches typically employ linear aggregations of membership, non-membership, and abstinence values. While computationally straightforward, such linear models may overlook subtle yet meaningful interactions inherent in Picture Hesitant Fuzzy Sets (PHFS). The proposed ωT_{S_M} measure, by contrast, introduces a weighted tangent-based transformation, which captures nonlinear relationships and amplifies small variations in the input data. This transformation enables a more refined and sensitive evaluation of similarity, particularly in cases involving hesitant or ambiguous information.

Moreover, the proposed measure demonstrates greater robustness and consistency across multiple comparisons. Unlike static linear models that treat all components uniformly, ωT_{S_M} incorporates adaptive weighting, allowing it to respond dynamically to the structural complexity of PHFS data. This adaptability enhances its discriminative power and reduces the likelihood of misclassification, especially in borderline cases.

From an application standpoint, these results underscore the practical value of ωT_{S_M} in domains such as material science, medical diagnosis, and pattern recognition. By accurately modeling nonlinear dependencies and accounting for hesitation, the proposed measure offers a reliable and theoretically sound framework for similarity assessment in high-stakes decision-making environments.

4.1. Advantages

(i) The enhanced sensitivity of the proposed measure enables it to effectively capture subtle variations in membership, non-membership, neutrality, and refusal levels. This enables more accurate differentiation between closely related cases, rendering it exceptionally beneficial in applications such as medical diagnosis and material classification.

(ii) The proposed similarity measure effectively models ambiguous and hesitant information, in contrast to conventional approaches that rely on linear assumptions. This feature enhances its suitability for decision-making in uncertain contexts, where precise differentiation is crucial.

(iii) Conventional measures, such as the Weighted Cosine Similarity Measure, Set-Theoretic Similarity Measure, PHFWHVSM, and Weighted Grey Similarity, often struggle to encapsulate the intricate, nonlinear interdependencies among features. The proposed approach addresses this limitation by effectively modeling these relationships in fuzzy data, ensuring more reliable and precise results.

5. Conclusion

This study proposed two novel similarity measures: the Tangent Similarity Measure (TSM) and the Weighted Tangent Similarity Measure (ωTSM), which are specifically designed for Picture Hesitant Fuzzy Sets (PHFS). These measures aim to overcome key limitations of traditional linear similarity models by incorporating the nonlinear characteristics of the tangent function. This approach improves sensitivity to subtle differences and captures complex relationships among membership, non-membership, and neutrality degrees.

Theoretical validation confirmed that both TSM and ωTSM satisfy the fundamental properties required for similarity measures. Empirical evaluations, which were conducted on two real-world decision-making problems, namely medical diagnosis and building material classification, demonstrated that ωTSM provides significantly higher accuracy and better discriminative performance compared to existing similarity measures.

These results confirm that ωTSM is capable of reliably distinguishing between closely related alternatives in uncertain and hesitant environments. Furthermore, its flexibility and adaptability make it a promising tool for a wide range of practical applications.

Nonetheless, the proposed methodology is not without limitations. First, the reliance on expert-defined weights may introduce subjectivity in some contexts. Second, while ωTSM's heightened sensitivity is generally advantageous, it may also increase vulnerability to noise in certain datasets. Finally, the computational complexity of the method could become a challenge for very large-scale problems compared to simpler linear models.

Future research will focus on extending the proposed similarity framework to advanced domains such as clustering, image segmentation, and multi-criteria decision-making (MCDM). Emphasis will be placed on improving robustness, scalability, and interpretability in complex fuzzy environments.

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