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Independent Neighborhood Polynomial of the Direct and Corona Products of Trees

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Abstract. Let G be a connected graph. We say that a given graph is a tree if every pair of vertices is connected by a unique path. The corona product of two graphs G and H is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. On the other hand, the direct product $G \times H$ is a graph such that the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$; and vertices (g, h) and (g', h') are adjacent in $G \times H$ if and only if g is adjacent to g' in G, and h is adjacent to h' in H.

Here, authors worked on the independent neighborhood sets of the direct and corona products of two trees are obtained. Also, corresponding independent neighborhood polynomialsare found.

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Key Words and Phrases: Corona Product, Direct Product, Independent Neighborhood Polynomial

1. Introduction

A graph G is a pair $(V(G), E(G))$ consisting of a nonempty finite set of vertices $V(G)$ and a set of edges $E(G)$ of unordered pairs of elements of $V(G)$. The cardinalities of $V(G)$ and $E(G)$ are called the order and size of G, respectively. We write $x = uv$ and say that u and v are adjacent vertices; vertex u and edge x are incident with each other, so are v and x. The two vertices incident with an edge are its end vertices or ends, and an edge joins its ends. Two vertices of a graph G are said to be neighbors if they are adjacent in G. The neighborhood of a vertex $v \in V(G)$ is the set $N(v) = \{w : w \in V(G) \text{ and } vw \in E(G)\}$. A vertex v is pendant if its neighborhood contains only one vertex; and edge $e = uv$ is pendant if one of its end vertices is a pendant vertex.

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A graph is *acyclic* if it has no cycles. A graph is said to be *connected* if every pair of vertices are joined by a path. A graph that is not connected is called *disconnected*. A *tree* is a connected acyclic graph.

In 2008, Brown et al. [1] studied the neighborhood polynomial of a graph. Later, in 2014, Alwardi et al. [2] also investigated the neighborhood polynomial of graphs. The following year, in 2015, Kulli [3] focused on the neighborhood graph of a graph. Motivated by these studies, the authors draw inspiration from Murthy's paper "The Independent Neighborhood Polynomial of Graphs." [4] The main objective of the present paper is to establish results on the independent neighborhood polynomial for the direct and corona products of two trees.

2. Preliminaries

This section presents some basic concepts and known results needed in this study.

Definition 1. [5] A graph is *acyclic* if it has no cycles. A graph is said to be *connected* if every pair of vertices are joined by a path. A graph that is not connected is called *disconnected*. A *tree* is a connected acyclic graph.

Definition 2. [6] Let G be a graph. The *distance* between two vertices x and y in a graph G, denoted by $d_G(x, y)$ or simply $d(x, y)$, is the length of the shortest path joining them, otherwise, $d(x, y) = \infty$.

Theorem 1. [7] *Let G be any tree. Then for any $u \in V(G)$, the set*

$$\Omega = \{v \in V(G) : d(u, v) = 2n, n \in \mathbb{N}^*\}$$

is an independent neighborhood set of G.

Corollary 1. [7] *Let F be the set of nonpendant vertices in a tree G and $f \in F$. Let*

$$A = \{a \in F : d_G(a, f) = 2n, n \in \mathbb{N}^*\}$$

and

$$B = F \setminus A.$$

Then the sets

$$\Omega = \{v : v \text{ is a pendant neighbor of } a, \text{ for some } a \in A\} \cup B$$

and

$$\Delta = \{u : u \text{ is a pendant neighbor of } b, \text{ for some } b \in B\} \cup A$$

are the only independent neighborhood sets of G.

Remark 1. [7] For any tree T with independent neighborhood sets Ω and Δ,

$$N_i(T, x) = x^{|\Omega|} + x^{|\Delta|}.$$

Corollary 2. [7] *For any tree T with independent neighborhood sets Ω and Δ, if $uv \in E(T)$, then $u \in \Omega$ and $v \in \Delta$.*

Definition 3. [8] The direct product (or conjunction) of two graphs G and H, denoted by $G \cdot H$, is the graph whose vertex set is $V(G) \times V(H)$ in which (u, v) is adjacent to (u', v') if and only if $uu' \in E(G)$ and $vv' \in E(H)$.

Example 1. Figure 1 shows two graphs G and H and their direct product.

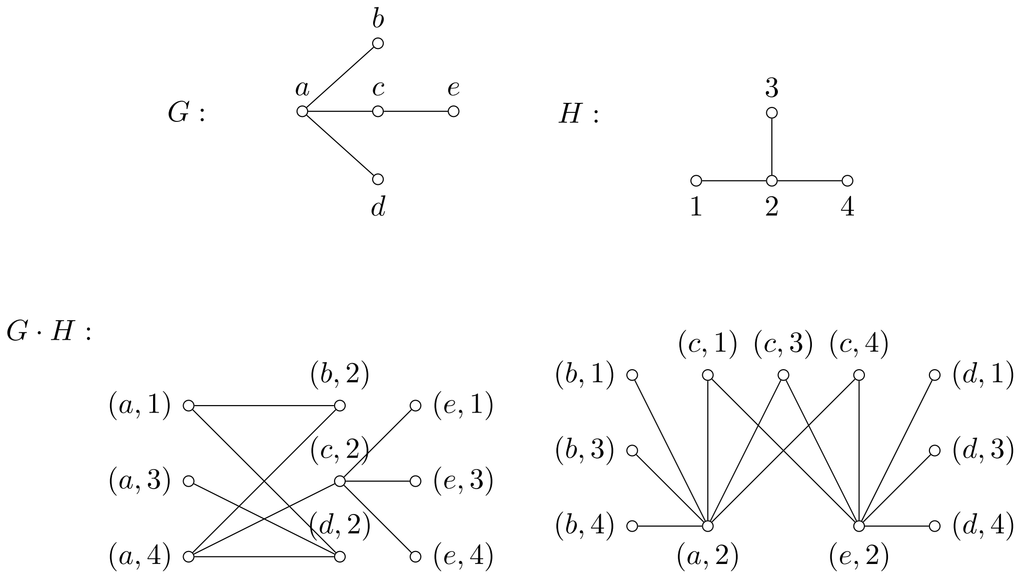


Figure 1: Graphs G and H and their direct product

Definition 4. [5] The corona of two graphs G and H, denoted $G \circ H$, is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.

It is customary to denote by H_v that copy of H whose vertices are adjoined with the vertex v of G. In effect, $G \circ H$ is composed of the subgraphs $H_v + v$ joined together by the edges of G. Moreover,

$$V(G \circ H) = \bigcup_{v \in V(G)} V(H_v + v).$$

Example 2. Figure 2 shows two graphs G and H and their corona products $G \circ H$ and $H \circ G$.

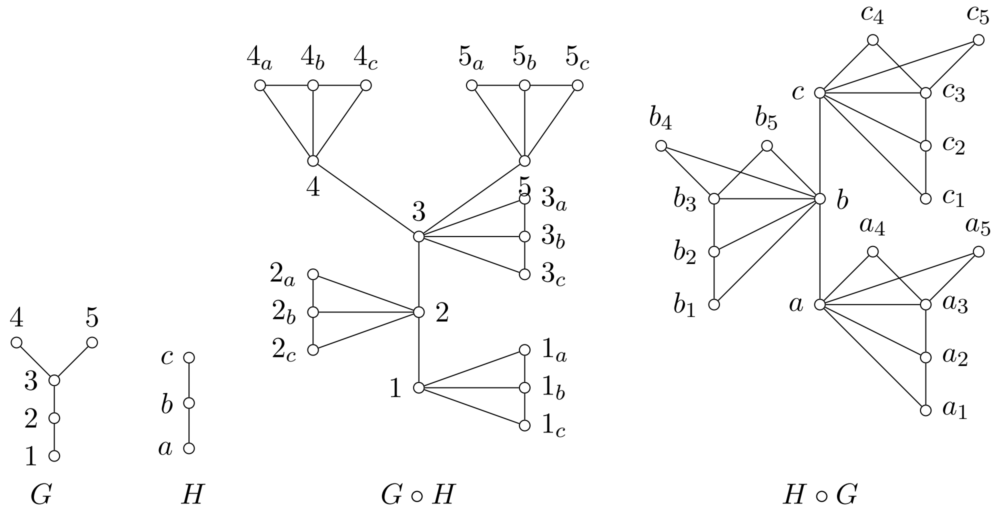


Figure 2: Two trees and their two coronas

Definition 5. [9] The *open neighborhood* of a vertex x, denoted by $N(x)$, is a set containing all vertices y which are adjacent to x, that is,

$$N(x) = \{y \in V(G) : xy \in E(G)\}.$$

In case $N(x)$ is a singleton, x is an end-vertex. The *closed neighborhood* of a vertex x of G is the set $N[x] = N(x) \cup \{x\}$.

Definition 6. [4] A set S of vertices in a graph G is a *neighborhood set* if $G = \bigcup_{v \in S} \langle N[v] \rangle$ where $\langle N[v] \rangle$ is the subgraph of G induced by v and all the vertices adjacent to v. The *neighborhood number* of G is the minimum cardinality of neighborhood sets, denoted by $n_i(G)$.

Example 3. Consider the graph H in Figure 3.

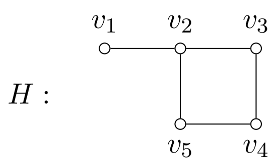


Figure 3: A graph with neighborhood number equals 2

The neighborhood sets of H are $\{v_2, v_4\}$, $\{v_1, v_3, v_5\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_2, v_3, v_4\}$, $\{v_1, v_2, v_3, v_5\}$,

$\{v_1, v_2, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}$ and $\{v_1, v_2, v_3, v_4, v_5\}$. Here, $n_i(H) = 2$.

Definition 7. [4] A set $S \subseteq V(G)$ is an *independent neighborhood set* of G, if S is a neighborhood set and no two vertices in S are adjacent.

Remark 2. Not all graphs have independent neighborhood set. If it has an independent neighborhood set, then it is called an independent neighborhood graph or an IN-graph.

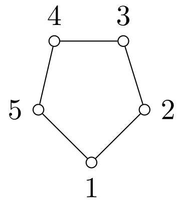


Figure 4: A graph that is not an IN-graph

Definition 8. [4] Let $G = (V, E)$ be a graph with m vertices. Then the *independent neighborhood polynomial* of G of order m is

$$N_i(G, x) = \sum_{j=\eta_i(G)}^{m} n_i(G, j)x^j,$$

where $n_i(G, j)$ is the number of independent neighborhood sets of G of cardinality j and $\eta_i(G)$ is the minimum cardinality of an independent neighborhood set which is called the *independent neighborhood number* of G.

Example 4. In Figure 3, the only independent neighborhood sets of H are $\{v_2, v_4\}$ and $\{v_1, v_3, v_5\}$. Therefore, the independent neighborhood polynomial of H is

$$N_i(H, x) = x^2 + x^3.$$

Proposition 1. [4] *Let $G = G_1 \cup G_2$ where G_1, G_2 be any two IN-graphs. Then $N_i(G, x) = N_i(G_1, x)N_i(G_2, x)$.*

Proposition 2. [4] *Let G be an IN-graph with $n+1$ vertices. Then $N_i(G, x) = x(1+x^{n-1})$ if and only if $G \cong k_{1,n}$.*

3. Main Results

3.1. Independent Neighborhood Polynomial of the Direct Product of Trees

Theorem 2. *For any trees G and H, the direct product of G and H, $G \cdot H$, is disconnected*

Proof. Label the vertices of G by x_i and of H by y_j. Since G and H are trees, by Corollary 1, G and H have independent neighborhood sets say Ω_1, Δ_1 and Ω_2, Δ_2, respectively. We now show that $G \cdot H$ is disconnected, that is, there exist $(x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2}) \in V(G \cdot H)$ such that (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_2}) are not connected by any path. Consider $(x_1, y_1), (x_2, y_2) \in V(G \cdot H)$ such that $x_1x_2 \notin E(G)$ or $y_1y_2 \notin E(H)$. Without loss of generality, let $x_{i_1}, x_{i_2} \in \Omega_1, y_{j_1} \in \Omega_2$ and $y_{j_2} \in \Delta_2$. Clearly, $(x_{i_1}, y_{j_1})(x_{i_2}, y_{j_2}) \notin E(G \cdot H)$ since $x_{i_1}, x_{i_2} \in \Omega_2$ which means x_{i_1} and x_{i_2} are nonadjacents. Observe that for any (x_i, y_i)-(x_n, y_n) path in $G \cdot H$, if $x_i \in \Omega_1, y_i \in \Omega_2$, then $x_{i+1} \in \Delta_1, y_{i+1} \in \Delta_2$ while if $x_i \in \Omega_1$ and $y_i \in \Delta_2$, then $x_{i+1} \in \Delta_1$ and $y_{i+1} \in \Omega_2$. Now, since $x_{i_1} \in \Omega_1, y_{j_1} \in \Omega_2$ and $y_{j_2} \in \Delta_2$ but $x_{i_2} \in \Delta_1$, it follows that there is no path connecting the vertices (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_2}) in $G \cdot H$. Therefore, $G \cdot H$ is disconnected.

Theorem 3. *For any trees G and H, the direct product of G and H has 2 components, say A and B, that is,*

$$G \cdot H = A \cup B.$$

Proof. Let G and H be any trees with independent neighborhood sets Ω_1, Δ_1 and Ω_2, Δ_2, respectively. Assume that $G \cdot H$ has more than 2 components, say $A_r, r \in \mathbb{Z}^+$. This means that for every $(u_r, v_r) \in A_r$, (u_r, v_r) are disconnected $\forall r$. We note that for any $(u, v) \in V(G \cdot H)$, either $u \in \Omega_1$ or $u \in \Delta_1$ and $v \in \Omega_2$ or $v \in \Delta_2$. Furthermore, for any (u_i, v_i)-(u_n, v_n) path in $G \cdot H$, $u_iu_{i+1} \in E(G)$ and $v_iv_{i+1} \in E(H)$. By Corollary 2, if $u_iu_{i+1} \in E(G)$ and $v_iv_{i+1} \in E(H)$, then $u_1 \in \Omega_1, u_{i+1} \in \Delta_1$ and $v_1 \in \Omega_2, v_{i+1} \in \Delta_2$. It follows that for (u_i, v_i)-(u_n, v_n) path in $G \cdot H$, if $u_i \in \Omega_1, v_i \in \Omega_2$ for i odd, then $u_j \in \Delta_1, v_j \in \Delta_2$ for j even while if $u_i \in \Omega_1, v_i \in \Delta_2$, then $u_j \in \Delta_1, v_j \in \Omega_2$. Now, consider

$$\begin{aligned}
(u_1, v_1) \in& A_1 \text{ such that } u_1 \in \Omega_1, v_1 \in \Omega_2,\\
(u_2, v_2) \in& A_2 \text{ such that } u_2 \in \Omega_1, v_2 \in \Delta_2,\\
(u_3, v_3) \in& A_3 \text{ such that } u_3 \in \Delta_1, v_3 \in \Omega_2,\\
(u_4, v_4) \in& A_4 \text{ such that } u_4 \in \Delta_1, v_4 \in \Delta_2.
\end{aligned}$$

We can easily see that (u_1, v_1) and (u_3, v_3) are disconnected, so are (u_1, v_1) and (u_2, v_2), (u_2, v_2) and (u_4, v_4), and (u_3, v_3) and (u_4, v_4). But observe that (u_1, v_1)-(u_4, v_4) is a path in $G \cdot H$. This implies (u_1, v_1) and (u_4, v_4) belong to 1 component. Similarly, (u_2, v_2) and (u_3, v_3) belong to 1 component. This is a contradiction. Thus, $G \cdot H$ cannot have more than 2 components. Therefore, $G \cdot H$ has exactly 2 components.

Theorem 4. *Suppose G has independent neighborhood sets Ω_1, Δ_1 and H has independent neighborhood sets Ω_2, Δ_2. If A and B are the components of $G \cdot H$, then*

$$V(A) = \{(x, y) : x \in \Omega_1, y \in \Omega_2\} \cup \{(w, z) : w \in \Delta_1, z \in \Delta_2\}$$

and

$$V(B) = \{(x, y) : x \in \Omega_1, y \in \Delta_2\} \cup \{(w, z) : w \in \Delta_1, z \in \Omega_2\}.$$

Proof. The proof is similar to the proof of Theorem 3.

Corollary 3. *Let A and B be the components of $G \cdot H$ such that*

$$V(A) = \{(x, y) : x \in \Omega_1, y \in \Omega_2\} \cup \{(w, z) : w \in \Delta_1, z \in \Delta_2\}$$

and

$$V(B) = \{(x, y) : x \in \Omega_1, y \in \Delta_2\} \cup \{(w, z) : w \in \Delta_1, z \in \Omega_2\}.$$

Then the independent neighborhood sets of A are the sets $\{(x, y) : x \in \Omega_1, y \in \Omega_2\}$ and $\{(w, z) : w \in \Delta_1, z \in \Delta_2\}$ while the independent neighborhood sets of B are the sets $\{(x, y) : x \in \Omega_1, y \in \Delta_2\}$ and $\{(w, z) : w \in \Delta_1, z \in \Omega_2\}$ where Ω_1, Δ_1 are the independent neighborhood sets of G and Ω_2, Δ_2 are the independent neighborhood sets of H.

Proof. Let A and B be the components of $G \cdot H$ with

$$V(A) = \{(x, y) : x \in \Omega_1, y \in \Omega_2\} \cup \{(w, z) : w \in \Delta_1, z \in \Delta_2\}$$

and

$$V(B) = \{(x, y) : x \in \Omega_1, y \in \Delta_2\} \cup \{(w, z) : w \in \Delta_1, z \in \Omega_2\}.$$

Suppose $A_1 = \{(x, y) : x \in \Omega_1, y \in \Omega_2\}, A_2 = \{(w, z) : w \in \Delta_1, z \in \Delta_2\}$, $B_1 = \{(x, y) : x \in \Omega_1, y \in \Delta_2\}$ and $B_2 = \{(w, z) : w \in \Delta_1, z \in \Omega_2\}$. We will show that A_1, A_2 are the independent neighborhood sets of A and B_1, B_2 are the independent neighborhood sets of B. Consider A_1. Let $(x_1, y_1), (x_2, y_2) \in V(A_1)$. Since $x_1, x_2 \in \Omega_1$, $y_1, y_2 \in \Omega_2$ where Ω_1 and Ω_2 are the independent neighborhood sets of G and H, respectively, implies x_1, x_2 are nonadjacent vertices, so are y_1, y_2. Thus, $(x_1, y_1)(x_2, y_2) \notin E(A)$. This means (x_1, y_1) and (x_2, y_2) are nonadjacent vertices. Next, we will show that $\bigcup\limits_{(x,y)\in A_1} \langle N[(x, y)]\rangle = A$. Assume to the contrary that $\bigcup\limits_{(x,y)\in A_1} \langle N[(x, y)]\rangle \neq A$. Then there exists $(x_3, y_3)(x_4, y_4) \in E(A)$ such that $(x_3, y_3)(x_4, y_4) \notin \bigcup\limits_{(x,y)\in A_1} \langle N[(x, y)]\rangle$. It follows that both $(x_3, y_3), (x_4, y_4) \notin V(A_1)$. Hence, $x_3, x_4 \in \Delta_1$ and $y_3, y_4 \in \Delta_2$. Since both Δ_1 and Δ_2 are independent neighborhood sets, x_3 and x_4 are nonadjacents, so are y_3 and y_4. Thus, $x_3x_4 \notin E(G)$ and $y_3y_4 \notin E(H)$. It follows that $(x_3, y_3)(x_4, y_4) \notin E(G\cdot H)$ which is a contradiction. Hence, $\bigcup\limits_{(x,y)\in A_1} \langle N[(x, y)]\rangle = A$. Therefore, A_1 is an independent neighborhood set of A. Similarly, we can show that A_2 is also an independent neighborhood set of A. Following the same argument for B_1 and B_2, we have B_1 and B_2 are the independent neighborhood sets of B.

Corollary 4. *Let G and H be trees with independent neighborhood sets Ω_1, Δ_1 and Ω_2, Δ_2, respectively. Then*

$$N_i(G \cdot H, x) = x^{|\Omega_1||\Omega_2|+|\Omega_1||\Delta_2|} + x^{|\Omega_1||\Omega_2|+|\Delta_1||\Omega_2|} + x^{|\Delta_1||\Delta_2|+|\Omega_1||\Delta_2|} + x^{|\Delta_1||\Delta_2|+|\Delta_1||\Omega_2|}.$$

Proof. Suppose G and H are trees with independent neighborhood sets Ω_1, Δ_1 and Ω_2, Δ_2, respectively. By Theorems 3 and 4, $G \cdot H = A \cup B$ where

$$V(A) = \{(x, y) : x \in \Omega_1, y \in \Omega_2\} \cup \{(w, z) : w \in \Delta_1, z \in \Delta_2\}$$

and

$$V(B) = \{(x, y) : x \in \Omega_1, y \in \Delta_2\} \cup \{(w, z) : w \in \Delta_1, z \in \Omega_2\}.$$

By Corollary 3, the sets

$$\{(x, y) : x \in \Omega_1, y \in \Omega_2\}$$

and

$$\{(w, z) : w \in \Delta_1, z \in \Delta_2\}$$

are the independent neighborhood sets of A while the sets

$$\{(x, y) : x \in \Omega_1, y \in \Delta_2\}$$

and

$$\{(w, z) : w \in \Delta_1, z \in \Omega_2\}$$

are the independent neighborhood sets of B. We note that

$$\begin{aligned}
|\{(x, y) : x \in \Omega_1, y \in \Omega_2\}| =& |\Omega_1||\Omega_2| \\
|\{(w, z) : w \in \Delta_1, z \in \Delta_2\}| =& |\Delta_1||\Delta_2| \\
|\{(x, y) : x \in \Omega_1, y \in \Delta_2\}| =& |\Omega_1||\Delta_2| \\
|\{(w, z) : w \in \Delta_1, z \in \Omega_2\}| =& |\Delta_1||\Omega_2|.
\end{aligned}$$

Thus,

$$N_i(A, x) = x^{|\Omega_1||\Omega_2|} + x^{|\Delta_1||\Delta_2|} \text{ and } N_i(B, x) = x^{|\Omega_1||\Delta_2|} + x^{|\Delta_1||\Omega_2|}.$$

Since $G \cdot H = A \cup B$, by Proposition 1,

$$N_i(G \cdot H, x) = N_i(A, x)N_i(B, x).$$

Therefore,

$$\begin{aligned}
N_i(G \cdot H, x) = & \left(x^{|\Omega_1||\Omega_2|} + x^{|\Delta_1||\Delta_2|}\right)\left(x^{|\Omega_1||\Delta_2|} + x^{|\Delta_1||\Omega_2|}\right) \\
=& x^{|\Omega_1||\Omega_2|}x^{|\Omega_1||\Delta_2|} + x^{|\Omega_1||\Omega_2|}x^{|\Delta_1||\Omega_2|} + x^{|\Delta_1||\Delta_2|}x^{|\Omega_1||\Delta_2|} + x^{|\Delta_1||\Delta_2|}x^{|\Delta_1||\Omega_2|} \\
=& x^{|\Omega_1||\Omega_2|+|\Omega_1||\Delta_2|} + x^{|\Omega_1||\Omega_2|+|\Delta_1||\Omega_2|} + x^{|\Delta_1||\Delta_2|+|\Omega_1||\Delta_2|} + x^{|\Delta_1||\Delta_2|+|\Delta_1||\Omega_2|}.
\end{aligned}$$

Example 5. Given the graphs G and H and their corresponding direct product in Figure 5.

We note that the independent neighborhood sets of G are $\Omega_1 = \{a, c, d\}$ and $\Delta_1 = \{b, e, f\}$ while the independent neighborhood sets of H are $\Omega_2 = \{1, 3, 4\}$ and $\Delta_2 =$

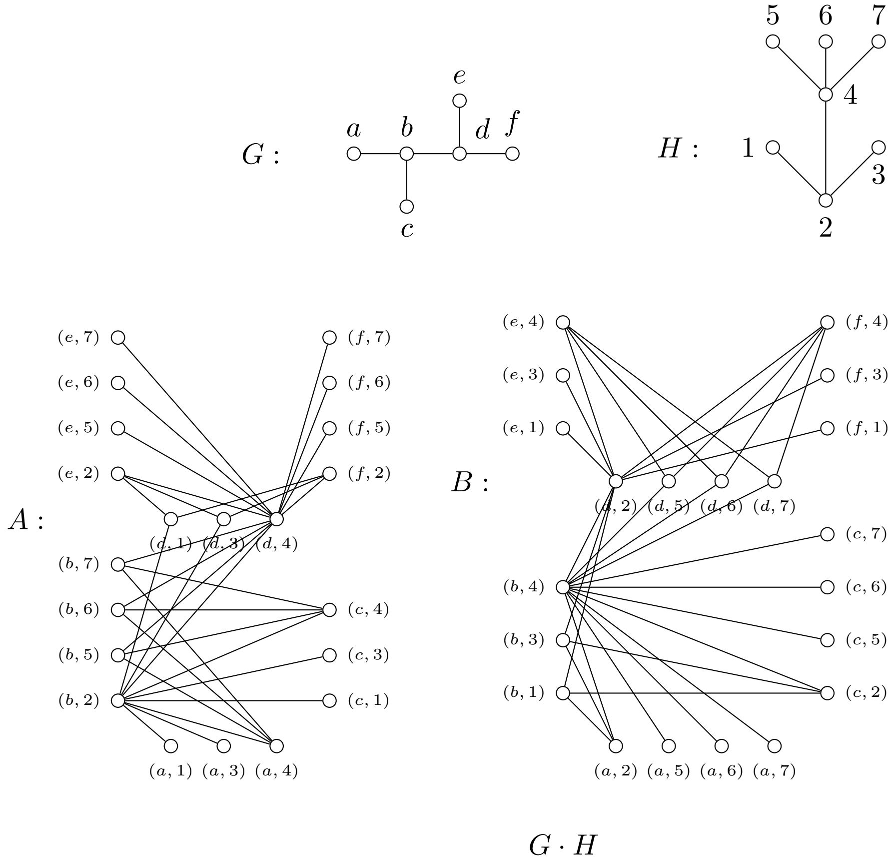


Figure 5: The graphs G and H and their direct product $G \cdot H$

$\{2, 5, 6, 7\}$. By Corollary 3, the sets

$$\{(x, y) : x \in \Omega_1, y \in \Omega_2\} = \{(a, 1), (a, 3), (a, 4), (c, 1), (c, 3), (c, 4), (d, 1), (d, 3), (d, 4)\}$$

and

$$\begin{aligned}\{(w, z) : w \in \Delta_1, z \in \Delta_2\} =& \{(b, 2), (b, 5), (b, 6), (b, 7), (e, 2), (e, 5), (e, 6), (e, 7), (f, 2), (f, 5),\\ & (f, 6), (f, 7)\}\end{aligned}$$

are the independent neighborhood sets of A while the sets

$$\begin{aligned}\{(x, y) : x \in \Omega_1, y \in \Delta_2\} =& \{(a, 2), (a, 5), (a, 6), (a, 7), (c, 2), (c, 5), (c, 6), (c, 7), (d, 2), (d, 5),\\ & (d, 6), (d, 7)\}\end{aligned}$$

and

$$\{(w, z) : w \in \Delta_1, z \in \Omega_2\} = \{(b, 1), (b, 3), (b, 4), (e, 1), (e, 3), (e, 4), (f, 1), (f, 3), (f, 4)\}$$

are the independent neighborhood sets of B. Furthermore,

$$|\Omega_1| = 3, |\Delta_1| = 3, |\Omega_2| = 3 \text{ and } |\Delta_2| = 4.$$

Therefore, by Corollary 4,

$$\begin{aligned} N_i(G \cdot H, x) =& x^{3(3)+3(4)} + x^{3(3)+3(3)} + x^{3(4)+3(4)} + x^{3(4)+3(3)} \\ =& x^{21} + x^{18} + x^{24} + x^{21} \\ =& x^{18} + 2x^{21} + x^{24}. \end{aligned}$$

Theorem 5. *For any graphs G and H, the direct product of G and H is commutative, that is,*

$$G \cdot H = H \cdot G.$$

Proof. Let G and H be any graphs. To show that the direct product of G and H is commutative, we will show that $G \cdot H$ and $H \cdot G$ are isomorphic. We note that $|V(G \cdot H)| = |V(G) \times V(H)| = |V(H) \times V(G)| = |V(H \cdot G)|$. Define the map $\beta : V(G \cdot H) \longrightarrow V(H \cdot G)$ by $\beta((g, h)) = (h, g)$ for all $(g, h) \in V(G \cdot H)$. For $(g_1, h_1), (g_2, h_2) \in V(G \cdot H)$ such that $\beta((g_1, h_1)) = \beta((g_2, h_2))$, we have

$$\begin{aligned} (h_1, g_1) = \beta((g_1, h_1)) = \beta((g_2, h_2)) = (h_2, g_2) \\ \Longrightarrow (h_1, g_1) = (h_2, g_2) \\ \Longrightarrow h_1 = h_2 \text{ and } g_1 = g_2. \end{aligned}$$

Thus, $(g_1, h_1) = (g_2, h_2)$ which shows β is one-to-one. Now, observe that for every $(h, g) \in V(H \cdot G)$, there exists $(g, h) \in V(G \cdot H)$ such that $\beta((g, h)) = (h, g)$, for all $(g, h) \in V(G \cdot H)$. This implies $\beta(V(G \cdot H)) = V(H \cdot G)$ and it follows that β is onto. Finally, we let $(g_1, h_1), (g_2, h_2) \in V(G \cdot H)$. Then $(g_1, h_1)(g_2, h_2) \in E(G \cdot H) \Leftrightarrow g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H) \Leftrightarrow (h_1, g_1)(h_2, g_2) \in E(H \cdot G)$. Hence, β preserves adjacency. Consequently, $G \cdot H \cong H \cdot G$. Therefore,

$$G \cdot H = H \cdot G.$$

Remark 3. For any trees G and H, the independent neighborhood polynomial of $G \cdot H$ is the same as the independent neighborhood polynomial of $H \cdot G$, that is

$$N_i(G \cdot H, x) = N_i(H \cdot G, x).$$

3.2. Independent Neighborhood Polynomial of the Corona Product of Two Trees

Theorem 6. *Let G and H be any trees. Suppose G has independent neighborhood sets Ω_1, Δ_1 and H has independent neighborhood sets Ω_2, Δ_2. Then*

$$N_i(G \circ H, x) = x^{|\Delta_1|} \left(x^{|\Omega_2|} + x^{|\Delta_2|} \right)^{|\Omega_1|} + x^{|\Omega_1|} \left(x^{|\Omega_2|} + x^{|\Delta_2|} \right)^{|\Delta_1|}.$$

Proof. Let G and H be any trees. Suppose G and H has independent neighborhood sets Ω_1, Δ_1 and Ω_2, Δ_2, respectively. Then by Remark 1,

$$N_i(G,x) = x^{|\Omega_1|} + x^{|\Delta_1|} \text{ and } N_i(H,x) = x^{|\Omega_2|} + x^{|\Delta_2|}.$$

By definition of corona of two graphs, every i^{th} vertex of G is joined to every vertex in the i^{th} copy of H. Observe that the independent neighborhood set Ω_1 of G and the independent neighborhood sets in each copy of H joined to every vertex in Δ_1 forms an independent neighborhood set in $G \circ H$. Similarly, the independent neighborhood set Δ_1 of G and the independent neighborhood sets in each copy of H joined to every vertex in Ω_1 is also an independent neighborhood set in $G \circ H$. Moreover, the number of combinations of the indepedent neighborhood sets of each copy of H joined to every $u \in \Delta_1$ is $|\Delta_1|$, that is, $N_i\left(\bigcup_{j\in\Delta_1} H_j, x\right) = \left(x^{|\Omega_2|} + x^{|\Delta_2|}\right)^{|\Delta_1|}$. Also, there are $|\Omega_1|$ combinations of independent neighborhood sets of each copy of H joined to each $v \in \Omega_1$, that is, $N_i\left(\bigcup_{r\in\Omega_1} H_r, x\right) = \left(x^{|\Omega_2|} + x^{|\Delta_2|}\right)^{|\Omega_1|}$. Hence,

$$N_i(G \circ H, x) = x^{|\Delta_1|}\left(x^{|\Omega_2|} + x^{|\Delta_2|}\right)^{|\Omega_1|} + x^{|\Omega_1|}\left(x^{|\Omega_2|} + x^{|\Delta_2|}\right)^{|\Delta_1|}.$$

Example 6. Consider the corona of two graphs $B(2,2)$ and $K_{1,3}$.

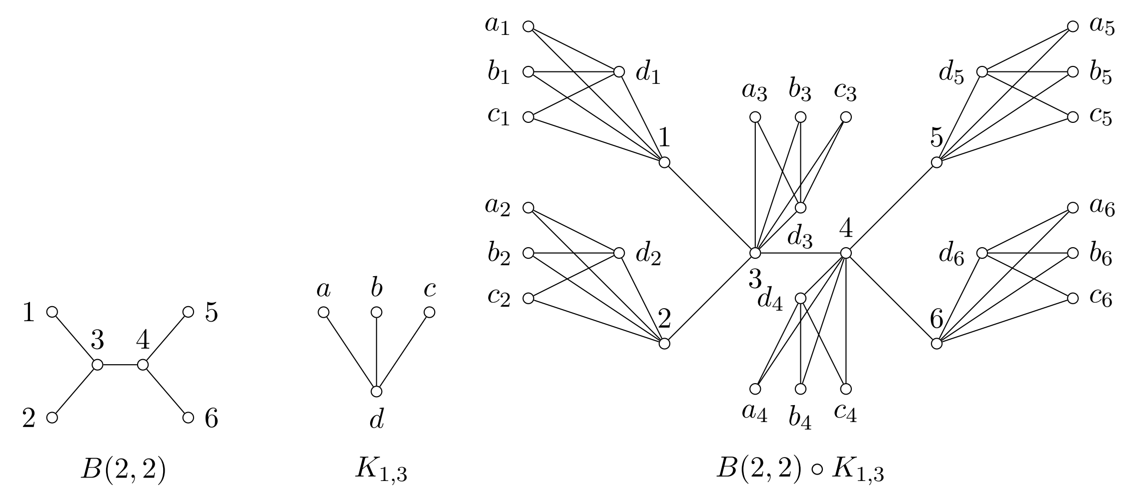
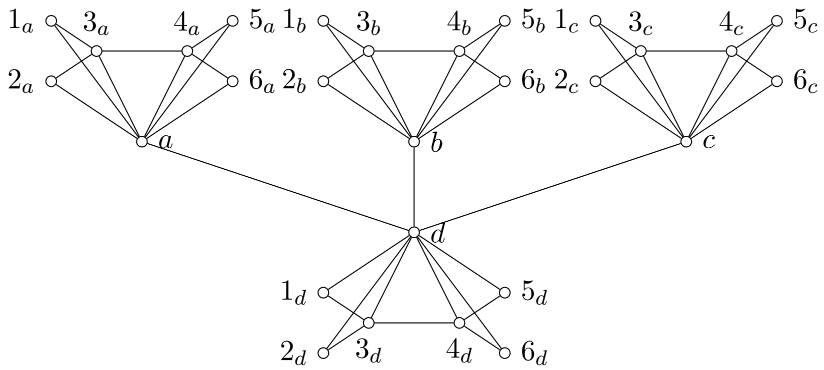


Figure 6: The graph $B(2,2), K_{1,3}$ and $B(2,2) \circ K_{1,3}$

Note that G has independent neighborhood sets $\Omega_1 = \{1, 2, 4\}, \Delta_1 = \{3, 5, 6\}$ and H has independent neighborhood sets $\Omega_2 = \{d\}$, $\Delta_2 = \{a, b, c\}$. We can see that the independent neighborhood sets of $G \circ H$ are $\{1, 2, 4, a_3, b_3, c_3, a_5, b_5, c_5, a_6, b_6, c_6\}$, $\{1, 2, 4, a_3, b_3, c_3, a_5, b_5, c_5, d_6\}$, $\{1, 2, 4, a_3, b_3, c_3, d_5, a_6, b_6, c_6\}$, $\{1, 2, 4, a_3, b_3, c_3, d_5, d_6\}$, $\{1, 2, 4, d_3, a_5, b_5, c_5, a_6, b_6, c_6\}$, $\{1, 2, 4, d_3, a_5, b_5, c_5, d_6\}$, $\{1, 2, 4, d_3, d_5, a_6, b_6, c_6\}$, $\{1, 2, 4, d_3, d_5, d_6\}$, $\{3, 5, 6, a_1, b_1, c_1, a_2, b_2, , c_2, a_4, b_4, c_4\}$, $\{3, 5, 6, a_1, b_1, c_1, a_2, b_2, , c_2, d_4\}$, $\{3, 5, 6, a_1, b_1, c_1, d_2, a_4, b_4, c_4\}$, $\{3, 5, 6, a_1, b_1, c_1, d_2, d_4\}$, $\{3, 5, 6, d_1, a_2, b_2, , c_2, a_4, b_4, c_4\}$, $\{3, 5, 6, d_1, d_2, a_4, b_4, c_4\}$, $\{3, 5, 6, d_1, a_2, b_2, , c_2, d_4\}$, $\{3, 5, 6, d_1, d_2, d_4\}$. Moreover, by Theorem 6,

$$\begin{aligned} N_i(G \circ H, x) &= x^{|\Delta_1|}\left(x^{|\Omega_2|} + x^{|\Delta_2|}\right)^{|\Omega_1|} + x^{|\Omega_1|}\left(x^{|\Omega_2|} + x^{|\Delta_2|}\right)^{|\Delta_1|} \\ &= x^3(x + x^3)^3 + x^3(x + x^3)^3 \\ &= 2x^3(x + x^3)^3 \\ &= 2x^3(x^3 + 3x^5 + 3x^7 + x^9) \\ &= 2x^6 + 6x^8 + 6x^{10} + 2x^{12}. \end{aligned}$$

On the other hand, the corona $H \circ G$ is given in Figure 7.



$K_{1,3} \circ B(2, 2)$

Figure 7: The graph $K_{1,3} \circ B(2, 2)$

We can see that the independent neighborhhood sets of $K_{1,3} \circ B(2, 2)$ are $\{a, b, c, 1_d, 2_d, 4_d\}$, $\{a, b, c, 3_d, 5_d, 6_d\}$, $\{d, 1_a, 2_a, 4_a, 1_b, 2_b, 4_b, 1_c, 2_c, 4_c\}$, $\{d, 1_a, 2_a, 4_a, 1_b, 2_b, 4_b, 3_c, 5_c, 6_c\}$, $\{d, 1_a, 2_a, 4_a, 3_b, 5_b, 6_b, 1_c, 2_c, 4_c\}$, $\{d, 1_a, 2_a, 4_a, 3_b, 5_b, 6_b, 3_c, 5_c, 6_c\}$, $\{d, 3_a, 5_a, 6_a, 1_b, 2_b, 4_b, 1_c, 2_c, 4_c\}$, $\{d, 3_a, 5_a, 6_a, 1_b, 2_b, 4_b, 3_c, 5_c, 6_c\}$, $\{d, 3_a, 5_a, 6_a, 3_b, 5_b, 6_b, 1_c, 2_c, 4_c\}$, $\{d, 3_a, 5_a, 6_a, 3_b, 5_b, 6_b, 3_c, 5_c, 6_c\}$. By Theorem 6,

$$N_i(K_{1,3} \circ B(2, 2)) = x^3(x^3 + x^3) + x(x^3 + x^3)^3$$

$$\begin{aligned} &= x^3(2x^3) + x(2x^3)^3 \\ &= 2x^6 + 8x^{10}. \end{aligned}$$

Theorem 7. *For any trees G and H, if $G \cong H$, then*

$$N_i(G \circ H, x) = N_i(H \circ G, x).$$

Proof. Let G and H be any trees. Suppose $G \cong H$. Then G and H has indepedent neighborhood sets Ω_1, Δ_1 and Ω_2, Δ_2, respectively. Since $G \cong H$, either $|\Omega_1| = |\Omega_2|$ and $|\Delta_1| = |\Delta_2|$ or $|\Omega_1| = |\Delta_2|$ and $|\Delta_1| = |\Omega_2|$. Without loss of generality, assume $|\Omega_1| = |\Omega_2|$ and $|\Delta_1| = |\Delta_2|$. Thus,

$$\begin{aligned} N_i(G \circ H, x) &= x^{|\Delta_1|}\left(x^{|\Omega_2|} + x^{|\Delta_2|}\right)^{|\Omega_1|} + x^{|\Omega_1|}\left(x^{|\Omega_2|} + x^{|\Delta_2|}\right)^{|\Delta_1|} \\ &= x^{|\Delta_2|}\left(x^{|\Omega_1|} + x^{|\Delta_1|}\right)^{|\Omega_2|} + x^{|\Omega_2|}\left(x^{|\Omega_1|} + x^{|\Delta_1|}\right)^{|\Delta_2|} \\ &= N_i(H \circ G, x). \end{aligned}$$

Example 7. Consider the corona of $K_{1,4}$ to itself as shown in Figure 8.

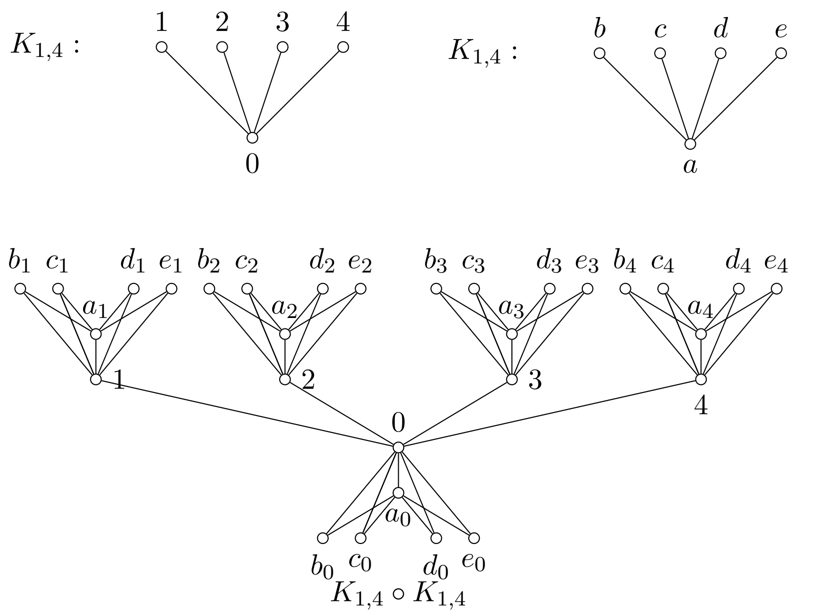


Figure 8: The graph $K_{1,4} \circ K_{1,4}$

By Proposition 2, $K_{1,4}$ has independent neighborhood sets of cardinalities 1 and 4, respectively. This implies $|\Omega_1| = 1, |\Delta_1| = 4, |\Omega_2| = 1$, and $|\Delta_2| = 4$. Therefore,

$$\begin{aligned} N_i(K_{1,4} \circ K_{1,4}, x) &= x^1(x + x^4)^4 + x^4(x + x^4)^1 \\ &= x(x^4 + 4x^3 \cdot x^4 + 6x^2 \cdot x^8 + 4x \cdot x^{12} + x^{16}) + x^5 + x^8 \\ &= 2x^5 + 5x^8 + 6x^{11} + 4x^{14} + x^{17}. \end{aligned}$$

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