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Stability, $\mathfrak{D}$-stability, Strong $\mathfrak{D}$-Stability of Positive Linear Time-invariant Systems with Applications

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Abstract. This paper presents new results on the analysis of positive linear time-invariant systems with non-negative state variables and output data for non-negative initial conditions and inputs. The positive linear time-invariant systems are characterized by the state-space equations which offer a formal mathematical structure of form

$$\frac{dx(t)}{dt} = Ax(t)$$

where the system matrix $A \in \mathbb{R}^{n,n}$ is Metzler, with non-negative off-diagonal components. These systems exhibit essential characteristics such as monotonicity, stability, and non-negativity, making them fundamental in applications such as biological systems, mathematical economics, chemical reaction networks, and transportation models. We present the theoretical foundations utilizing a mathematical framework from algebraic systems, matrix theory, and stability analysis to investigate stability, $\mathfrak{D}$-stability, and strong $\mathfrak{D}$-stability of positive linear time-invariant systems in the presence of Metzler and Hurwitz matrices. The numerical testing supports the spectrum analysis and ϵ-pseudospectrum of Metzler matrices.

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Key Words and Phrases: Singular values, structured singular values, strong $\mathfrak{D}$-stable matrix, positive linear systems, Metzler matrices

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1. Introduction

The stability theory concerning positive linear time-invariant (LTI) systems is well-documented and widely accessible in the literature. However, when extending this theory to nonlinear systems, systems with uncertainties, or positive LTI systems, the framework remains less developed and not well established. The stability characteristics of positive LTI systems have been generalized to include positive descriptor systems and were further analyzed in [1]. Additionally, in [2], stability theory for positive LTI systems was expanded to switched positive linear systems. Moreover, the concept of $\mathfrak{D}$-stability for positive switched linear systems was further refined, leading to the derivation and presentation of new findings in [2].

The time-invariant linear system characterized by

$$\frac{dx(t)}{dt} = Ax(t),$$

is classified as positive if and only if $A \in \mathbb{R}^{n,n}$ is a Metzler matrix, meaning that all its off-diagonal elements are non-negative. As demonstrated in [3], such a system attains global asymptotic stability if and only if the system

$$\frac{dx(t)}{dt} = DAx(t),$$

is asymptotically stable, where D represents a positive diagonal matrix. In [4], it was demonstrated that a delayed positive linear system of the form

$$\frac{dx(t)}{dt} = Ax(t) + Bx(t-\tau), \quad \tau \geq 0,$$

where A is a Metzler matrix and B is a non-negative matrix, exhibits global asymptotic stability. Furthermore, [5] explored theoretical developments in cooperative systems that maintain homogeneity of arbitrary degree under certain dilation mappings. Considering the linear time-invariant system:

$$\frac{dx(t)}{dt} = Ax(t),$$

where A is an $n \times n$ real Metzler matrix and $x(t) \in \mathbb{R}^{n,1}$, the system is deemed stable if the real parts of all eigenvalues of A are strictly negative. Such a system satisfies the Hurwitz condition, ensuring that all eigenvalues reside within the open left half-plane of the complex domain. For a system of this type, one can deduce that stability implies the existence of a positive matrix P such that:

$$A^T P + PA < 0.$$

Additionally, the stability condition ensures that there exists a positive diagonal matrix D.

The notion of $\mathfrak{D}$-stability for real-valued matrices was given by Arrow and McManus [6] in their classical paper. It was then introduced by Enthoven and Arrow [7] for the analysis and interpretations of stability of competitive markets. The concept of $\mathfrak{D}$-stability holds significant importance across various fields, for instance, in the study of models in economics, see [8–14]. An n-dimensional real matrix A is considered $\mathfrak{D}$-stable if there exists a positive diagonal matrix D such that either DA or AD remains positive stable. Characterizing $\mathfrak{D}$-stability is highly challenging, and in certain cases, it leads to computationally intractable (NP-hard) problems; see [15, 16].

The problems concerning $\mathfrak{D}$-stability for the linear systems have received a great attention over the last few decades. Numerous significant findings and mathematical approaches were established in the literature; refer to [17–25] and the associated references. The selection of an appropriate region(s) in the complex plane, for instance, $D(\alpha, r)$ is a disk, centered at $\alpha + i0$, having radius r (see [26]) plays a vital role for the characterization of $\mathfrak{D}$-stability.

The analysis on $\mathfrak{D}$-stability for singular systems is indeed very complicated and a challenging problem as compared to normal state space systems. The establishment of numerous significant results for $\mathfrak{D}$-stability, regularity, and causality is documented in [27–30] and the cited sources.

For a given stable matrix $A \in \mathbb{R}^{n,n}$, it is well established that the matrix $A+M$ remains stable, provided that $\|M\| < \gamma$ for some $\gamma > 0$. see [31, 32]. This further ensures that the stability is a property that is very much robust to some small admissible perturbations appearing across the system.

For given D, a positive diagonal matrix, assume that $A \in \mathbb{R}^{n,n}$, is $\mathfrak{D}$-stable matrix, then one can easily observe that matrix products DA, and $D(A+M)$ are also stable. The important question that arises at this stage is then to ask if it is possible to search for a suitable $\gamma > 0$, so that $A + M$ is a $\mathfrak{D}$-stable matrix for $|M| < \gamma$. But, Unfortunately, in general, the response is a **NO**. However, in certain special cases, it is indeed feasible to identify a class of $\mathfrak{D}$-stable matrices where small, permissible perturbations result exclusively in $\mathfrak{D}$-stable matrices.

The given $A \in \mathbb{R}^{n,n}$ is strongly $\mathfrak{D}$-stable if there exist a $\gamma > 0$ such that $A + M$ is a $\mathfrak{D}$-stable for each $M \in \mathbb{R}^{n,n}$, with $|M| < \gamma$. Furthermore, one can easily notice that each strongly $\mathfrak{D}$-stable matrix must be a $\mathfrak{D}$-stable matrix.

The mathematical problem of optimization of spectral abscissa over the families of Metzler matrices is helpful to study and analyze the stabilization of the Metzler matrices. The objective function is neither convex not concave, not Lipschitz and this cause the problem to be very hard. Further, from this one might obtained many local extrema which in turn are very hard to localize, see [33]. The necessary and sufficient condition for a given matrix to be $\mathfrak{D}$-stable by using Kalman-Yacubovich-Popov lemma were given in [34]. It was shown that obtained condition is mathematically equivalent to requirement that pair of linear-time-invariant systems have common Lyapunov function in the lower dimensions. Furthermore, the simple conditions for Hurwitz stability of a given Metzler matrix were derived.

In recent years, the study of fractional differential equations (FDEs) has gained significant attention due to their ability to model memory and hereditary properties in various physical and engineering systems. Analytical methods such as the Laplace transform method, Adomian decomposition method (ADM), and the homotopy analysis method (HAM) have been widely applied to obtain exact or approximate solutions to FDEs. These approaches are especially useful for linear or weakly nonlinear problems, offering insight into the qualitative behavior of solutions. On the numerical side, methods like the finite difference method (FDM), finite element method (FEM), and spectral methods have proven effective in handling more complex or strongly nonlinear problems, particularly when closed-form solutions are not attainable. More recent advancements include the development of Grünwald–Letnikov and Caputo-based numerical schemes, which are particularly suited for time-fractional models, as well as predictor–corrector algorithms and adaptive mesh techniques that improve accuracy and efficiency. For a comprehensive overview, the works of [35–40] provide foundational insights into both the theoretical and practical aspects of these methods.

The spectra and pseudo-spectra of structured matrices to different linear and nonlinear mathematical problems helps to analyze their behavior. The spectrum and pseudo-spectrum of D-stable matrices for an economic model was the subject of some recent novel mathematical studies, see [41]. In [42] a characterization of D-stable matrices from transportation problems were analyzed. A in-depth mathematical analysis on stability, D-stability, and pseudo-spectrum of structured matrices arising from economic models was studied in [43]. The most recent mathematical findings on interaction between μ-values and Schur stability were studied and analyzed in [44].

Our main objective in this article is to extend the results on stability, $\mathfrak{D}$-stability, and strong $\mathfrak{D}$-stability theory. Mainly, we target problems of linear time-invariant systems which are positive and have following mathematical formulations:

Problem-I: To extend and construct some new insights on stability, $\mathfrak{D}$-stability and strong $\mathfrak{D}$-stability analysis of LTI system

$$\frac{dx(t)}{dt} = Ax(t);\ x(t) \in \mathbb{R}^{n,1},\ A \in \mathbb{R}^{n,n},$$

where A is Metzler and Hurwitz.

Problem-II: To extend and construct some new findings on stability, $\mathfrak{D}$-stability and strong $\mathfrak{D}$-stability analysis of LTI system

$$\frac{dx(t)}{dt} = A(t)x(t);\ x(t) \in \mathbb{R}^{n,1},\ A(t) \in \{A_1, A_2\},$$

with A_1, A_2 being asymptotically stable matrices.

Problem-III: To extend and construct some new findings on stability, $\mathfrak{D}$-stability and strong $\mathfrak{D}$-stability analysis of LTI system

$$\frac{dx(t)}{dt} = A(t)x(t);\ x(t) \in \mathbb{R}^{n,1},\ A(t) \in \{D_1A_1, D_2A_2\},$$

with A_1, A_2 being irreducible matrices, and $D_1, D_2 > 0$.

Overview of article: Section 2 is about notations and the preliminary findings on stability, $\mathfrak{D}$-stability, and strong $\mathfrak{D}$-stability. In section 3, we recall some basic and fundamental outcomes on Metzler matrices and their spectrum. In section 4, we present some new results on the extension of concepts to stability, $\mathfrak{D}$-stability, and strong $\mathfrak{D}$-stability of Metzler and Hurwitz matrices. For this purpose, we apply different approaches drawn from linear algebra, matrix analysis, and system theory. The numerical illustration is presented in section 5. The applications to positive linear time-invariant systems having Metzler and Hurwitz as their coefficient matrices are presented in section 6, and then finally in section 7, we present the conclusion to our paper.

2. Preliminaries

The problem on the analysis of stability and its properties for linear time-invariant system with positive constraints has been thoroughly investigated and examined in the literature in much greater detail. The positive system stability is an important focus of many disciplines, for instance, applied mathematics, engineering, and computational sciences. We study and analyze the positive system of linear time-invariant of the form $\frac{dx(t)}{dt} = Ax(t),\ A \in \mathbb{R}^{n,n},\ x(t) \in \mathbb{R}^{n,1}$, with A being Metzler, and Hurwitz.

Definition 1. *[45] A matrix $A \in \mathbb{R}^{n,n}$ is classified as Metzler if all of its off-diagonal entries $a_{ij},\ \forall\ i \neq j$ are non-negative.*

Definition 2. *The matrix $A \in \mathbb{R}^{n,n}$ is Hurwitz if $\lambda_i(A)$ $\forall i$ is such that $Re(\lambda_i(A)) < 1\ \forall\ i$.*

The $\mathfrak{D}$-stability analysis aims to study and analyze the stability of an equilibrium of dynamic models appearing in competitive market. A given n-dimensional real-valued matrix A is a $\mathfrak{D}$-stable matrix if for any diagonal matrix $D = diag(d_{ii}),\ d_{ii} > 0,\ \forall\, i$, the matrix DA or AD is a stable matrix.

Remark 1. *The classification of $\mathfrak{D}$-stability is generally difficult, except in the case of matrices with size 3, or less, see [15, 46].*

The concept of strong $\mathfrak{D}$-stability for n-dimensional real-valued matrix A, introduced in [47], seeks to investigate and characterize the properties of $\mathfrak{D}$-stability.

Definition 3. *The matrix $A \in \mathbb{R}^{n,n}$ is stable if real-part of all of its eigenvalues are strictly positive (sometime in literature it maybe consider as strictly negative).*

Definition 4. *[6] The matrix $A \in \mathbb{R}^{n,n}$ is $\mathfrak{D}$-stable if real-part of all eigenvalues of DA or AD are strictly positive (sometime in literature it maybe consider as strictly negative) for a positive diagonal matrix D.*

Definition 5. *[47] The matrix $A \in \mathbb{R}^{n,n}$ is strongly $\mathfrak{D}$-stable if there exists $\gamma > 0$ so that $A + M$ is a $\mathfrak{D}$-stable matrix, for every $M \in \mathbb{R}^{n,n}$ with $|M| < \gamma$.*

Remark 2. *The $\mathfrak{D}$-stability of a matrix needs to be a robust property. This can be easily understand with an example (see [47]), for instance, the matrix* $A(0) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$*, is a $\mathfrak{D}$-stable matrix, but the perturbed matrix, that is,* $A(\gamma) = \begin{pmatrix} \gamma & 1 \\ -1 & -1 \end{pmatrix}$*, is not a $\mathfrak{D}$-stable matrix for $\gamma > 0$.*

The following Lemma is taken from [48].

Lemma 1. *If A is a $\mathfrak{D}$-stable matrix, then A is non-singular and each of following matrices are D-stable.*
1. A^T
2. A^{-1}
3. P^TAP for a permutation matrix P
4. DAE for positive diagonal matrices D, E.

The following Theorem is taken from [48].

Theorem 1. *Let $A \in \mathbb{C}^{n,n}$ be a stable matrix. Then following statements are equivalent.*
1. A is a D-stable matrix
2. $det(A \pm iD) \neq 0$ for each positive diagonal matrix D.

Let $\mathbb{B}$ denotes the set of block-diagonal matrices and is defined as

$$\mathbb{B} = \{diag(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_S I_{r_S}; \Delta_1, \Delta_2, \cdots, \Delta_F) : \delta_i \in \mathbb{K}, \Delta_j \in \mathbb{K}^{m_j,m_j}, \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}\}.$$

The structured singular value, denoted as $\mu_{\mathbb{B}}(A)$, of a real matrix $A \in \mathbb{R}^{n,n}$ with respect to $\mathbb{B}$ is defined such that $\mu_{\mathbb{B}}(A) = 0$ if and only if there exists no $\Delta \in \mathbb{B}$ for which the matrix $I_n - A\Delta$ has at least one eigenvalue equal to zero, where I_n represents the $n \times n$ identity matrix. Otherwise, $\mu_{\mathbb{B}}(A)$ takes a nonzero (positive) value, that is

$$\mu_{\mathbb{B}}(A) := \frac{1}{min\{||\Delta||_2 : det(I_n - A\Delta) = 0\}},$$

where **min** is taken over all $\Delta \in \mathbb{B}$.

Remark 3. *The necessary condition for μ-value is that if $det(I_n - A\Delta) \neq 0$ for any $\Delta \in \mathbb{B}$. Then, $0 \leq \mu_{\mathbb{B}}(A) < 1$.*

Remark 4. *The matrix $A \in \mathbb{R}^{n,n}$ is $\mathfrak{D}$-stable iff it is stable, and $det(A+iD) \neq 0,\ i = \sqrt{-1}$ for some specific positive diagonal matrix D, see [13].*

Remark 5. *The sufficient condition for μ-value is that if $0 \leq \mu_{\mathbb{B}}(A) < 1$, then $det(I_n - A\Delta) \neq 0$, $\forall\, \Delta \in \mathbb{B}$.*

The analysis and investigation on the behaviour of $A \in \mathbb{R}^{n,n}$ subject to an admissible perturbation Δ_p, for $\epsilon > 0$ such that $||\Delta_p|| \leq \epsilon$, is to study the pseudo-spectrum [49]. For given $A \in \mathbb{R}^{n,n}$ and $\epsilon > 0$, the ϵ-pseudospectrum is given by

$$\Lambda_\epsilon(A) := \{z \in \mathbb{C} : ||(A - zI_n)^{-1}||^{-1} < \epsilon\},$$

where $||\cdot||$ is matrix-norm.

Remark 6. *If* $||\cdot||$ *is being Euclidean norm, then*

$$\Lambda_\epsilon(A) := \{z \in \mathbb{C} : ||(A - zI_n)^{-1}||^{-1} < \epsilon\} = \{z \in \mathbb{C} : \sigma_{min}(A - zI_n) < \epsilon\},$$

where $\sigma_{min}(\cdot)$ *denotes smallest singular-value of a matrix.*

2.1. Sufficient conditions for D-stability:

We review C.R. Johnson's [15] 13 sufficient conditions for the D-stability for a given n-dimensional real-valued matrix A.
C_1 : All the eigenvalues $\lambda_i\left(DA + A^T D\right) > 0,\ \forall i$, D is a diagonal matrix which is positive.
C_2 : The A-matrix means that all of the principal minors are positive and all of the off-diagonal entries are non-positive.
C_3 : There exists a positive diagonal matrix D such that $AD = B = (b_{ij})$ which satisfies the condition that

$$Re(b_{ii}) > \sum_{j=1}^{n} |b_{ij}|;\quad i = 1 : n,\ j \neq i.$$

C_4 : Given $A \in \mathbb{R}^{n,n}$ is a triangular matrix and the real part of all the off-diagonal entries a_{ii} is strictly positive.
C_5 : Given $A \in \mathbb{R}^{n,n}$ is a sign stable matrix.
C_6 : Every principal minor of $A \in \mathbb{R}^{n,n}$ is positive, and A is a tri-diagonal matrix.
C_7 : Given $A \in \mathbb{R}^{n,n}$ is an oscillatory matrix, that is, A is totally non-negative matrix.
C_8 : For each $x \in \mathbb{R}^{n,1}$, $x \neq 0$, a positive diagonal matrix D exists such that the real part of $x^T DAx$ is positive.
C_9 : For given $A \in \mathbb{R}^{n,n}$, and for every positive definite matrix P, the Hadamard product of P and A is a stable matrix.
C_{10} : The given $A \in \mathbb{R}^{n,n}$ is a strictly sign symmetric matrix and its every minor is positive.
C_{11} : Given $A \in \mathbb{R}^{n,n}$ such that $A \in \mathbb{R}^{2,2} \cap P_0^+$.
C_{12} : Given $A \in \mathbb{R}^{n,n}$ such that $A \in \mathbb{R}^{3,3} \cap P_0^+$, and $A = \begin{pmatrix} x & a & b \\ \alpha & y & c \\ \beta & \alpha & z \end{pmatrix}$.
C_{13} : Given $A \in \mathbb{R}^{n,n}$ such that $A \in \mathbb{R}^{n,n} \cap P_0^+$ satisfies GKK condition with $n \leq 4$.

2.2. Sufficient condition for strong D-stability [50]:

For a given $A \in \mathbb{R}^{n,n}$, the sufficient conditions for the strong D-stability are:
C_1 : For a positive diagonal matrix D, all the eigenvalues $\lambda_i\left(DA + A^T D\right) < 0,\ \forall i$.
C_2 : Given $A \in \mathbb{R}^{n,n}$ is an A-matrix, that is, all the off-diagonal entries are non-positive and all the principal minors are positive.
C_3 : There exists a positive diagonal matrix D such that $AD = B = (b_{ij})$ which satisfies the condition that

$$Re(b_{ii}) < -\sum_{1 \leq j \leq n}^{n} |b_{ij}|;\quad 1 \leq i \leq n,\ j \neq i.$$

C_4 : Given $A \in \mathbb{R}^{n,n}$ is a sign triangular matrix, and $a_{ii} < 0,\ i = 1 : n.$.
C_5 : Given $A \in \mathbb{R}^{n,n}$ is a sign stable matrix without having a any of non-zero entry.
C_6 : For given $A \in \mathbb{R}^{n,n}$ is a jocabi matrix, and each of jth-order principal minor is of sign $(-1)^j$.
C_7 : Given $A \in \mathbb{R}^{n,n}$ is an oscillatory matrix, that is, A is totally non-negative matrix.
C_8 : For each $x \in \mathbb{R}^{n,1}$, $x \neq 0$, there exists a positive diagonal matrix D such that real part of $x^T DAx$ is strictly positive.
C_9 : For given $A \in \mathbb{R}^{n,n}$, the Hadamard product $(H \circ (A + G))$ is Schur stable matrix for each positive definite symmetric matrix H, and a perturbation matrix G such that $||G||_2 < \alpha,\ \alpha \in \mathbb{R}$.
C_{10} : For given $A \in \mathbb{R}^{n,n}$ each jth-order principal minor is of sign $(-1)^j$.
C_{11} : Given $A \in \mathbb{R}^{2,2}$ is strongly D-stable iff its jth-order principal minors are of sign $(-1)^j$.
C_{12} : Given $A \in \mathbb{R}^{3,3}$ with all of its jth-order principal minors are with sign $(-1)^j$, and

$$a_{11}a_{22}a_{33} < \frac{a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}}{2}.$$

C_{13} : Given Given $A \in \mathbb{R}^{n,n}$ is strongly D-stable matrix if for $n \leq 4$, and A satisfies GKK condition.

3. Metzler Matrices and their spectrum

In this section, we provide an overview of Metzler matrices and their spectral properties, specifically the set of eigenvalues. Metzler matrices are fundamentally linked to the study and analysis of continuous-time linear systems. The spectrum of a Metzler matrix can be determined by shifting the spectrum of a non-negative matrix by αI_n, where I_n represents the n-dimensional identity matrix, as noted in [51].

It is well established that the spectrum of a positive matrix is positive and corresponds exactly to its spectral radius ρ, see [52]. Moreover, as demonstrated in [53], the spectrum of a non-negative matrix is confined to specific regions within the complex plane. These regions, referred to as Kerpelevich regions, define the spectral behavior of such matrices. In [45], the Metzler matrix spectrum has been proven to reside within the Kerpelevich region, shaped like a cone in the complex plane. Additionally, it has been demonstrated that for 3×3 matrices, both necessary and sufficient conditions for the spectrum of Metzler matrices are established. Metzler matrices share a strong connection with positive matrices, particularly in the context of analyzing linear time-invariant systems. In [45], following dynamical system was considered for a demonstration.

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) \\ x(0) = x_0 \\ x \in \mathbb{R}^{n,1},\ A \in \mathbb{R}^{n,n}. \end{cases}$$

Lemma 2. *[45] The dynamical system (given as above) is positive iff A is a non-negative matrix, and for all $t \geq 0$, the matrix e^{At} is non-negative.*

Lemma 3. *[45] For $A \in \mathbb{R}^{n,n}$, the following statements are equivalent:*
(i) $e^{At} \geq 0, \ \forall \ t \geq 0$.
(ii) The matrix A is Metzler matrix.

Remark 7. *The Metzler matrtix $A \in \mathbb{R}^{n,n}$ can be expressed as the sum of a non-negative matrix N, and αI_n, where arbitrary number $\alpha \in \mathbb{R}$ satisfies $\alpha \geq diag(N)$.*

For $S_n^r := \{\lambda \in \mathbb{C} \text{ such that } \exists \ N \in \mathbb{R}^{n,n}, \ \lambda \in \sigma(N), \ N \geq 0, \ \rho(N) = r\}$, one may easily find that the set S_n^r can be obtained by the following Theorem 2.

Theorem 2. *[45] The set S_n^r is characterized as:*
(i) The set S_n^r is in a unit disk of complex plane, and it's spectrum is symmetric about real-axis.
(ii) The set S_n^r may intersect the unit circle in a finite number of vertices.

4. New Results

In this section, we provide recent findings on the stability, $\mathfrak{D}$-stability and strong $\mathfrak{D}$-stability analysis on positive dynamical systems whose coefficient matrices are Metzler matrices. Mainly, We analyze positive linear time-invariant systems, where the development of these findings incorporates various principles from linear algebra, matrix theory, and system theory. We use results on the interconnection between μ-theory and $\mathfrak{D}$-stability theory to formulate and bring forth new findings on stability, $\mathfrak{D}$-stability and strong $\mathfrak{D}$-stability.

4.1. The stability of positive linear time-invariant systems:

We present some recent findings on stability analysis of positive linear time-invariant systems having the appearance of Metzler, and Hurwitz matrices. Theorem 3 gives the conditions under which linear time-invariant system with n-dimensional real-valued Metzler, and Hurwitz matrices, is stable.

Theorem 3. *Let $A_1, A_2 \in \mathbb{R}^{n,n}$ be Metzler matrices and Hurwitz. The linear system:*

$$\frac{dx(t)}{dt} = A(t)x(t); \, A(t) \in \{D_1A_1, D_2A_2\} : D_1, D_2 > 0, \, x(t) \in \mathbb{R}^{n,1},$$

is stable if

$$Re\left(\lambda_1(D_1(A_1 + \gamma D_1^{-1}D_2A_2))\right) > \left|Re\left(\lambda_k(D_1(A_1 + \gamma D_1^{-1}D_2A_2))\right|\right.,$$

where λ_1 is largest eigenvalue, and λ_k denotes all the remaining eigenvalues other than λ_1, and $\gamma \geq 0$.

Proof. For the largest eigenvalue λ_1, we have that, $Re\left(\lambda_1(D_1(A_1+\gamma D^{-1}D_2A_2))\right) > 0$ because $\lambda_1 \in \mathbb{R}$, and

$$\sum_{i=1} \lambda_i\left(D_1(A_1+\gamma D^{-1}D_2A_2)\right) = T_r\left(D_1(A_1+\gamma D_1^{-1}D_2A_2)\right),$$

where **trace** of the matrix is denoted by $T_r(\cdot)$. The above expression is strictly positive, and from this it follows that $\lambda_1\left(D_1(A_1+\gamma D_1^{-1}D_2A_2)\right) > 0$. Next, we aim to show that $\lambda_1 \neq \lambda_k$, and

$$Re\left(\lambda_1(D_1(A_1+\gamma D_1^{-1}D_2A_2))\right) > \left|Re\left(\lambda_k(D_1(A_1+\gamma D_1^{-1}D_2A_2)\right)\right|.$$

Consider for $\lambda_1 > \lambda_k, \vec{v_1}$ be normalized eigenvector then,

$$\sum_i a_{i,j}\vec{v_1} = \lambda_k\vec{v_1},$$

and let $|\vec{v_1}| = x_1$, then

$$0 < Re\left(\lambda_1(D_1(A_1+\gamma D_1^{-1}D_2A_2))\right) = \sum_{ij} a_{ij}\vec{v_1}\vec{v_k} = \left|\sum_{ij}\vec{v_1}\vec{v_k}\right| \leq \sum_{ij} a_{ij}x_1x_2.$$

Suppose that x_2 be an eigenvector corresponding to $\lambda_1\left(D_1(A_1+\gamma D_1^{-1}D_2A_2)\right)$, then we have

$$\sum_{ij} a_{ij}x_2 = \lambda_1 x_1, \quad x_2 \neq \vec{0}.$$

The condition that $x_2 \neq \vec{0}$ is a non-degeneracy condition, and it then allows to have

$$Re\left(\lambda_1(D_1(A_1+\gamma D_1^{-1}D_2A_2)\right) > \sum_{ij} a_{ij}|v_1||v_k| \geq \left|\sum_{ij} a_{ij}v_1v_k\right| = \left|\lambda_k\left(D_1(A_1+\gamma D_1^{-1}D_2A_2)\right)\right|$$

$$= \left|Re\left(\lambda_k(D_1(A_1+\gamma D_1^{-1}D_2A_2))\right)\right| > 0.$$

Theorem 4 shows that the positive linear time-invariant system with coefficient matrices $A_1, A_2 \in \mathbb{R}^{n,n}$ being Metzler and Hurwitz, is stable system for $x(t) \in \mathbb{R}^{n,1},\ \gamma > 0$, the quantity $x^t(t)\left(D_1(A_1+\gamma D_1^{-1}D_2A_2)\right)x(t)$ is strictly positive if and only if $Re\left(\lambda_i(D_1(A_1+\gamma D_1^{-1}D_2A_2))\right)$ is strictly positive.

Theorem 4. *Let $A_1, A_2 \in \mathbb{R}^{n,n}$ be Metzler matrices and Hurwitz. The linear system*

$$\frac{dx(t)}{dt} = A(t)x(t),\ \textit{where}\ A(t) \in \{D_1A_1, D_2A_2\},\ D_1, D_2 > 0,\ \textit{and}\ x(t) \in \mathbb{R}^{n,1}$$

is stable then $x^T(t)\left(D_1(A_1+\gamma D_1^{-1}D_2A_2)\right)x(t) > 0 \iff Re\left(\lambda_i(D_1(A_1+\gamma D_1^{-1}D_2A_2))\right) > 0$.

Proof. To ensure stability, it is sufficient to demonstrate that for $z = x+iy : x, y \in \mathbb{R}^{n,1}$, the quadratic form

$$z^T \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) z > 0,\ \ z \in \mathbb{C}^{n,1}.$$

Further

$$y^T \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)x\right)^T = x^t \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right)^T y.$$

Also,

$$z^T \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) z$$

$$=$$

$$x^T \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) x + y^T \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) y + i \left(x^T (D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) y$$

$$-y^T \left((D_1(A_1 + \gamma D_1^{-1} D_2 A_2))x\right) = x^T \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2\right) x + y^T \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) y > 0.$$

This is true if $x, y \neq \vec{0}$, and hence this implies that $Re\left(\lambda_i(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) > 0, \forall\, i.$

Theorem 5. *Let $A_1, A_2 \in \mathbb{R}^{n,n}$ be Metzler matrices and Hurwitz. The linear system*

$$\frac{dx(t)}{dt} = A(t)x(t),\ \ where\ \ A(t) \in \{D_1A_1, D_2A_2\}, D_1, D_2 > 0,\ \ and\ x(t) \in \mathbb{R}^{n,1},$$

is stable then $x^T \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) x > 0 \iff Re\left(\lambda_i(D_1(A_1 + \gamma D_1^{-1} D_2 A_2))\right) > 0, \forall\ i = 1 : n$.

Proof. The quantity $x^T \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) x$ is real, and positive for the given matrix $(D_1(A_1+\gamma D_1^{-1} D_2 A_2))$. Then for $x \in \mathbb{R}^{n,1}$, the quadratic form $x^T \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) x$, $x \in \mathbb{R}^{2,1}$. Also, $Re\left(\lambda_i(D_1(A_1 + \gamma D_1^{-1} D_2 A_2))\right) > 0, \forall i = 1 : 2$ because for $v_1 \in \mathbb{R}^{n,1}$,

$$Re\left(\lambda_i(D_1(A_1 + \gamma D_1^{-1} D_2 A_2))\right) = v_1^T(\lambda_i v_1) = v_1^T \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) v_1,$$

where v_1 is unit eigenvector corresponding to λ_i $\forall i = 1 : n$. If we consider the matrix $\left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right)$ such that it has only one positive eigenvalue, then

$$\left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) = U \Lambda U^T, U^T U = U U^T = I_n \ and\ \Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n).$$

This yield

$$x^T \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) x = x^T U \Lambda U^T x = (U^t x)^T \Lambda (U^T x) = \sum_i Re(\lambda_i) |v_1^T x|^2 \geq 0,$$

for $Re\left(\lambda_i(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) > 0$, and some $v_1^T x \neq 0$, with $x \neq 0$.

Theorem 6 shows that the positive linear time-invariant system with coefficient matrices $A_1, A_2 \in \mathbb{R}^{n,n}$ being Metzler and Hurwitz, is a stable system for $S^T = S, \gamma > 0$, the perturbed matrix

$$D_1 \left(A_1 + \gamma D_1^{-1} D_2 A_2)\right)^T S \left(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)\right) - S,$$

has strictly positive real part for all of it's eigenvalues.

Theorem 6. *Let $A_1, A_2 \in \mathbb{R}^{n,n}$ be Metzler matrices and Hurwitz. The linear system*

$$\frac{dx(t)}{dt} = A(t)x(t), \text{ where } A(t) \in \{D_1A_1, D_2A_2\}, D_1, D_2 > 0, \text{ and } x(t) \in \mathbb{R}^{n,1},$$

is stable if

$$Re\left[\lambda_i \left(D_1(A_1 + \gamma D_1^{-1}D_2A_2)\right)^T S \left(D_1(A_1 + \gamma D_1^{-1}D_2A_2)\right) - S\right] > 0 \;\; \forall i = 1 : n$$

with $S^t = S$ and $S > 0$ is positive definite matrix.

Proof. Consider $\alpha \in \{1, 2, \cdots, n\}$, and let $x(t) \in \mathbb{R}^{n,1}, t \in \mathbb{R}^+$. Also, assume that $x[\alpha] \neq 0$. For the proof of our result, we take a non-zero vector $x(t)$, so that

$$x[\alpha]^T \left(D_1(A_1 + \gamma D_1^{-1}D_2A_2\right)^T S \left(D_1(A_1 + \gamma D_1^{-1}D_2A_2) - S)\right) x[\alpha]$$

$$=$$

$$x^T(t) \left((D_1(A_1 + \gamma D_1^{-1}D_2A_2\right) S \left(D_1(A_1 + \gamma D_1^{-1}D_2A_2) - S)x(t)\right) > 0.$$

Further, we have

$$x^T(t) Re \left(\lambda_i(D_1(A_1 + \gamma D_1^{-1}D_2A_2)\right) x(t) > 0, \;\; \forall i = 1 : n.$$

In turn, this yields

$$Re(\lambda_i \left(D_1(A_1 + \gamma D_1^{-1}D_2A_2)\right) > 0, \;\; \forall i = 1 : n \text{ and } ||x(t)|| = 1.$$

Thus, finally we have that given linear system is stable.

Theorem 7. *Let $A_1, A_2 \in \mathbb{R}^{n,n}$ be Metzler matrices and Hurwitz.*
The linear system

$$\frac{dx(t)}{dt} = A(t)x(t), \text{ where } A(t) \in \{D_1A_1, D_2A_2\}, D_1, D_2 > 0, \text{ and } x(t) \in \mathbb{R}^{n,1}$$

is stable if

$$Re\left[\lambda_i \left((D_1(A_1 + \gamma D_1^{-1}D_2A_2)\right)^T S \left(D_1(A_1 + \gamma D_1^{-1}D_2A_2)) - S\right)\right] > 0 \;\; \forall i = 1 : n.$$

Let $(\lambda_i(t), x_i(t))$ be an eigenpair, and assume that

$$\rho = \left|Re \left(\lambda_i(D_1(A_1 + \gamma D_1^{-1}D_2A_2\right)^T S \left(D_1(A_1 + \gamma D_1^{-1}D_2A_2)\right) - S\right|,$$

then $|x(t)| > 0, t \in \mathbb{R}^+$ and $\left((D_1(A_1 + \gamma D_1^{-1}D_2A_2))^T S(D_1(A_1 + \gamma D_1^{-1}D_2A_2)) - S\right) |x(t)|$

$$=$$

$$\rho \left((D_1(A_1 + \gamma D_1^{-1}D_2A_2))^T S(D_1(A_1 + \gamma D_1^{-1}D_2A_2)) - S)\right) |x(t)|.$$

Proof. Consider that

$$\tilde{x}(t) = \left((D_1(A_1 + \gamma D_1^{-1} D_2 A_2))^T S(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)) - S)\right) |x(t)| > 0.$$

This further implies that

$$\tilde{x}(t) = Re\left[\lambda_i\left((D_1(A_1 + \gamma D_1^{-1} D_2 A_2))^T S(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)) - S)\right)\right] |x(t)|.$$

Also, from this we have that,

$$\tilde{x}(t) = \rho\left((D_1(A_1 + \gamma D_1^{-1} D_2 A_2))^T S(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)) - S)\right) |x(t)|.$$

Let $\hat{x}(t) = x(t) - \rho\left((D_1(A_1 + \gamma D_1^{-1} D_2 A_2))^T S(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)) - S)\right) |x(t)| \geq 0.$ Also,

$$\rho\left((D_1(A_1 + \gamma D_1^{-1} D_2 A_2))^T S(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)) - S)\right) \geq 0, \text{ and } |x(t)| > 0, \ \hat{x}(t) = 0.$$

For $\tilde{x}(t) \neq 0$, we have

$$\left((D_1(A_1 + \gamma D_1^{-1} D_2 A_2))^T S(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)) - S)\right) x(t)$$

$$>$$

$$\rho\left((D_1(A_1 + \gamma D_1^{-1} D_2 A_2))^T S(D_1(A_1 + \gamma D_1^{-1} D_2 A_2)) - S)\right) x(t),$$

which is only possible if we don't consider $x(t)$, and this is not possible, and hence $\tilde{x}(t) \neq 0$.

4.2. The $\mathfrak{D}$-stability of positive linear time-invariant systems:

We derive some new results on $\mathfrak{D}$-stability analysis of positive time-invariant linear systems having the presence of Metzler, and Hurwitz matrices. The characterization of D-stability [54] for a given real-valued n-dimensional matrix A in terms of the real structured singular values is given by the following Theorem 8.

Theorem 8. *Let $A \in \mathbb{R}^{n,n}$ be the given matrix. Then A is a D-stable matrix if and only if it is stable and none of the eigenvalues of $A \pm iD$ is exactly equal to zero, and*

$$0 \leq \mu_{\mathbb{B}}\left((iI + A)^{-1}(iI - A)\right) < 1.$$

Theorem 9 gives the conditions under which linear time-invariant system with n-dimensional real-valued Metzler, and Hurwitz matrices, is $\mathfrak{D}$-stable.

Theorem 9. *Let $A_1, A_2 \in \mathbb{R}^{n,n}$ be Metzler matrices and Hurwitz.The time-invariant linear system*

$$\frac{dx(t)}{dt} = A(t)x(t), \text{ where } A(t) \in \{A_1, A_2\}, \text{ and } x(t) \in \mathbb{R}^{n,1},$$

is $\mathfrak{D}$-stable if $Re\left(\lambda_i(A_1 + DA_2)\right) \neq 0, \forall i = 1 : n$ with $D = diag(d_{ii}) : \ d_{ii} > 0$.

Proof. Let $A(t) \in \{A_1, A_2\}$ is $\mathfrak{D}$-stable, which means that $Re\left(\lambda_i(A_1 + DA_2)\right) > 0, \forall i = 1 : n$. Further $(A_1+DA_2)+iD = D\left(D^{-1}(A_1 + A_2D) + iI_n\right)$ so that $\lambda_i\left(D(D^{-1}(A_1 + A_2D) + iI_n)\right) \neq 0, \forall i = 1 : n$, with I_n being n-dimensional identity matrix. Assume that $A(t) \in \{A_1, A_2\}$ is not $\mathfrak{D}$-stable matrix, which implies that $Re(\lambda_i(D(A_1+DA_2))) \ngtr 0, \forall i = 1 : n$. For each positive diagonal matrix $\tilde{D}$, the matrix $\tilde{D}D(A_1+A_2D)$ is not stable, but $\tilde{D}(A_1+A_2D)$ is a stable matrix for $0 < t \leq 1$ and for some $\gamma > 0$, we have that $\frac{1}{\gamma}(tD+(1-t)I_n)\tilde{D}(A_1+DA_2)$ with $D = \gamma(tD + (1 - t)I_n)^{-1}$ has an eigenvalue $i = \sqrt{-1}$.

In Theorem 10, it has been proven that the positive linear time-invariant system is $\mathfrak{D}$-stable if the product of all eigenvalues of the perturbed matrix $(A_1 + DA_2)D^{-1} + D(A_1 + DA_2)^{-1}$ is strictly positive.

Theorem 10. *Let $A_1, A_2 \in \mathbb{R}^{n,n}$ be Metzler, and Hurwitz matrices. The positive linear time-invariant system*

$$\frac{dx(t)}{dt} = A(t)x(t), \text{ where } A(t) \in \{A_1, A_2\} \text{ and } x(t) \in \mathbb{R}^{n,1},$$

is $\mathfrak{D}$-stable if $\prod_{i=1}^{n} \lambda_i\left((A_1 + DA_2)D^{-1} + D(A_1 + DA_2)^{-1}\right) > 0, \forall\ i = 1 : n$.

Proof. We start with the partitioned matrix of the form

$$\begin{pmatrix} A_1 + DA_2 & -D \\ D & A_1 + DA_2 \end{pmatrix}.$$

The Schur complement of the partitioned matrix is given as

$$(A_1 + DA_2) + D(A_1 + DA_2)^{-1}D = D(A_1 + DA_2)D^{-1}D + D(A_1 + DA_2)^{-1}D$$
$$= ((A_1 + DA_2)^{-1} + D(A_1 + DA_2)^{-1}D).$$

The eigenvalues $\lambda_i\ \forall\ i = 1 : n$ are with

$$\prod_{i=1}^{n} \lambda_i((A_1 + DA_2) + D(A_1 + DA_2)^{-1}) = \prod_{i=1}^{n} \lambda_i((A_1 + DA_2)D^{-1} + D(A_1 + DA_2)^{-1})D$$
$$= \prod_{i=1}^{n} \lambda_i((A_1 + DA_2) \prod_{i=1}^{n} \lambda_i((A_1 + DA_2)^{-1}D^{-1} + D(A_1 + DA_2)^{-1}) \prod_{i=1}^{n} \lambda_i(D).$$

We know that

$$\prod_{i=1}^{n} \lambda_i\left((A_1 + DA_2) + D(A_1 + DA_2)^{-1}D\right) > 0$$
$$\Leftrightarrow$$
$$\prod_{i=1}^{n} \lambda_i\left((A_1 + DA_2)D^{-1} + D(A_1 + DA_2)^{-1}\right) > 0,$$

because

$$\lambda_i(A_1 + DA_2) > 0, \quad \lambda_i(D) > 0, \quad \forall i = 1 : n.$$

$$\text{Also, } \lambda_i((A_1 + DA_2)^{-1}) > 0, \text{ which implies } \quad \lambda_i(D(A_1 + DA_2)^{-1}) > 0, \quad \forall i = 1 : n.$$

The results established in Theorem 11 show that positive linear time-invariant system is $\mathfrak{D}$-stable for n-dimensional real-valued Metzler, and Hurwitz matrices A_1, A_2 if $\lambda_i((A_1 + DA_2) + iD) \neq 0, \forall i = 1 : n$.

Theorem 11. *The dynamical system* $\frac{dx(t)}{dt} = A(t)x(t), \quad A(t) \in \{A_1, A_2\}$ *where* $A_1, A_2 \in \mathbb{R}^{n,n}$ *are Metzlern and Hurwitz, is* $\mathfrak{D}$*-stable if*

$$\lambda_i((A_1 + DA_2) + iD) \neq 0, \forall i = 1 : n, with\, D = diag(d_{ii}),\ where\ d_{ii} > 0.$$

Proof. Suppose that $A(t) \in \{A_1, A_2\}$ is a $\mathfrak{D}$-stable matrix. This implies that

$$Re(\lambda_i(D(A_1 + DA_2))) > 0, \forall i \ = 1 : n.$$

The eigenvalue $i = \sqrt{-1}$ is not an eigenvalue of the matrix product $D(A_1 + DA_2)$. Furthermore, we have that

$$(A_1 + DA_2) + iD = D(D^{-1}(A_1 + DA_2) + iI_n),$$

such that $\lambda_i(D(D^{-1}(A_1 + DA_2) + iI_n)) \neq 0, \forall i = 1 : n$ where I_n is $n \times n$ identity matrix. This also hold for a positive diagonal matrix $\tilde{D}$ so that the matrix $\tilde{D}D(A_1 + DA_2)$ is not stable, but $\tilde{D}(A_1 + DA_2)$ is stable, means that

$$Re(\lambda_i(\tilde{D}(A_1 + DA_2)) > 0, \forall i = 1 : n.$$

This follow from fact that if $0 < t \leq 1$, and for $\gamma > 0$, the matrix $\frac{1}{\gamma}(tD + (1-t)I_n)\tilde{D}(A_1 + DA_2))$, with $D = \gamma(tD + (1-t)I_n)^{-1}$ has an eigenvalue $i = \sqrt{-1}$.

Theorem 12 provides insights into the $\mathfrak{D}$-stability of positive linear time-invariant systems. It has been established that a positive linear time-invariant system, where $A_1, A_2 \in \mathbb{R}^{n,n}$ are Metzler and Hurwitz matrices, is $\mathfrak{D}$-stable if the μ-value of $(A_1 + DA_2)^{-2}$ greater than or equal to 0 and less than 1, for some positive diagonal matrix D.

Theorem 12. *The dynamical system*

$$\frac{dx(t)}{dt} = A(t)x(t),\ where\ A(t) \in \{A_1, A_2\}\ and\ x(t) \in \mathbb{R}^{n,1},$$

where $A_1, A_2 \in \mathbb{R}^{n,n}$ *are* $\mathfrak{D}$*-stable Metzler and Hurwitz matrices iff* $0 \leq \mu_{\mathbb{B}}((A_1 + DA_2)^{-2}) < 1$, *with* $D = diag(d_{ii})$; $d_{ii} > 0$.

Proof. From [55], it is evident that the matrix $(A_1 + DA_2)$ is $\mathfrak{D}$-stable iff $(A_1 + DA_2)$ is stable and

$$det \begin{pmatrix} A_1 + DA_2 & -D \\ D & A_1 + DA_2 \end{pmatrix} \neq 0.$$

As we know that for a stable matrix $(A_1 + DA_2)$, we have that

$$det \begin{pmatrix} A_1 + DA_2 & -D \\ D & A_1 + DA_2 \end{pmatrix} \neq 0.$$

This implies that $0 \leq \mu_{\mathbb{B}}((A_1+DA_2)^{-2}) < 1$. Also from [56], we have $det \begin{pmatrix} A_1 + DA_2 & -D \\ D & A_1 + DA_2 \end{pmatrix} \neq 0$, in turn this implies

$$det\left((A_1 + DA_2)^2 - D(A_1 + DA_2)^{-1}D(A_1 + DA_2)\right) \neq 0,$$

which further yields $det\left((A_1 + DA_2)^2 - D(A_1 + DA_2)^{-1}D(A_1 + DA_2)\right) \neq 0$. This implies that $det\left(I_2 - (A_1 + DA_2)^{-2}\tilde{D}\right) \neq 0$, where $\tilde{D} = D = diag(d_{11}, d_{22}); d_{ii} > 0$. Thus finally, $det(I_n - (A_1 + DA_2)^{-2}\tilde{D}) \neq 0 \implies 0 \leq \mu_{\mathbb{B}}((A_1 + DA_2)^{-2}) < 1$.

Theorem 13 establishes results regarding the $\mathfrak{D}$-stability of positive linear time-invariant systems. It has been demonstrated that a positive linear time-invariant system with $A_1, A_2 \in \mathbb{R}^{n,n}$, where both are Metzler and Hurwitz matrices, is $\mathfrak{D}$-stable if the real parts of all eigenvalues of the matrix $D(A_1 + DA_2) + (A_1 + DA_2)^T D$ are strictly positive. Additionally, for a positive diagonal matrix D, the system remains $\mathfrak{D}$-stable if the μ-value of the matrix M, derived from A_1, A_2, and D, satisfies $0 \leq \mu(M) < 1$.

Theorem 13. *The dynamical system*

$$\frac{dx(t)}{dt} = A(t)x(t), \text{ where } A(t) \in \{A_1, A_2\}, \ x(t) \in \mathbb{R}^{n,1},$$

where $A_1, A_2 \in \mathbb{R}^{n,n}$ *are Metzler, and Hurwitz matrices, is* $\mathfrak{D}$*-stable if and only if*

$$Re\left(\lambda_i(D(A_1 + DA_2) + (A_1 + DA_2)^T D)\right) > 0, \quad \forall i = 1 : n$$

with $D = diag(d_{ii})$, $d_{ii} > 0$, *and* $0 \leq \mu_{\mathbb{B}}(M) < 1$. *The matrix* M *is defined as*

$$M = \left(iI_n + D(A_1 + DA_2) + (A_1 + DA_2)^T D\right)^{-1} (iI_n - D(A_1 + DA_2) - (A_1 + DA_2)D).$$

Proof. To prove that $(A_1 + DA_2)$ is $\mathfrak{D}$-stable iff $0 \leq \mu_{\mathbb{B}}(M) < 1$, we assume that $(A_1 + DA_2)$ is $\mathfrak{D}$-stable matrix, means that for all $D = diag(d_{ii})$, $\lambda_i((A_1 + DA_2) + iD) \neq 0, \forall\, i = 1 : n$. Let $\Delta \in \mathbb{B}$ has block-diagonal structure, that is, $\Delta = (iI_n - D)(iI_n + D)^{-1}$. Then, $D = (iI_n + \Delta)^{-1}(iI_n - \Delta), \Delta \in B$. Since $\lambda_i((A_1 + DA_2) + iD) \neq 0, \forall\, i = 1 : n$ for some D, a positive diagonal matrix. This further implies that

$$\lambda_i((A_1 + DA_2) + i(iI_n + \Delta)^{-1}(iI_n - \Delta)) \neq 0 \ \forall \Delta \in \mathbb{B}, \forall\, i = 1 : n.$$

We also observe that,

$$rank\left((A_1 + DA_2) + i(iI_n + \Delta)^{-1}(iI_n - \Delta)\right) \approx rank\left((iI_n + (A_1 + DA_2) - (iI_n - (A_1 + DA_2)\Delta)\right).$$

This allows us to arrive at following expression, that is,

$$(iI_n+(A_1+DA_2))-(iI_n-(A_1+DA_2)\Delta)) = (I_n-(iI_n+(A_1+DA_2)^{-1})(iI_n-(A_1+DA_2)\Delta)), \forall\Delta \in \mathbb{B}.$$

Thus

$$\lambda_i((I_n - (iI_n + (A_1 + DA_2)^{-1}(iI_n - (A_1 + DA_2))\Delta) \neq 0, \forall\Delta \in \mathbb{B}, \forall i = 1 : n$$

which yield that $0 \leq \mu_{\mathbb{B}}(M) < 1$. Conversely, we assume that $0 \leq \mu_{\mathbb{B}}(M) < 1$ and we show that $(A_1 + DA_2)$ is $\mathfrak{D}$-stable matrix. For $0 \leq \mu_{\mathbb{B}}(M) < 1$, a sufficient condition is that

$$\lambda_i((A_1 + DA_2) + iD) \neq 0, \forall\, i = 1 : n,$$

and for some $D = diag(d_{ii})$, a positive diagonal matrix and this shows that $(A_1 + DA_2)$ is a $\mathfrak{D}$-stable matrix.

4.3. The strong $\mathfrak{D}$-stability of positive linear time-invariant systems:

We present some recent findings on strong $\mathfrak{D}$-stability analysis of positive linear time-invariant systems having the presence of Metzler, and Hurwitz matrices. The characterization of strong D-stability [57] for a given real-valued n-dimensional matrix A in terms of the real structured singular values is given by the following Theorem 14.

Theorem 14. *Let $A \in \mathbb{R}^{n,n}$ be the given matrix. Then A is a strongly D-stable matrix if and only if it is stable and*

$$0 \leq \mu_{\mathbb{B}}\left((iI + A)^{-1}(iI - A)\right) < 1.$$

Theorem 15 gives the conditions under which linear time-invariant system with n-dimensional real-valued Metzler, and Hurwitz matrices, is strongly $\mathfrak{D}$-stable.

Theorem 15. *Let $M \in \mathbb{R}^{n,n}$ and let $A_1, A_2, ..., A_r \in \mathbb{R}^{n,n}$.*
The linear system

$$\frac{dx(t)}{dt} = MA(t)x(t)\ ; A(t) \in \{A_1, A_2, ..., A_r\};\ x(t) \in \mathbb{R}^{n,1}$$

is strongly $\mathfrak{D}$-stable if there exist $\gamma_i > 0$, the matrix

$$\log(M)+\left((\log(M) \otimes (A_1 + \gamma_2 A_2 + ... + \gamma_r A_r)^T \Delta + \Delta((\log(M) \otimes (A_1 + \gamma_2 A_2 + ... + \gamma_r A_r)\right)$$

is a $\mathfrak{D}$-stable matrix with $\Delta \in \mathbb{B}$ and $\otimes$ denotes entry-wise product of matrices.

Proof. Consider that $\Delta = \Delta(t) \in \mathbb{B}$ be an admissible perturbation and $\mathbb{B}$ is the set of block diagonal matrices with real or complex uncertainties. Let $\lambda(t) = |\lambda(t)|(\cos\theta + i\sin\theta)$ for $0 < \theta \leq 2\pi$ be the largest eigenvalue. Assume that $\tilde{x}(t)$, $\tilde{y}(t)$ be the right and left eigenvectors and let

$$z = \Big((\log(M) \otimes (A_1 + \gamma_2 A_2 + \cdots + \gamma_r A_r))^T \Delta + \Delta\left(\log(M) \otimes (A_1 + \gamma_2 A_2 + \cdots + \gamma_r A_r)\right)\Big)\, y(t).$$

We make use of an eigenvalue perturbation result by [32] on the largest and simple eigenvalue $\lambda(t)$ to have that

$$\frac{d}{dt}|\lambda(t)|\bigg|_{t=0}^{2} = 2\epsilon\,\frac{|\lambda(t)|}{\alpha}\mathrm{Re}(z^T\Delta(t)x), \quad \text{with} \quad \alpha = (\cos\theta + i\sin\theta)y^T x > 0,\ \epsilon > 0.$$

In turn, this implies that

$$(\log(M) \otimes (A_1 + \gamma_2 A_2 + \cdots + \gamma_r A_r))^T \Delta + \Delta\left(\log(M) \otimes (A_1 + \gamma_2 A_2 + \cdots + \gamma_r A_r)\right) > 0,$$

which further yields that

$$\log(M) + \Big((\log(M) \otimes (A_1 + \gamma_2 A_2 + \cdots + \gamma_r A_r))^T \Delta + \Delta\left(\log(M) \otimes (A_1 + \gamma_2 A_2 + \cdots + \gamma_r A_r)\right)\Big)$$

is $\mathfrak{D}$-stable.

Theorem 16. *Let $M \in \mathbb{R}^{n,n}$, and let $M = e^{\{A_1+\gamma_2 A_2+\ldots+\gamma_r A_r\}}$ where $A_1, A_2, \ldots, A_r$ are n-dimensional Hermition matrices, then M is strongly $\mathfrak{D}$-stable matrix if M is stable and for some $\alpha > 0$ the matrix $(M + G)$ is $\mathfrak{D}$-stable where*

$$G = \left(M \otimes (A_1 + \gamma_2 A_2 + \ldots + \gamma_r A_r)^T)\Delta + \Delta(M \otimes (A_1 + \gamma_2 A_2 + \ldots + \gamma_r A_r)\right), \Delta \in \mathbb{B}, \quad \|G\| < \alpha.$$

Proof. Assume that $\Delta = \Delta(t) \in \mathbb{B}$, where $\mathbb{B}$ is set of block diagonal matrices. Let $\lambda(t) = |\lambda(t)|(\cos\theta + i\sin\theta)$, $0 < \theta \leq 2\pi$ be the largest eigenvalue. In addition, assume that $x(t), y(t)$ are left- and right-eigenvectors. Consider $z = G^T y$. The eigenvalues perturbation findings by [32] on $\lambda(t)$ yield

$$\frac{d}{dt}|\lambda(t)|^2\bigg|_{t=0} = 2\epsilon\,\frac{|\lambda(t)|}{\alpha}\mathrm{Re}(z^T\dot{\Delta}(t)x)$$

with $\alpha = (\cos\theta + i\sin\theta)y^t x$; $\alpha \geq 0$. As we know that, $\mathrm{Re}(z^T\dot{\Delta}(t)x) > 0$, and in turn this implies that

$$\left(M \otimes (A_1 + \gamma_2 A_2 + \ldots + \gamma_r A_r)^T)\Delta + \Delta(M \otimes (A_1 + \gamma_2 A_2 + \ldots + \gamma_r A_r)\right), \Delta \in \mathbb{B}$$

is PD-matrix. Finally for $D = diag(d_{ii})$; $d_{ii} > 0$, the matrix $D\left(e^{\{A_1+\gamma_2 A_2+\cdots+\gamma_r A_r\}} + G\right)$ and from this it follows that

$$M + G = M + \left(M \otimes (A_1 + \gamma_2 A_2 + \ldots + \gamma_r A_r)^T)\Delta + \Delta(M \otimes (A_1 + \gamma_2 A_2 + \ldots + \gamma_r A_r)\right)$$

is $\mathfrak{D}$-stable matrix.

5. Numerical Testing

The numerical experimentation illustrate the computation, and to visualize the spectrum, the singular values, the structured singular values, and the pseudo-spectra for Metzler matrices appearing across the positive dynamical systems. The graphical representations of the ϵ-pseudospectrum are the level sets which are corresponding to resolvent norm $||(zI_n - A)^{-1}||$.

Example 1. Consider $\frac{dx(t)}{dt} = Ax(t)$ is a positive dynamical system with A Metzler matrix give by

$$A = \begin{bmatrix} -10^{-9} & 10^9 & 0 & 0 & 0 \\ 0 & -10^{-9} & 0 & 0 & 10^9 \\ 0 & 10^9 & -10^{-9} & 0 & 0 \\ 10^9 & 10^9 & 0 & -10^{-9} & 10^9 \\ 0 & 0 & 0 & 0 & -10^{-9} \end{bmatrix}.$$

The spectral properties like the computation of spectrum, singular, structured singular values, and pseudo-spectrum of Metzler matrix A are presented in Figure 1.

In Figure 2, we plot the eigenmode corresponding to the eigenvalues. The top plot in the figure (left) shows an envelope which is produced by plotting the absolute value of an eigenmode and minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level show absolute value of eigenmode being ploted at a log scale. Further, it shows that how quickly an eigenmode is decaying with time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations. We plot the value of inverse of the resolvent norm (right). We show real part of pseudomode in magenta.The right singular vector corresponding to the smallest singular value of the matrix $zI_n - A$ is depicted in pseudomode.

Example 2. Consider a positive Frobenius matrix [58]

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.4714 & 0.1953 & 0.3333 \end{bmatrix},$$

and a characteristic polynomial for F given as

$$P(\beta, F) = \beta^3 - 0.3333\beta^2 - 0.1953\beta - 0.4714,$$

with $\beta = \lambda + \eta,\ \eta = 1.1 > \rho(F) = 1$. The matrix $A = F - \eta I_3$, is a Metzler matrix with

$$A = \begin{bmatrix} -1.1 & 1 & 0 \\ 0 & -1.1 & 1 \\ 0.4714 & 0.1953 & 0.7667 \end{bmatrix}.$$

The spectral properties like the computation of spectrum, singular, structured singular values and pseudo-spectrum of Metzler matrix A are presented in Figure 3.

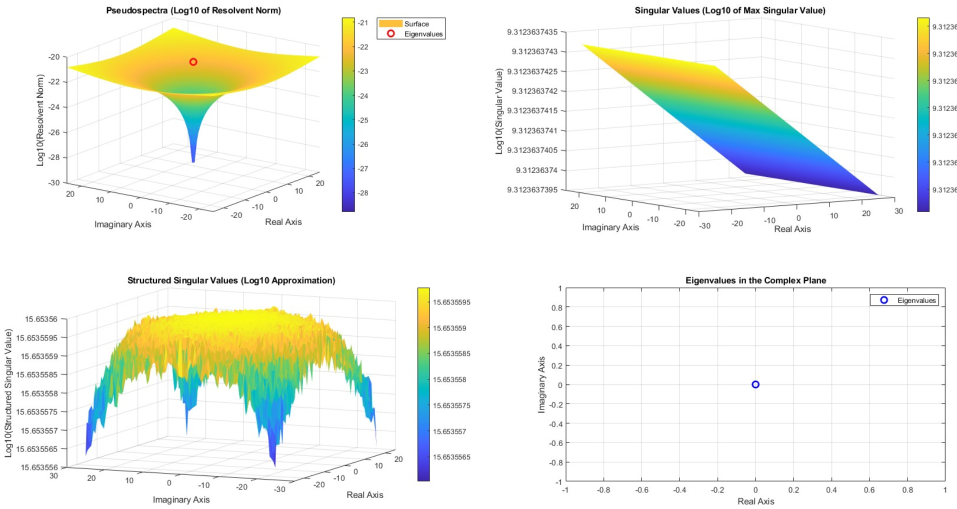

Figure 1: Spectral properties of Metzler matrix A in Example-1.

In Figure 4, we plot the eigenmode corresponding to the eigenvalues. The top plot (left) in the figure shows an envelope which is produced by plotting the absolute value of an eigenmode and minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level show absolute value of eigenmode being ploted at a log scale. Further, it shows that how quickly an eigenmode is decaying with time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations. We plot the value of inverse of the resolvent norm (right). We show real part of pseudomode in magenta. The right singular vector corresponding to the smallest singular value of the matrix $zI_n - A$ is depicted in pseudomode.

Example 3. We consider Metzler matrices with sizes $50, 100$, and 500, respectively. The spectral properties like the computation of spectrum, singular, structured singular values and pseudo-spectrum of Metzler matrix A are presented in Figures 5,7, and 9.

In Figures 6,8, and 10, we plot (left) the eigenmode corresponding to the eigenvalues. The top plot in the figure shows an envelope which is produced by plotting the absolute value of an eigenmode and minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level show absolute value of eigenmode being ploted at a log scale. Further, it shows that how quickly an eigenmode is decaying with time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations. We plot the value of inverse of the resolvent norm (right). We show real part of pseudomode in magenta. The right singular vector corresponding to the smallest singular value of the matrix $zI_n - A$ is depicted in pseudomode.

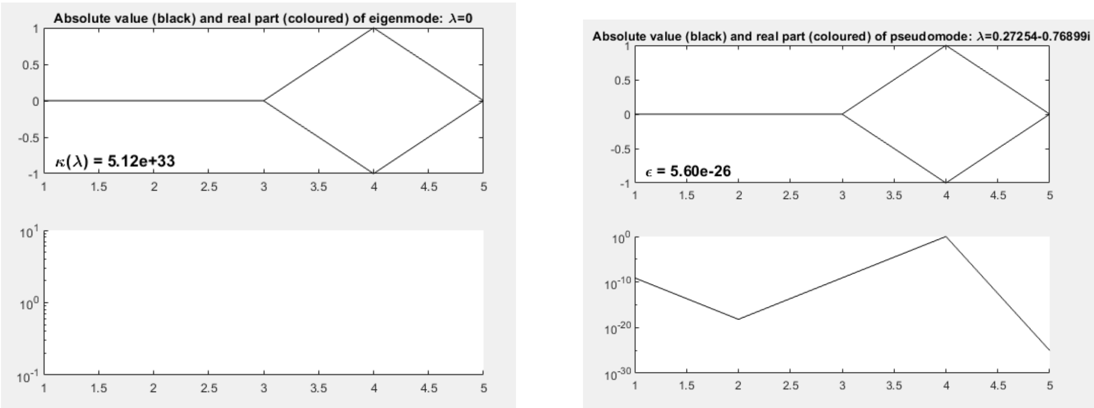

Figure 2: Eigenmode (left) and inverse of resolvent norm (right) of Metzler matrix A in Example-1

Example 4. We consider Hurwitz matrices with sizes $50, 100$, and 500, respectively. The spectral properties like the computation of spectrum, singular, structured singular values and pseudo-spectrum of Hurwitz matrix A are presented in Figures 11,13, and 15. In Figures 12,14, and 16, we plot (left) the eigenmode corresponding to the eigenvalues. The top plot in the figure shows an envelope which is produced by plotting the absolute value of an eigenmode and minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level show absolute value of eigenmode being ploted at a log scale. Further, it shows that how quickly an eigenmode is decaying with time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations. We plot the value of inverse of the resolvent norm (right). We show real part of pseudomode in magenta. The right singular vector corresponding to the smallest singular value of the matrix $zI_n - A$ is depicted in pseudomode.

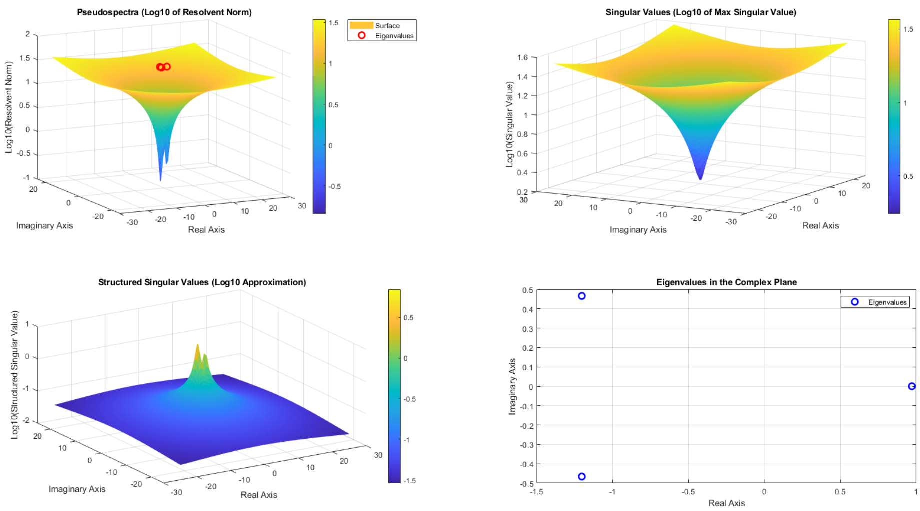

Figure 3: Spectral properties of Metzler matrix A in Example-2.

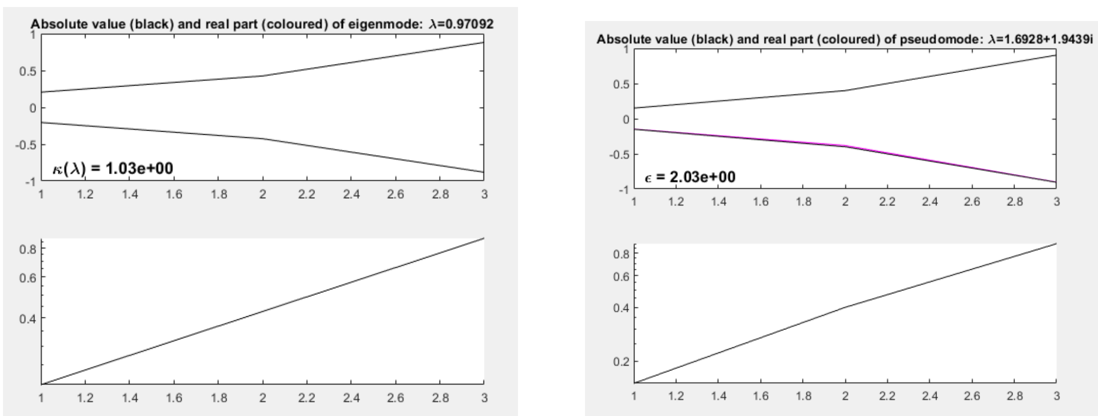

Figure 4: Eigenmode (left) and inverse of resolvent norm (right) of Metzler matrix A in Example-2

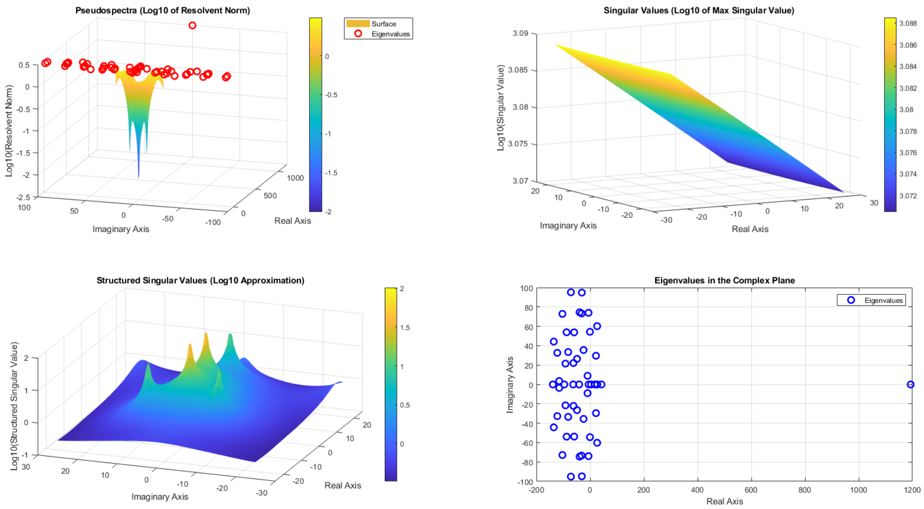

Figure 5: Spectral properties of Metzler matrix (size = 50) in Example-3.

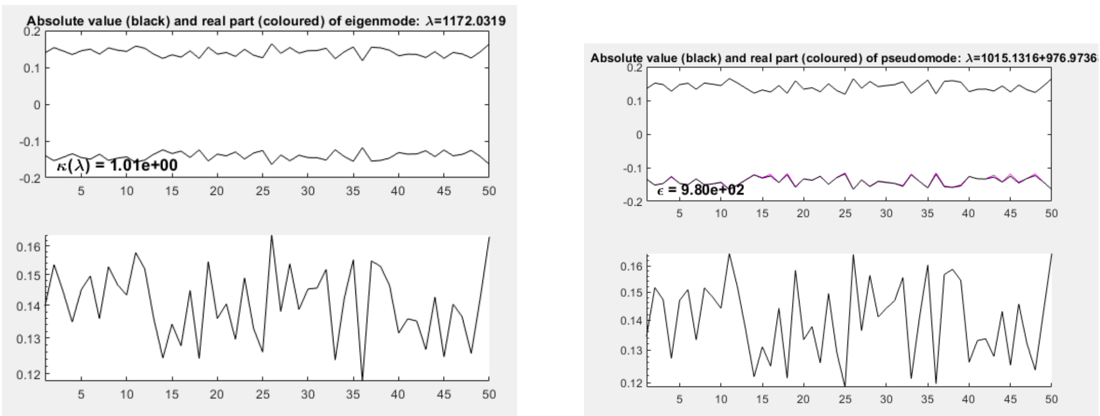

Figure 6: Eigenmode (left) and inverse of resolvent norm (right) of Metzler matrix (size = 50) in Example-3

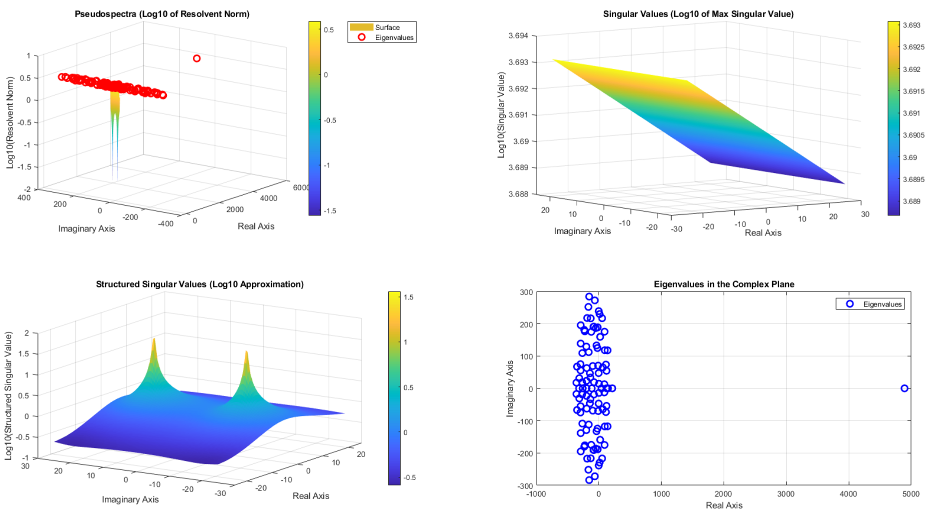

Figure 7: Spectral properties of Metzler matrix (size = 100) in Example-3.

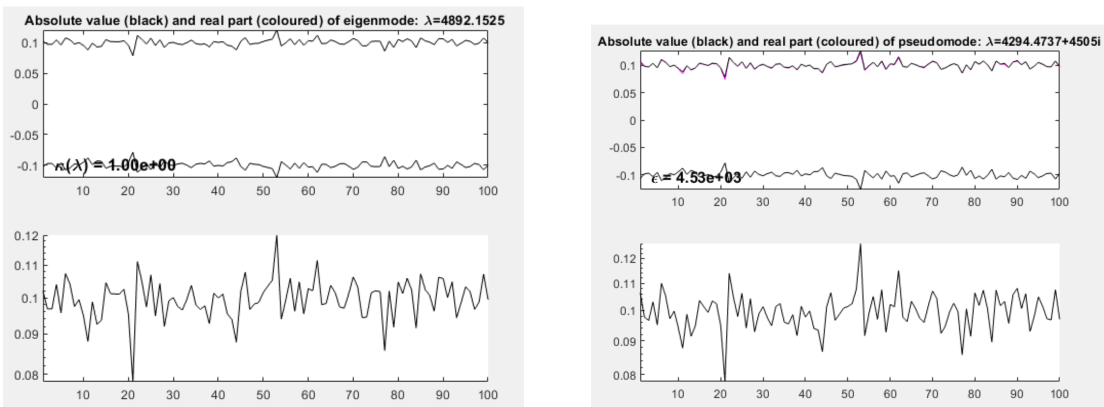

Figure 8: Eigenmode (left) and inverse of resolvent norm (right) of Metzler matrix (size = 100) in Example-3

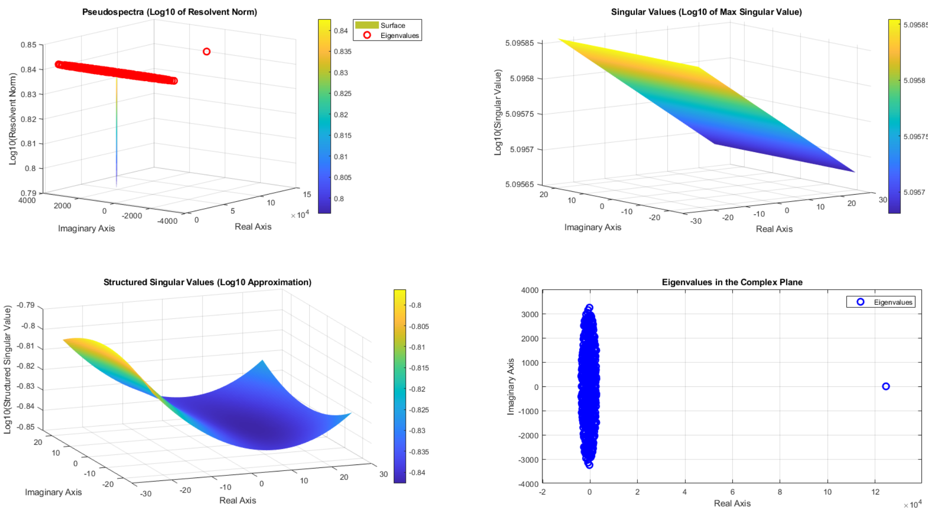

Figure 9: Spectral properties of Metzler matrix (size = 500) in Example-3.

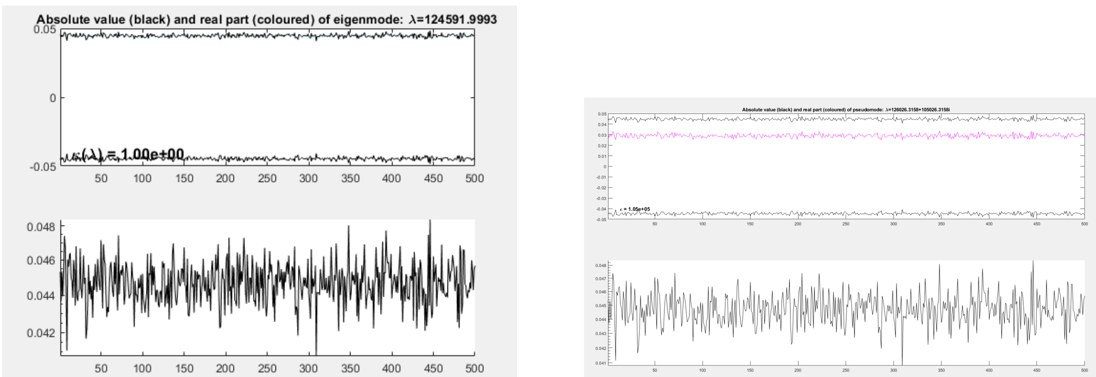

Figure 10: Eigenmode (left) and inverse of resolvent norm (right) of Metzler matrix (size = 500) in Example-3

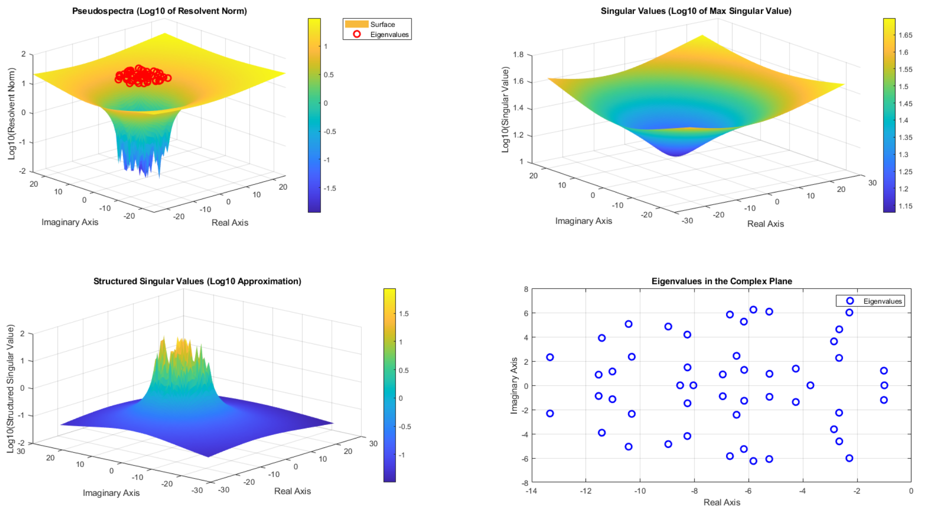

Figure 11: Spectral properties of Hurwitz matrix (size = 50) in Example-4.

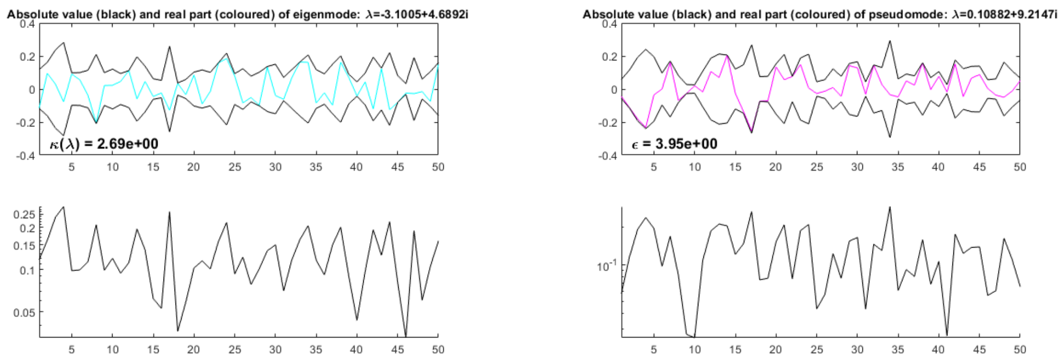

Figure 12: Eigenmode (left) and inverse of resolvent norm (right) of Hurwitz matrix (size = 50) in Example-4

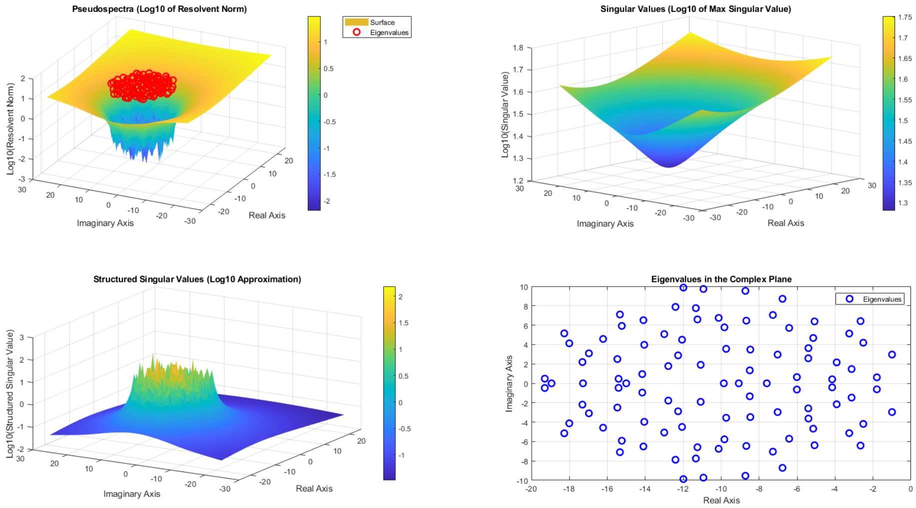

Figure 13: Spectral properties of Hurwitz matrix (size = 100) in Example-4.

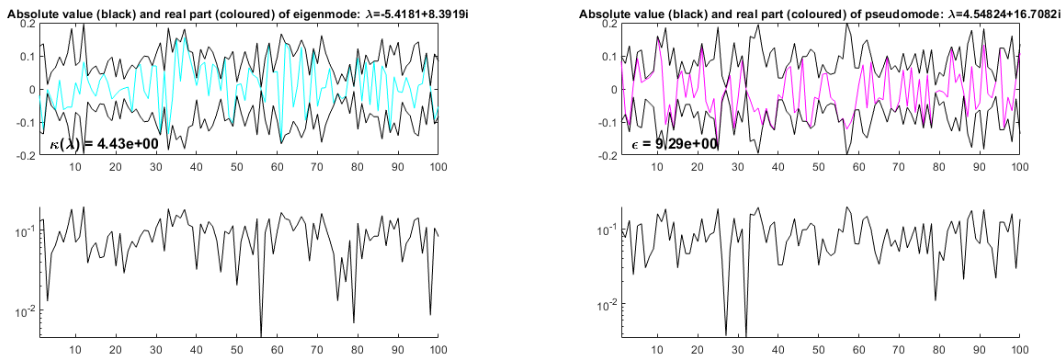

Figure 14: Eigenmode (left) and inverse of resolvent norm (right) of Hurwitz matrix (size = 100) in Example-4

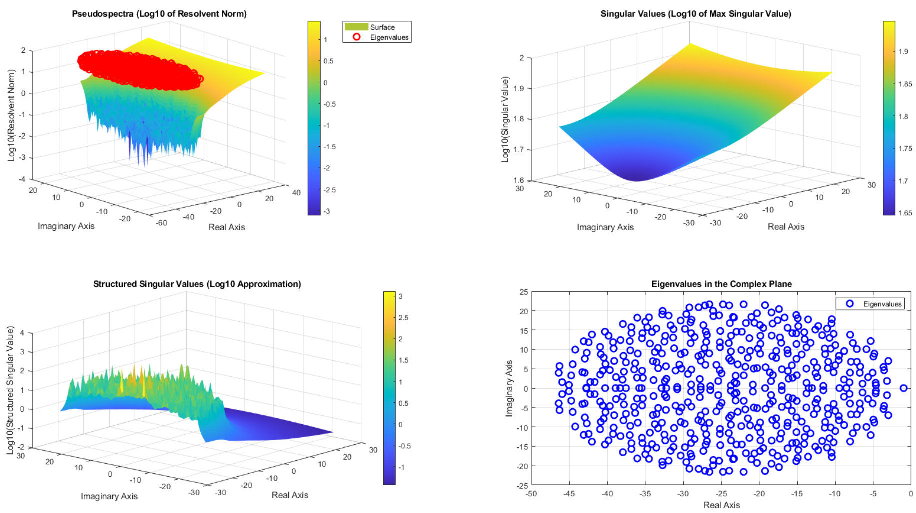

Figure 15: Spectral properties of Hurwitz matrix (size = 500) in Example-4.

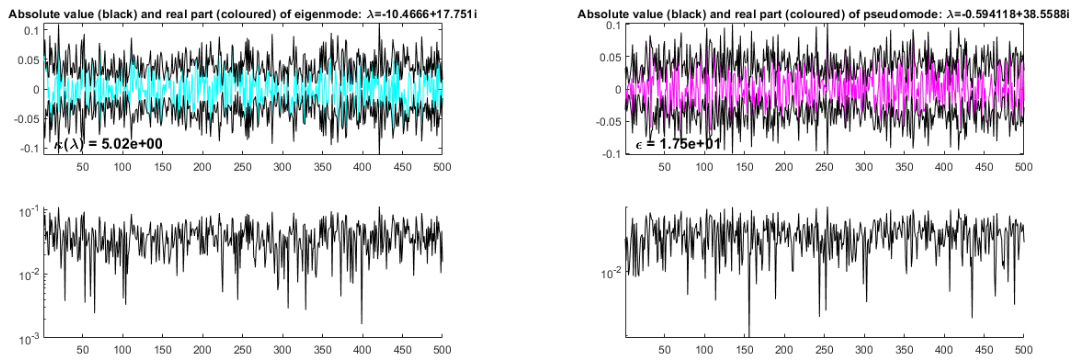

Figure 16: Eigenmode (left) and inverse of resolvent norm (right) of Hurwitz matrix (size = 500) in Example-4

6. Applications

1. Mathematical Modeling: The Metzler matrices play an important role for mathematical modeling of various problems, see [59–61]. For an input data vector $x(t) \in \mathbb{R}^{n,1}$, the most common problem is to analyze the given linear system having the form

$$\frac{dx(t)}{dt} = Ax(t),$$

with A being a Metzler matrix. The analysis on positive linear switching systems also play an important and vital role to the mathematical modeling. An extensive amount of research work has already been done in this directions, see [18, 62, 63].

2. Dynamics of Love: A mathematical model was proposed by S. Rinaldi [64] to describe the dynamics of love of Laura de Noves and Petrarch, which leads to a periodic dynamics. For the implementation of model, 23 dated poems out of 366 poems Petrarch wrote in **Canzoniere** during 21 years were analyzed, see [65]. Rinaldi's model describes love dynamic in three variables, L denotes love of Laura for Petrarch, P denotes love of Petrarch for Laura, and Z denotes the poetic inspiration. The mathematical model is:

$$\begin{cases} \frac{d}{dt}(L(t)) = -\alpha_1 L(t) + R_L(P(t)) + A_p \\ \frac{d}{dt}(P(t)) = -\alpha_2 P(t) + R_P(L(t)) + \beta_2 A_L(Z(t)) \\ \frac{d}{dt}Z(t) = -\alpha_3 Z(t) + \beta_4 P(t), \end{cases}$$

where all parameters involved are defined and described in [65]. This love model can be expressed as

$$\frac{dx(t)}{dt} = A_0 x(t) + b = \phi(t),$$

where

$$A_0 = \begin{bmatrix} -\alpha_1 & 0 & 0 \\ 0 & -\alpha_2 & -\beta_3 \\ 0 & \beta_4 & -\alpha_3 \end{bmatrix};\ b = \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{bmatrix};\ \phi(t) = [R_L(P(t), 0, 0)]^t.$$

For a matrix $\tilde{Q}$, and using matrix A_0, one can have that $A = \tilde{Q}A_0\tilde{Q}$, where the matrix A is same as Metzler matrix. Thus, finally, the analysis on A can lead to interesting results for the love dynamic model.

7. Conclusion

In this paper, we have studied and analyzed mathematical problems related to positive linear time-invariant systems whose coefficient matrices are Metzler and Hurwitz matrices. We have established and presented some new theoretical results on stability, $\mathfrak{D}$-stability, and strong $\mathfrak{D}$-stability to the following three mathematical problems:

Problem-I: To extend and construct some recent findings on stability, $\mathfrak{D}$-stability and strong $\mathfrak{D}$-stability analysis of LTI system $\frac{dx(t)}{dt} = Ax(t)$, where $x(t) \in \mathbb{R}^{n,1}$, $A \in \mathbb{R}^{n,n}$, with A a Metzler and Hurwitz matrix.

Problem-II: To extend and construct some new findings on stability, $\mathfrak{D}$-stability and strong $\mathfrak{D}$-stability analysis of LTI system $\frac{dx(t)}{dt} = A(t)x(t)$, where $x(t) \in \mathbb{R}^{n,1}$, $A(t) \in \{A_1, A_2\}$, with A_1, A_2 being asymptotically stable matrices.
Problem-III: To extend and construct some new findings on stability, $\mathfrak{D}$-stability and strong $\mathfrak{D}$-stability analysis of LTI system $\frac{dx(t)}{dt} = A(t)x(t)$, where $x(t) \in \mathbb{R}^{n,1}$, $A(t) \in \{D_1A_1, D_2A_2\}$, with A_1, A_2 being irreducible matrices, and $D_1, D_2 > 0$.

Conflicts of Interest

The authors confirm that they have no conflicts of interest concerning the publication of this paper.

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