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An Analytical Study of Two-Dimensional Bell Polynomials and Their Properties

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Abstract. This study investigates two-dimensional Bell polynomials, emphasizing their fundamental properties and applications in mathematical analysis. Utilising the framework of generating functions, we derive explicit representations, summation formulae, recurrence relations, and addition formulas for these polynomials. Furthermore, we introduce the $2D$ Bell-based Stirling polynomials of the second kind and explore their associated properties. This research aims to enhance the theoretical understanding of Bell polynomials and their broader applications in mathematical analysis.

2020 Mathematics Subject Classifications: 33E20, 33C45, 33B10, 33E30, 11T23

Key Words and Phrases: $2D$ Special Polynomials, Generating function, Explicit form, Series representation

1. Introduction and preliminaries

A fascinating class of mathematical functions, namely special polynomials, are characterized by unique properties and find specific significance in various mathematical contexts, for example [1–5]. These polynomials encompass well-known families such as Legendre polynomials, Chebyshev polynomials, Hermite polynomials, Bell polynomials, and Touchard polynomials. Legendre polynomials, for example, arise in problems involving

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electrostatics and fluid dynamics, while Chebyshev polynomials have applications in numerical analysis and signal processing. On the other hand, Hermite polynomials frequently emerge in quantum mechanics and probability theory. Bell and Touchard polynomials also play important roles in combinatorics and number theory. Studying these special polynomials and their applications is essential in mathematical physics, engineering, computer science, and other scientific disciplines. On the topic of polynomial families and their various extensions, much research has appeared in the literature (see, for example, [8–10, 13]).

The exploration of special polynomials in general cases has revealed new properties and applications, greatly expanding our mathematical understanding of these polynomials. Mathematicians have discovered unique characteristics, relationships, and applications that were previously unknown, enriching the field of mathematics. The study of special polynomials is crucial due to their frequent appearance in solving "differential equations, orthogonal polynomial theory, numerical analysis, and various other mathematical and computational problems". These polynomials exhibit specific algebraic structures and recurrence relations, which make them particularly amenable to analysis, enhancing the broader study of algebra and mathematical structures. Additionally, special polynomials have profound connections to other areas of mathematics, such as "combinatorics, number theory, and analysis". These interconnections promote interdisciplinary research and the advancement of mathematical theories, leading to broader implications across scientific and applied fields. Consequently, the ongoing research and discoveries in the properties of special polynomials continue to play a vital role in advancing both pure and applied mathematics.

One of the most intriguing and significant classes of polynomial sequences and numbers is the Stirling numbers. These numbers form a family that is essential in combinatorics, particularly in problems related to permutations, combinations, and partitions, for example, [6, 7, 14–19]. There are two primary types of Stirling numbers, each serving a distinct purpose in combinatorial mathematics: the Stirling numbers of the first kind, denoted as $S_1(n, \epsilon)$, and the Stirling numbers of the second kind, denoted as $S_2(n, \epsilon)$. The Stirling numbers of the first kind, $S_1(n, \epsilon)$, represent the number of permutations of n elements that contain exactly ϵ cycles. This means they count the number of ways to arrange n distinct elements into ϵ cyclic groups. On the other hand, the Stirling numbers of the second kind, $S_2(n, \epsilon)$, quantify the number of ways to partition a set of n distinct elements into ϵ non-empty, indistinguishable subsets, where the order of these subsets does not matter. These numbers are significant in various mathematical and applied fields, including combinatorics, where they facilitate the understanding and solving problems related to permutations and partitions. Their utility extends to areas such as algebra, probability, and the analysis of algorithms, underscoring their importance in both theoretical and practical applications of mathematics. Stirling numbers are widely used in combinatorics for counting permutations, combinations, and partitions.

"Stirling Polynomials of the Second Kind", denoted as $S_2(n, \epsilon; q_1)$, are associated with exponential generating functions. The exponential generating function for Stirling polynomials of the second kind is given by:

$$\sum_{n=0}^{\infty} \frac{S_2(n,\epsilon;q_1)\ \xi^n}{n!} = \frac{(e^{\xi q_1}-1)^\epsilon}{\epsilon!}. \tag{1}$$

Certainly, the exponential generating function for Stirling numbers of the second kind simplifies to the following expression when $q_1 = 1$:

$$\sum_{n=0}^{\infty} \frac{S_2(n,\epsilon)\ \xi^n}{n!} = \frac{(e^{\xi}-1)^\epsilon}{\epsilon!}. \tag{2}$$

Further, the recurrence relation for Stirling numbers of the second Kind $S_2(n,\epsilon)$ can be computed using the recurrence relation:

$$q_1^n = \sum_{n=0}^{\infty} S_2(n,\epsilon)\ (q_1)_\epsilon \tag{3}$$

or

$$(q_1)_n = \sum_{\epsilon=0}^{n} S_2(n,\epsilon)\ q_1^\epsilon, \tag{4}$$

where, the falling factorial is given by $(q_1)_\epsilon = q_1(q_1-1)(q_2-2)\cdots(q_1-(\epsilon-1))$.

Additionally, for every non-negative integer ϵ in the set of natural numbers, the following expression holds true:

$$S_\epsilon(n) = \sum_{l=0}^{n} l^\epsilon.$$

The sum of integer powers is referred to as the "sum of integer powers", and the exponential generating function for $S_\epsilon(n)$ is as follows:

$$\sum_{\epsilon=0}^{\infty} S_\epsilon(n)\ \frac{\xi^\epsilon}{\epsilon!} = \frac{e^{(n+1)\xi-1}}{e^\xi-1}. \tag{5}$$

The concepts mentioned are crucial in combinatorics and are widely used to solve counting problems involving the organisation of distinguishable objects into partitions and subsets.

The incredible power of exponential operators shines through, especially when solving differential equations. These operators simplify the analysis and offer a convenient way to express solutions. Bell's groundbreaking work (Bell, 2010) presents a comprehensive exploration of the foundational formalism. It brilliantly demonstrates that by making a suitable change of variable, the effect of the operator on a given function of q_1 can be viewed as that of a traditional shift operator. In other words, for any parameter μ, applying the shift operator $\exp(\mu\partial_{q_1})$ to any function of q_1 yields the following remarkable result:

$$\exp(\mu\partial_{q_1})\{f(q_1)\} = \sum_{n=0}^{\infty} \partial_{q_1}^n f(q_1)\frac{\mu^n}{n!} = \sum_{n=0}^{\infty} f^n(q_1)\frac{\mu^n}{n!} = f(q_1+\mu), \tag{6}$$

where $\partial_{q_1}^n = \frac{\partial^n}{\partial q_1^n}$. This result illustrates that the operator $\exp(\mu\partial_{q_1})$ effectively shifts the argument of the function f by μ, simplifying the manipulation and solution of differential equations by transforming them into algebraic problems.

The following identities are exploited from (6):

$$\exp(\mu\ q_1^2\partial_{q_1})\{f(q_1)\} = f\Big(\frac{q_1}{1-\mu q_1}\Big), \tag{7}$$

$$\exp(\mu\partial_{q_1})\{q_1^n\} = \Big(q_1+\mu\Big)^n,$$

$$\exp(\mu\partial_{q_1}^n)\{e^{q_1}\} = e^{q_1+\mu},$$

$$\exp(\mu q_1\ \partial_{q_1})f\{q_1\} = f(e^{q_1}\mu).$$

One important class of special polynomials is the Bell polynomials [1], named after mathematician Eric Temple Bell. Bell polynomials play a crucial role in representing the partial Bell polynomials, which correspond to the partial derivatives of the exponential generating function. These polynomials have wide-ranging applications in fields such as combinatorics, probability theory, and the analysis of algorithms. Bell polynomials are particularly valuable for counting and enumerating various combinatorial structures in combinatorics. They describe partitions of sets, which involve dividing a set into non-overlapping subsets, and compositions of integers, where an integer is expressed as the sum of ordered integers. The utility of Bell polynomials extends to the analysis of algorithms, where they help in understanding the performance and behaviour of combinatorial algorithms. Notable works that explore these applications include references [11, 12, 14–19], which delve into the diverse and significant uses of Bell polynomials in these mathematical domains. Through these applications, Bell polynomials demonstrate their versatility and importance in solving complex problems and providing insights across various areas of mathematical research. These polynomials are a sequence that arises in combinatorics and is denoted as $\mathscr{B}_n^{[j]}(q_1)$. The following exponential generating function defines them:

$$\sum_{n=0}^{\infty} \mathscr{B}_n^{[j]}(q_1)\frac{\xi^n}{n!} = e^{q_1(e^{\xi}-1)}. \tag{8}$$

For the case where $q_1 = 1$, the Bell polynomials simplify to the Bell numbers according to the following relation:

$$\sum_{n=0}^{\infty} \mathscr{B}_n^{[j]}\frac{\xi^n}{n!} = e^{e^{\xi}-1}.$$

Bell polynomials are an incredibly versatile and powerful framework that offers deep insights into combinatorial structures, particularly in partitioning, generating functions, and algebraic combinatorics. Their significance in mathematics cannot be overstated, as they provide an effective tool for counting problems and analyzing discrete structures.

Here, we have made a significant advancement in the field by developing a new formulation for the generating relation of $2D$ Bell polynomials given by the generating expression:

$$\sum_{n=0}^{\infty} \mathscr{B}_n^{[j]}(q_1, q_2)\frac{\xi^n}{n!} = e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}. \tag{9}$$

For, $q_2 = 0$ in (9), the $2D$ Bell polynomials $\mathscr{B}_n^{[j]}(q_1, q_2)$ reduce to the Bell polynomials given by (8). In the upcoming sections, we will explore these mathematical marvels, uncovering their intricate properties and unveiling their remarkable applications. In Section 2, we dive headfirst into the realm of generating functions, where we introduce $2D$ Bell polynomials. Further, we derive explicit representations and unveil summation formulae, recurrence relations, and addition formulas, all while shedding light on their profound connection to Stirling polynomials of the second kind. In Section 3, we delve deeper into these polynomials' matrix form and product formula, unveiling their structural insights into their inner workings. In Section 4, where we introduce the $2D$ Bell-based Stirling polynomials of the second kind, expanding the horizons of our understanding even further. The conclusion is provided last.

For $j = 3$, the first five $2D$ Bell polynomials are as follows:

$$\begin{aligned}
\mathscr{B}_0^{[3]}(q_1, q_2) &= 1,\\
\mathscr{B}_1^{[3]}(q_1, q_2) &= q_1,\\
\mathscr{B}_2^{[3]}(q_1, q_2) &= q_1^2 + q_1,\\
\mathscr{B}_3^{[3]}(q_1, q_2) &= q_1^3 + 3q_1^2 + q_1 + 6q_2,\\
\mathscr{B}_4^{[j]}(q_1, q_2) &= q_1^4 + 6q_1^3 + 7q_1^2 + q_1 + 36q_2 + 24q_1q_2.
\end{aligned}$$

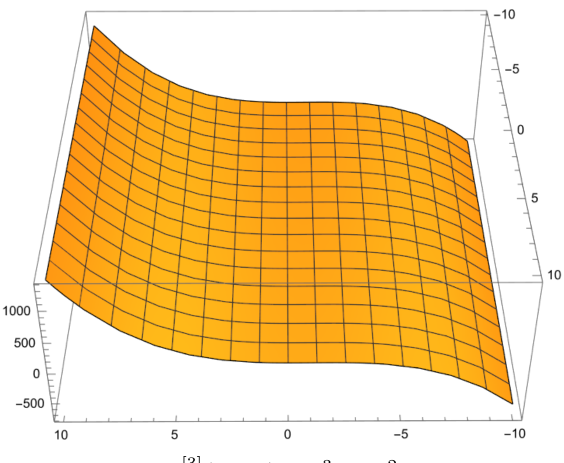

Figure 1: $\mathscr{B}_3^{[3]}(q_1, q_2) = q_1^3 + 3q_1^2 + q_1 + 6q_2$

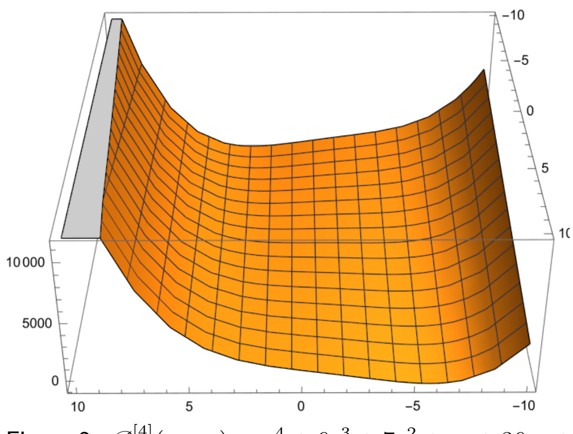

Figure 2: $\mathscr{B}_3^{[4]}(q_1, q_2) = q_1^4 + 6q_1^3 + 7q_1^2 + q_1 + 36q_2 + 24q_1q_2$

2. 2D Bell polynomials

The 2D special Bell polynomials play a crucial role in combinatorial analysis, number theory, and statistical mechanics. They express multivariate exponential generating functions and find applications in studying combinatorial structures and problems in discrete mathematics. Additionally, they are used in number theory to investigate properties of partitions, compositions, and other combinatorial objects. In statistical mechanics, these polynomials are employed to analyze the behavior of systems with two-dimensional degrees of freedom, providing insights into the thermodynamic properties and phase transitions of physical systems. Furthermore, in applied mathematics and engineering, 2D special Bell polynomials are utilized in problems involving complex systems with multiple variables, offering a systematic framework for modelling and analysis. Overall, the significance of 2D special Bell polynomials lies in their ability to bridge theoretical concepts with practical applications across diverse disciplines, making them invaluable tools for mathematical research and problem-solving in various fields.

Here, in this section, we derive the explicit forms and certain other properties of 2D Bell polynomials denoted by $\mathscr{B}_n^{[j]}(q_1, q_2)$ as follows:

Theorem 1. *The 2D Bell polynomials denoted by* $\mathscr{B}_n^{[j]}(q_1, q_2)$ *satisfy the listed explicit form:*

$$\mathscr{B}_n^{[j]}(q_1, q_2) = \sum_{s=0}^{[n]} \binom{n}{s} \mathscr{B}_{n-s}^{[j]}(q_1) \frac{S_2(s,k)\ (e^{\xi}-1)^{jk-k}}{1-q_2}. \tag{10}$$

Proof. The expression denoted by (9) can be expressed in view of the identity $e^{A+B} = e^A e^B$, as

$$e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j} = e^{q_1(e^{\xi}-1)} e^{q_2(e^{\xi}-1)^j} = \left(\sum_{n=0}^{\infty} \mathscr{B}_n^{[j]}(q_1) \frac{\xi^n}{n!}\right) \left(\sum_{r=0}^{\infty} \frac{q_2^r}{r!}(e^{\xi}-1)^{jr}\right) = \sum_{n=0}^{\infty} \mathscr{B}_n^{[j]}(q_1, q_2) \frac{\xi^n}{n!}.$$

Expand the second exponential using Stirling numbers given by expression (2), and $(e^{\xi} - 1)^{jr}$ can be written using a convolution involving Stirling numbers: $(e^{\xi}-1)^{jr} = \sum_{s=0}^{\infty} S_2(s,k)\ (e^{\xi}-1)^{jk-k}\ \frac{\xi^s}{s!}$, , it follows that

$$e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j} = \sum_{n=0}^{\infty} \mathscr{B}_n^{[j]}(q_1, q_2) \frac{\xi^n}{n!} = \left(\sum_{n=0}^{\infty} \mathscr{B}_n^{[j]}(q_1) \frac{\xi^n}{n!}\right) \left(\sum_{s=0}^{\infty} \frac{S_2(s,k)\ (e^{\xi}-1)^{jk-k}}{1-q_2} \frac{\xi^s}{s!}\right).$$

Applying the Cauchy product for power series:

$$\left(\sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!}\right) \left(\sum_{s=0}^{\infty} b_s \frac{\xi^s}{s!}\right) = \sum_{n=0}^{\infty} \left(\sum_{s=0}^{n} \binom{n}{s} a_{n-s} b_s\right) \frac{\xi^n}{n!}.$$

in preceding expression, it follows that

$$\sum_{n=0}^{\infty} \mathscr{B}_n^{[j]}(q_1, q_2) \frac{\xi^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{s=0}^{n} \binom{n}{s} \mathscr{B}_{n-s}^{[j]}(q_1) \frac{S_2(s,k)\ (e^{\xi}-1)^{jk-k}}{1-q_2}\right) \frac{\xi^n}{n!}.$$

By comparing the coefficients of $\frac{\xi^n}{n!}$ on both sides of the original generating function, we conclude:

$$\mathscr{B}_n^{[j]}(q_1,q_2)=\sum_{s=0}^{n}\binom{n}{s}\mathscr{B}_{n-s}^{[j]}(q_1)\frac{S_2(s,k)\ (e^{\xi}-1)^{jk-k}}{1-q_2}.$$

This completes the proof.

Theorem 2. *The Bell polynomials $\mathscr{B}_n^{[j]}(q_1,q_2)$, have a series representation as listed below:*

$$\mathscr{B}_n^{[j]}(q_1,q_2)=\sum_{s=0}^{[n]}\binom{n}{s}\ \frac{S_2(n-s,l)}{1-q_1}\frac{S_2(s,k)\ (e^{\xi}-1)^{jk-k}}{1-q_2}. \tag{11}$$

Proof. The expression denoted by (9) can be expressed in the form

$$e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}=e^{q_1(e^{\xi}-1)}e^{q_2(e^{\xi}-1)^j}.$$

Using the expression (1)–(4) into the right-hand side of the previous expression, we determine

$$e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}=\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}{q_1}^lS_2(n,l)\frac{\xi^n}{n!}(e^{\xi}-1)^{jk-k}n!\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}{q_2}^rS_2(s,k)\frac{\xi^s}{s!}(e^{\xi}-1)^{jk-k}.$$

Placing the right-hand side of the equation (9) into the left-hand side of the preceding expression and then simplifying the right-hand side, we can conclude that

$$\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1,q_2)\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\frac{S_2(n,l)}{1-q_1}\frac{S_2(s,k)\ (e^{\xi}-1)^{jk-k}}{1-q_2}\frac{\xi^{n+s}}{n!\ s!}.$$

Rearranging the series, we can substitute $n-s$ for n into the right-hand side of the preceding expression and then simplifying the right-hand side, we can conclude that

$$\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1,q_2)\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}\sum_{s=0}^{[n}\binom{n}{s}\frac{S_2(n-s,l)}{1-q_1}\frac{S_2(s,k)(e^{\xi}-1)^{jk-k}}{1-q_2}\frac{\xi^n}{n!}.$$

While comparing the similar abilities of $\frac{\xi^n}{n!}$ in the preceding statement, we arrive at statement (11).

Theorem 3. *The 2D Bell polynomials denoted by $\mathscr{B}_n^{[j]}(q_1,q_2)$. Then the following summation formulas hold.*

$$\mathscr{B}_n^{[j]}(q_1+q_3,q_2)\ =\ \sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k}^{[j]}(q_1,q_3)\mathscr{B}_k^{[j]}(q_2).$$

Proof. By (9) and (8), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1+q_3,q_2)\frac{\xi^n}{n!} &= e^{(q_1+q_2)(e^{\xi}-1)+q_3(e^{\xi}-1)^j}\\ &= \sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1,q_3)\frac{\xi^n}{n!}\sum_{k=0}^{\infty}\mathscr{B}_k^{[j]}(q_2)\frac{\xi^k}{k!}\\ &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k}^{[j]}(q_1,q_3)\mathscr{B}_k^{[j]}(q_2)\right]\frac{\xi^n}{n!}.\end{aligned}$$

Finally equating the coefficients of $\frac{\xi^n}{n!}$ of both sides, we get the asserted Theorem 8.

Theorem 4. *For any arbitrary $n \in \mathbb{N}$, the following relation hold true:*

$$\mathscr{B}_n^{[j]}(q_1+1,q_2)-\mathscr{B}_n^{[j]}(q_1,q_2)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k}^{[j]}(q_1,q_2)\mathscr{B}_k^{[j]}-\mathscr{B}_n^{[j]}(q_1,q_2). \tag{12}$$

Proof. Utilizing the expression (9), we find

$$\begin{aligned}\sum_{n=0}^{\infty}\left[\mathscr{B}_n^{[j]}(q_1+1,q_2)-\mathscr{B}_n^{[j]}(q_1,q_2)\right]\frac{\xi^n}{n!} &= e^{(q_1+1)(e^{\xi}-1)+q_2(e^{\xi}-1)^j}-e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}\\ &= e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}\left[e^{e^{\xi}-1}-1\right]\\ &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k}^{[j]}(q_1,q_2)B_n(q_1,q_2)B_k\right]\frac{\xi^n}{n!}.\end{aligned}$$

By equating both sides, we obtained the result (12).

Theorem 5. *For $n \geq 1$, let $\mathscr{B}_n^{[j]}(q_1,q_2)$ be the 2D Bell polynomials. Then we have*

$$\frac{\partial}{\partial q_1}\mathscr{B}_n^{[j]}(q_1,q_2)=\frac{1}{(n^2+n)}\sum_{k=0}^{n}\binom{n+1}{k}\mathscr{B}_k^{[j]}(q_1,q_2) \tag{13}$$

and

$$\frac{\partial}{\partial q_2}\mathscr{B}_n^{[j]}(q_1,q_2)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k}^{[j]}(q_1,q_2)j!S_2(k,j). \tag{14}$$

Proof. (See (13)). Differentiating partially with respect to the variable q_1 on both sides of the generating function:

$$\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1,q_2)\frac{\xi^n}{n!}=e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j},$$

we get

$$\frac{\partial}{\partial q_1}\left[\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1,q_2)\frac{\xi^n}{n!}\right] = \frac{\partial}{\partial q_1}\left[e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}\right]$$
$$= e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}(e^{\xi}-1).$$

Using the generating function of $\mathscr{B}_n^{[j]}(q_1,q_2)$, we substitute:

$$e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}(e^{\xi}-1) = \left[\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1,q_2)\frac{\xi^n}{n!}\right]\left[\sum_{r=1}^{\infty}\frac{\xi^r}{r!}\right]$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\mathscr{B}_k^{[j]}(q_1,q_2)\frac{\xi^{n+1}}{(n+1)!}.$$

Equating the coefficients of $\frac{\xi^n}{n!}$ on both sides gives:

$$\frac{\partial}{\partial q_1}\mathscr{B}_n^{[j]}(q_1,q_2) = \frac{1}{(n^2+n)}\sum_{k=0}^{n}\binom{n+1}{k}\mathscr{B}_k^{[j]}(q_1,q_2).$$

Proof. (See (14)). Differentiating partially with respect to the variable q_2 on both sides of the generating function:

$$\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1,q_2)\frac{\xi^n}{n!} = e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j},$$

we get

$$\frac{\partial}{\partial q_2}\left[\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1,q_2)\frac{\xi^n}{n!}\right] = \frac{\partial}{\partial q_2}\left[e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}\right]$$
$$= e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}(e^{\xi}-1)^j.$$

Recall that the exponential generating function of $j!S_2(n,j)$ is:

$$(e^{\xi}-1)^j = \sum_{n=0}^{\infty} j!S_2(n,j)\frac{\xi^n}{n!}.$$

So, we compute:

$$e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}(e^{\xi}-1)^j = \left[\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1,q_2)\frac{\xi^n}{n!}\right]\left[\sum_{m=0}^{\infty} j!S_2(m,j)\frac{\xi^m}{m!}\right]$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k}^{[j]}(q_1,q_2)j!S_2(k,j)\frac{\xi^n}{n!}.$$

Equating coefficients of $\frac{\xi^n}{n!}$ yields:

$$\frac{\partial}{\partial q_2}\mathscr{B}_n^{[j]}(q_1,q_2)=\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k}^{[j]}(q_1,q_2)j!S_2(k,j).$$

Theorem 6. *For $n\geq 0$, let $\left\{\mathscr{B}_n^{[j]}(q_1,q_2)\right\}_{n\geq 0}$ be the sequences of 2D Bell polynomials in the variable q_1,q_2 and q_3, they satisfy the following relation*

$$\sum_{k=0}^{n}\binom{n}{k}\left[\mathscr{B}_k^{[j]}(q_1+q_3,q_2)\mathscr{B}_{n-k}^{[j]}(2q_2)-\mathscr{B}_{n-k}^{[j]}(q_1,q_2)\mathscr{B}_n^{[j]}(q_3,q_2)\right]=0.$$

Proof. Let's consider the following expressions

$$e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j} = \sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1,q_2)\frac{\xi^n}{n!} \tag{15}$$

and

$$e^{q_3(e^{\xi}-1)+q_2(e^{\xi}-1)^j} = \sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_3,q_2)\frac{\xi^n}{n!}. \tag{16}$$

From (15) and (16), we have

$$\begin{aligned}
e^{(q_1+q_3)(e^{\xi}-1)+q_2(e^{\xi}-1)^j}e^{2q_2(e^{\xi}-1)^j} &= \left(\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1,q_2)\frac{\xi^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_3,q_2)\frac{\xi^n}{n!}\right)\\
\left(\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1+q_3,q_2)\frac{\xi^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(0,2q_2)\frac{\xi^n}{n!}\right) &= \left(\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1,q_2)\frac{\xi^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_3,q_2)\frac{\xi^n}{n!}\right)\\
\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_k^{[j]}(q_1+q_3,q_2)\mathscr{B}_{n-k}^{[j]}(2q_2)\frac{\xi^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k}^{[j]}(q_1,q_2)\mathscr{B}_n^{[j]}(q_3,q_2)\frac{\xi^n}{n!}\\
\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_k^{[j]}(q_1+q_3,q_2)\mathscr{B}_{n-k}^{[j]}(2q_2) &= \sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_{n-k}^{[j]}(q_1,q_2)\mathscr{B}_n^{[j]}(q_3,q_2).
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{n}\binom{n}{k}\left[\mathscr{B}_k^{[j]}(q_1+q_3,q_2)\mathscr{B}_{n-k}^{[j]}(2q_2)-\mathscr{B}_{n-k}^{[j]}(q_1,q_2)\mathscr{B}_n^{[j]}(q_3,q_2)\right]=0.$$

3. The 2D Bell-based Stirling polynomials of the second kind

Definition 1.

$$\sum_{n=0}^{\infty} {}_{\mathscr{B}}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2)\frac{\xi^n}{n!} = \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\ e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}. \tag{17}$$

This description serves as a foundational concept, establishing the basis for additional exploration and comprehension of these polynomials' important implications and possible applications in the broader realm of mathematics.

For $\epsilon, j = 2$, the first four two-variable Bell-based Stirling polynomials are as follows:

$$\begin{aligned}
{}_{\mathscr{B}}\mathcal{S}_2^{[2]}(0,2;q_1,q_2) &= \frac{1}{2},\\
{}_{\mathscr{B}}\mathcal{S}_2^{[2]}(1,2;q_1,q_2) &= \frac{1}{2}q_1 + \frac{1}{2},\\
{}_{\mathscr{B}}\mathcal{S}_2^{[2]}(2,2;q_1,q_2) &= \frac{1}{2}q_1^2 + \frac{3}{2}q_1 + q_2 + \frac{7}{12},\\
{}_{\mathscr{B}}\mathcal{S}_2^{[2]}(3,2;q_1,q_2) &= \frac{1}{2}q_1^3 + 3q_1^2 + 3q_1q_2 + 6q_2 + \frac{3}{4}.
\end{aligned}$$

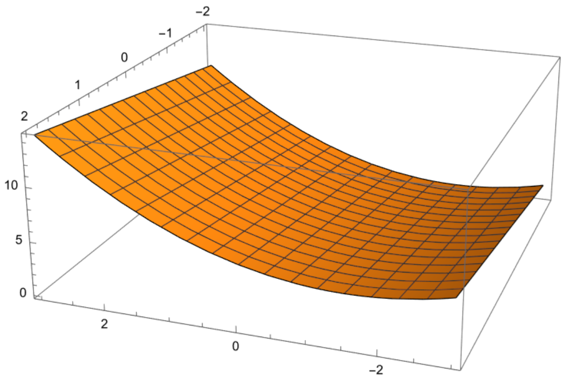

Figure 3: $\frac{1}{2}q_1^2 + \frac{3}{2}q_1 + q_2 + \frac{7}{12}$

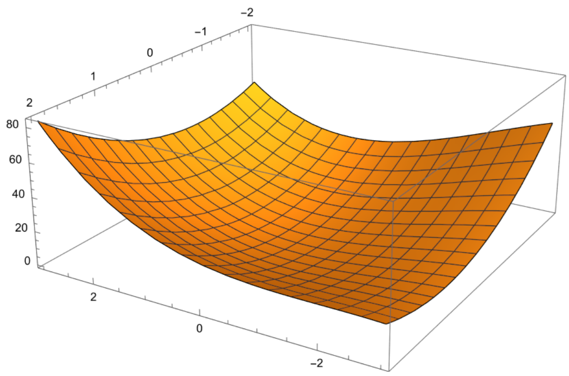

Figure 4: $\frac{1}{2}q_1^3 + 3q_1^2 + 3q_1q_2 + 6q_2 + \frac{3}{4}$

Remark 1. *The expression given by* (17) *yields a set of polynomials called the Bell-Stirling polynomials of the second kind when we substitute* $q_2 = 0$*. This set of polynomials is expressed as:*

$$\sum_{n=0}^{\infty} {}_{\mathscr{B}}\mathcal{S}_2(n,\epsilon;q_1)\frac{\xi^n}{n!} = \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\ e^{q_1(e^{\xi}-1)}.$$

Remark 2. *After substituting* $q_1 = q_2 = 0$ *into the expression from* (17)*, a group of polynomials called the Stirling numbers of the second kind, as shown in* (2)*, is derived.*

Theorem 7. *For any non-negative integer n, the Stirling polynomials of the second kind based on 2D Bell numbers exhibit the following correlation:*

$$\sum_{l=0}^{n}\binom{n}{l}\mathcal{S}_2(l,\epsilon)\mathscr{B}_{n-l}(q_1,q_2)={}_{\mathscr{B}}\mathcal{S}_2(n,\epsilon;q_1,q_2).$$

Proof. The expression labelled as (17) can be expressed as:

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{\mathscr{B}}\mathcal{S}_2(n,\epsilon;q_1,q_2)\frac{\xi^n}{n!} &= \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\ e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}\\ &= \sum_{n=\epsilon}^{\infty}\mathcal{S}_2(n,\epsilon)\frac{\xi^n}{n!}\ \sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_1,q_2)\frac{\xi^n}{n!},\end{aligned}$$

the above expression can be expressed in another form as

$$\sum_{n=0}^{\infty}{}_{\mathscr{B}}\mathcal{S}_2(n,\epsilon;q_1,q_2)\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\mathcal{S}_2(l,\epsilon)\ \mathscr{B}_{n-l}^{[j]}(q_1,q_2)\frac{\xi^n}{n!}. \tag{18}$$

We achieve the expected outcome by contrasting the exponents of identical powers of ξ.

Remark 3. *The correlation satisfied by the Bell-based Stirling polynomials of the second kind is obtained by substituting $q_2=0$ into the expression given by* (17) *as:*

$$\sum_{l=0}^{n}\binom{n}{l}\mathcal{S}_2(l,\epsilon)\mathscr{B}_{n-l}(q_1)={}_{\mathscr{B}}\mathcal{S}_2(n,\epsilon;q_1),$$

for a non-negative integer n.

Theorem 8. *The 2D Bell-based Stirling polynomials of the second kind can be obtained for a non-negative integer n. There are applicable summation formulas for these polynomials:*

$${}_{\mathscr{B}}\mathcal{S}_2^{[j]}(n,\epsilon;q_1+q_3,q_2+q_4)=\sum_{k=0}^{n}\binom{n}{k}{}_{\mathscr{B}}\mathcal{S}_2^{[j]}(n-k,\epsilon;q_1,q_2)\mathscr{B}_k^{[j]}(q_3,q_4).$$

Proof. Let's examine the generating functions provided in equations (17) and (9). As a result, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{\mathscr{B}}\mathcal{S}_2^{[j]}(n,\epsilon;q_1+q_3,q_2+q_4)\frac{\xi^n}{n!} &= \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\ e^{(q_1+q_3)(e^{\xi}-1)+(q_2+q_4)(e^{\xi}-1)^j}\\ &= \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\ e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}e^{q_3(e^{\xi}-1)+q_4(e^{\xi}-1)^j}\end{aligned}$$

$$= \sum_{n=0}^{\infty} {}_{\mathscr{B}}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2)\frac{\xi^n}{n!}\sum_{n=0}^{\infty}\mathscr{B}_n^{[j]}(q_3,q_4)\frac{\xi^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} {}_{\mathscr{B}}\mathcal{S}_2^{[j]}(n-k,\epsilon;q_1,q_2)\mathscr{B}_k^{[j]}(q_3,q_4)\frac{\xi^n}{n!}.$$

At last, by setting the coefficients of $\frac{\xi^n}{n!}$ equal on both sides, we prove Theorem 8 as claimed.

Theorem 9. *The 2D Bell-based Stirling polynomials of the second kind should be considered for a non-negative integer n. The following relation is valid:*

$${}_{\mathscr{B}}\mathcal{S}_2^{[j+\beta]}(n,\epsilon;q_1,q_2) = \sum_{k=0}^{n}\binom{n}{k} {}_{\mathscr{B}}\mathcal{S}_2^{[j]}(n-k,\epsilon;q_1,q_2)\mathscr{B}_k^{[\beta]}.$$

Proof. By (17), we have

$$\sum_{n=0}^{\infty} {}_{\mathscr{B}}\mathcal{S}_2^{[j+\beta]}(n,\epsilon;q_1,q_2)\frac{\xi^n}{n!} = \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\, e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^{j+\beta}}$$
$$= \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\, e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^{j}}e^{(e^{\xi}-1)^{\beta}}$$
$$= \sum_{n=0}^{\infty} {}_{\mathscr{B}}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2)\frac{\xi^n}{n!}\sum_{n=0}^{\infty}\mathscr{B}_n^{[\beta]}\frac{\xi^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} {}_{\mathscr{B}}\mathcal{S}_2^{[j]}(n-k,\epsilon;q_1,q_2)\mathscr{B}_k^{[\beta]}\frac{\xi^n}{n!}.$$

Finally, by setting the coefficients of $frac\xi^n n!$ equal on both sides, we prove Theorem 9 as claimed.

Theorem 10. *For every $n \geq 1$, if we let $\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2)$ represent the 2D Stirling polynomials of the second kind based on the Bell numbers, then the following holds:*

$$\frac{\partial}{\partial q_1}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{S}_2^{[j]}(n-k,\epsilon;q_1,q_2) - \mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2) \tag{19}$$

and

$$\frac{\partial}{\partial q_2}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{S}_2^{[j]}(n-k,\epsilon;q_1,q_2)j!S_2(k,j). \tag{20}$$

Proof. (see (19)). When we partially differentiate both sides of the equation (17) with respect to the variable q_1, we get

$$\frac{\partial}{\partial q_1}\left[\sum_{n=0}^{\infty}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2)\frac{\xi^n}{n!}\right] = \frac{\partial}{\partial q_1}\left[\frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\, e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^{j}}\right]$$

$$
\begin{aligned}
&= \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\, e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}(e^{\xi}-1) \\
&= \sum_{n=0}^{\infty}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2)\frac{\xi^n}{n!}\sum_{n=0}^{\infty}\frac{\xi^n}{n!}-\sum_{n=0}^{\infty}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2)\frac{\xi^n}{n!} \\
&= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\sum_{n=0}^{\infty}\mathcal{S}_2^{[j]}(n-k,\epsilon;q_1,q_2)-\sum_{n=0}^{\infty}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2)\right]\frac{\xi^n}{n!}.
\end{aligned}
$$

So,

$$
\frac{\partial}{\partial q_1}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{S}_2^{[j]}(n-k,\epsilon;q_1,q_2)-\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2).
$$

Proof. (See (20)). When we take the partial derivative with respect to the variable q_2 of both sides of the equation (17), we get

$$
\begin{aligned}
\frac{\partial}{\partial q_2}\left[\sum_{n=0}^{\infty}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2)\frac{\xi^n}{n!}\right] &= \frac{\partial}{\partial q_1}\left[\frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\, e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}\right] \\
&= \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}e^{q_1(e^{\xi}-1)+q_2(e^{\xi}-1)^j}(e^{\xi}-1)^j \\
&= \sum_{n=0}^{\infty}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2)\frac{\xi^n}{n!}\sum_{n=0}^{\infty}j!S_2(n,j)\frac{\xi^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2)j!S_2(k,j)\frac{\xi^n}{n!}.
\end{aligned}
$$

So,

$$
\frac{\partial}{\partial q_2}\mathcal{S}_2^{[j]}(n,\epsilon;q_1,q_2)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{S}_2^{[j]}(n-k,\epsilon;q_1,q_2)j!S_2(k,j).
$$

4. Conclusion

In this article, we present $2D$ Bell polynomials using generating functions and thoroughly examine their various associated properties. Our exploration includes explicit representations, summation formulae, recurrence relations, and addition formulas, providing valuable insights into their mathematical foundations. Lastly, we have introduced the $2D$ Bell-based Stirling polynomials of the second kind, broadening the scope of our study to include related concepts and results. Through this comprehensive analysis, our research contributes to a deeper comprehension of the properties and applications of Bell polynomials in mathematical analysis. This lays a solid groundwork for further exploration and practical use in diverse fields.

Future research in the realm of $2D$ Bell polynomials could focus on several avenues to further expand our understanding and applications of these mathematical entities. One potential direction is the exploration of higher-dimensional generalizations beyond the $2D$ case, investigating how Bell polynomials can be extended to three or more variables and uncovering their properties and relationships in multi-dimensional spaces.

Additionally, there is room for research into the development of more efficient computational algorithms and numerical techniques for handling $2D$ Bell polynomials, especially in scenarios involving large datasets or complex systems. Improving computational efficiency could open up new possibilities for applying Bell polynomials in practical fields such as data analysis, optimization, and machine learning.

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