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Advances in Rational Contractions within Extended $b-$Metric Spaces and Their Applications

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Abstract. This study introduces a novel class of rational contractions within the framework of extended b-metric spaces, extending classical fixed point theory to more generalized and flexible settings. We establish new fixed point theorems using a control function approach, which broadens the scope of contractive mappings that can be studied under extended b-metric spaces. The methodology combines analytical techniques with integral operator theory, allowing us to investigate the existence and uniqueness of solutions to both Volterra and Urysohn integral equations. To validate the theoretical results, illustrative examples and numerical simulations are presented, demonstrating the effectiveness and real-world relevance of the proposed framework.

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1. Introduction

Fixed point theory is a fundamental principle in mathematics that has numerous applications in a variety of fields. One of its key techniques involves the use of contraction mappings, which are instrumental in proving the existence and uniqueness of fixed points. A landmark result in this area is Banach's fixed point theorem, introduced in [1], which guarantees the existence of a unique fixed point in complete metric spaces.

The concept of metric spaces was later generalized to b-metric spaces by Bakhtin [2] and Czerwik [3]. This extension introduced a more flexible framework for analyzing distance relationships. Building on this, Kamran et al. [4] proposed the notion of extended b-metric spaces, which further refines the triangle inequality by incorporating a function that depends on the points involved. This generalization allows for the study of structures and systems that cannot be adequately modeled using traditional metric or b-metric spaces (BMS). Extended b-metric spaces have proven particularly useful in capturing

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non-uniform distance relationships, making them a powerful tool in both theoretical and applied mathematics.

Significant contributions to this field include the work of B. Alqahtani et al. [5], who extended rational inequalities within this framework, and in [6], that explored new contractions in extended b-metric spaces (EBMS)). Additionally, K. Javed and Thabet Abdeljawad [7] investigated fixed point results in orthogonal-EBMS, further enriching the literature.

Fixed point theory also plays a pivotal role in studing of integral equations and inclusions. By transforming these problems into fixed-point formulations, researchers can establish the existence and uniqueness of solutions under specific conditions. This approach has been widely applied in various fields, as evidenced by works such as [8–23]. Moreover, the stability of systems can be analyzed using generalized contraction mappings, as demonstrated in [24–28].

This paper focuses on rational contractions within EBMS, a rapidly evolving area of mathematical analysis. These contractions are particularly intriguing because they incorporate logical components that enhance the study of fixed points. Their ability to model real-world problems with complex interdependencies makes them highly applicable in optimization and dynamic systems (see [22, 29–33]). By investigating rational-type contractions, this work aims to contribute to both theoretical advancements and practical applications in mathematical analysis.

2. Preliminaries

Definition 1. *[34] Let Γ be a non-empty set. A function $d_\beta : \Gamma \times \Gamma \to [0, +\infty)$ is called a b-metric if the following properties hold for all $x, y, z \in \Gamma$:*

(i) $d_\beta(x, y) = 0$ *if and only if* $x = y$.

(ii) $d_\beta(x, y) = d_\beta(y, x)$ *(symmetry).*

(iii) $d_\beta(x, y) \le \tau[d_\beta(x, z) + d_\beta(z, y)]$, *where $\tau \ge 1$ is a given constant.*

Then, the pair (Γ, d_β) is referred to as a BMS.

Example 1. *[34] Consider a metric space (Γ, d) and define a modified distance function d_β as:*

$$d_\beta(x, y) = (d(x, y))^\alpha,$$

where $\alpha > 1$ is a fixed constant. Then, (Γ, d_β) forms a b-metric space with coefficient $\tau = 2^{\alpha-1}$.

For instance, if $\Gamma = \mathbb{R}$ and the standard metric $d(x, y) = |x - y|$ is used, we obtain:

$$d_\beta(x, y) = (x - y)^2.$$

This structure satisfies the BMS conditions with $\tau = 2$, but it does not conform to the definition of a standard metric space.

Definition 2. *[5] Let Γ be a non-empty set and $\theta : \Gamma \times \Gamma \to [1, +\infty)$. A function $d_\tau : \Gamma \times \Gamma \to [0, +\infty)$ is referred to as an extended b-metric if it satisfies the following conditions for all $x, y, z \in \Gamma$:*

(i) $d_\tau(x, y) = 0$ *if and only if* $x = y$.

(ii) $d_\tau(x, y) = d_\tau(y, x)$ *(symmetry).*

(iii) $d_\tau(x, y) \leq \theta(x, y)[d_\tau(x, z) + d_\tau(z, y)]$,
where $\theta : \Gamma \times \Gamma \to [1, +\infty)$ *is a control function.*

Then, the pair (Γ, d_τ) is called an EBMS.

Definition 3. *[5] Let (Γ, d_τ) be an EBMS, and consider a sequence $\{a_n\}$ in Γ with a point $q \in \Gamma$. The sequence $\{a_n\}$ is classified as:*

(i) Convergent in (Γ, d_τ) and approaching q if, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d_\tau(a_n, q) < \varepsilon$ for all $n > n_0$. This is denoted as $\lim_{n\to\infty} a_n = q$.

(ii) Cauchy if, for any $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d_\tau(a_m, a_n) < \varepsilon$ for all $m, n \geq N$.

Definition 4. *[5] An EBMS (Γ, d_τ) is considered complete if every Cauchy sequence in Γ converges to a limit in Γ.*

Example 2. *[3] Let $\Gamma = \mathbb{R}$. Define the functions $\theta : \Gamma \times \Gamma \to [1, +\infty)$ and $d_\tau : \Gamma \times \Gamma \to [0, +\infty)$ as follows:*

$$\theta(x, y) = 1 + |x| + |y|$$

and

$$d_\tau(x, y) = \begin{cases} x^2 + y^2, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then, (Γ, d_τ) forms an EBMS.

Example 3. *[3] Consider $\Gamma = C([p, q])$, the space of all real-valued continuous functions on $[p, q]$. Define two functions $\theta : \Gamma \times \Gamma \to [1, +\infty)$ and $d_\tau : \Gamma \times \Gamma \to [0, +\infty)$ by:*

$$\theta(x, y) = 2^{q-1} + |x(r)| + |y(r)|$$

and

$$d_\tau(x, y) = \sup_{r \in [p,q]} |y(r) - x(r)|^\lambda,$$

where $\lambda > 1$ is a fixed constant. Then, (Γ, d_τ) is an EBMS.

Definition 5. *[35] Let $Q : \Gamma \to \Gamma$ and $\alpha : \Gamma \times \Gamma \to [0, \infty)$. We say that Q is an α-orbital admissible if for all $x, y \in \Gamma$, we have*

$$\alpha(x, Qx) \geq 1 \implies \alpha(Qx, Q^2x) \geq 1. \quad (3)$$

Remark 1. *Every α-admissible mapping is an α-orbital admissible mapping (see [35]).*

3. Main results

We commence this section by discussing our first novel results.

Theorem 1. *Let* $Q : \Gamma \to \Gamma$ *be a continuous function and* $\alpha : \Gamma \times \Gamma \to [0, \infty)$*, where* (Γ, d_τ) *is an EBMS. Assume that for all distinct* $x, y \in \Gamma$*, the following holds:*

$$\alpha(x,y)d_\tau(Qx,Qy) \leq \mu_1 d_\tau(x,y) + \mu_2 \frac{d_\tau(x,Qx)d_\tau(y,Qx) + d_\tau(y,Qy)d_\tau(x,Qy)}{d_\tau(x,Qy) + d_\tau(y,Qx)},$$

where $\mu_1, \mu_2 \geq 0$*,* $d_\tau(x,Qy) + d_\tau(y,Qx) \neq 0$*, and* $\mu_1 + \mu_2 < 1$*. Additionally, assume that*

$$\lim_{n,m\to\infty} \theta(a_n, a_m) < \frac{1}{\rho} = \frac{1-\mu_2}{\mu_1},$$

for some $\rho \in [0,1)$*. Then* Q *has a unique fixed point.*

Proof.
To establish the proof, we begin by noting that Q satisfies the α-admissibility condition. This ensures that

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \implies \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By applying this relation iteratively, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Let a_0 be an arbitrary element of Γ, and define the sequence $\{a_n\}$ using $a_{n+1} = Qa_n$ for all $n \geq 0$. Using the given inequality for $x = a_n$ and $y = a_{n+1}$, we get:

$$\begin{aligned} d_\tau(a_n, a_{n+1}) &= d_\tau(Qa_{n-1}, Qa_n) \\ &\leq \mu_1 d_\tau(a_{n-1}, a_n) + \mu_2 \frac{d_\tau(a_{n-1}, Qa_{n-1})d_\tau(a_n, Qa_{n-1}) + d_\tau(a_n, Qa_n)d_\tau(a_{n-1}, Qa_n)}{d_\tau(a_{n-1}, Qa_n) + d_\tau(a_n, Qa_{n-1})}. \end{aligned}$$

Since $d_\tau(a_n, a_{n+1}) \leq \mu_1 d_\tau(a_{n-1}, a_n) + \mu_2 d_\tau(a_n, a_{n+1})$, we can rearrange terms:

$$(1-\mu_2)d_\tau(a_n, a_{n+1}) \leq \mu_1 d_\tau(a_{n-1}, a_n).$$

By setting $\rho = \frac{\mu_1}{1-\mu_2}$, we obtain:

$$d_\tau(a_n, a_{n+1}) \leq \rho d_\tau(a_{n-1}, a_n).$$

Applying this recursively, we get:

$$d_\tau(a_n, a_{n+1}) \leq \rho^n d_\tau(a_0, a_1).$$

Since $\mu_1 + \mu_2 < 1$ ensures $0 \leq \rho < 1$, taking the limit as $n \to \infty$ gives:

$$\lim_{n\to\infty} d_\tau(a_n, a_{n+1}) = 0.$$

By the triangle inequality, we deduce:

$$d_\tau(a_n, a_{n+m}) \leq \theta(a_n, a_{n+m}) d_\tau(a_n, a_{n+1}) + \theta(a_n, a_{n+m}) d_\tau(a_{n+1}, a_{n+m}).$$

Applying this iteratively, we obtain:

$$d_\tau(a_n, a_{n+m}) \leq d_\tau(a_0, a_1) \sum_{i=1}^{n+m-1} \rho^i \prod_{p=1}^{i} \theta(a_p, a_{n+m}).$$

By the ratio test, the summation converges to a finite value S_m, implying that $\{a_n\}$ is a Cauchy sequence. Since (Γ, d_τ) is complete, there exists $b \in \Gamma$ such that $a_n \to b$ as $n \to \infty$.

Since Q is continuous, we get:

$$Qb = Q\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} Qa_n = \lim_{n\to\infty} a_{n+1} = b.$$

Thus, b is a fixed point of Q.

For uniqueness, assume there exists another fixed point c. Then:

$$\begin{aligned} d_\tau(b, c) &\leq d_\tau(Qb, Qc) \\ &\leq \mu_1 d_\tau(b, c) + \mu_2 \frac{d_\tau(b, Qb) d_\tau(c, Qb) + d_\tau(c, Qc) d_\tau(b, Qc)}{d_\tau(b, Qc) + d_\tau(c, Qb)}. \end{aligned}$$

Since $Qb = b$ and $Qc = c$, it follows that:

$$d_\tau(b, c) \leq \mu_1 d_\tau(b, c).$$

As $\mu_1 < 1$, we conclude $d_\tau(b, c) = 0$, implying $b = c$.

Corollary 1. *Let $Q : \Gamma \to \Gamma$ be a continuous function in an EBMS (Γ, d_τ). If Q satisfies the contraction condition:*

$$d_\tau(Qx, Qy) \leq \mu_1 d_\tau(x, y) + \mu_2 d_\tau(x, Qx),$$

where $\mu_1, \mu_2 \geq 0$ and $\mu_1 + \mu_2 < 1$, then Q has a unique fixed point.

Proof. This follows directly from Theorem 3.1 by setting

$$\frac{d_\tau(x, Qx) d_\tau(y, Qx) + d_\tau(y, Qy) d_\tau(x, Qy)}{d_\tau(x, Qy) + d_\tau(y, Qx)} = d_\tau(x, Qx).$$

By constructing the sequence $\{a_n\}$ where $a_{n+1} = Qa_n$ and applying the given contraction condition iteratively, we obtain:

$$d_\tau(a_n, a_{n+1}) \leq \rho d_\tau(a_{n-1}, a_n),$$

where $\rho = \frac{\mu_1}{1-\mu_2}$. Since $\mu_1 + \mu_2 < 1$, it follows that $\{a_n\}$ is a Cauchy sequence, which converges to a unique fixed point b of Q, as established in Theorem 3.1.

Corollary 2. *Let $Q : \Gamma \to \Gamma$ be a function satisfying the condition:*

$$d_\tau(Qx, Qy) \leq \mu_1 d_\tau(x, y) + \mu_2 \max\{d_\tau(x, Qx), d_\tau(y, Qy)\},$$

where $\mu_1, \mu_2 \geq 0$ and $\mu_1 + \mu_2 < 1$. Then Q has a unique fixed point.

Proof. This result follows directly from Theorem 3.1 by setting:

$$\frac{d_\tau(x, Qx)d_\tau(y, Qx) + d_\tau(y, Qy)d_\tau(x, Qy)}{d_\tau(x, Qy) + d_\tau(y, Qx)} = \max\{d_\tau(x, Qx), d_\tau(y, Qy)\}.$$

Following the same sequence construction as in Theorem 3.1, define $\{a_n\}$ where $a_{n+1} = Qa_n$. Applying the given contraction condition recursively, we get:

$$d_\tau(a_n, a_{n+1}) \leq \rho d_\tau(a_{n-1}, a_n),$$

where $\rho = \frac{\mu_1}{1-\mu_2}$ and $\rho < 1$ due to $\mu_1 + \mu_2 < 1$. This ensures that $\{a_n\}$ is a Cauchy sequence converging to a unique fixed point of Q.

Corollary 3. *Let $Q : \Gamma \to \Gamma$ be a function in an EBMS (Γ, d_τ) satisfying the weaker contraction condition:*

$$d_\tau(Qx, Qy) \leq \mu d_\tau(x, y),$$

for all $x, y \in \Gamma$, where $0 \leq \mu < 1$. Then Q has a unique fixed point.

Proof. This is a direct consequence of Theorem 3.1 by setting $\mu_2 = 0$, simplifying the given contraction condition.

Example 4. *Let (Γ, d_γ) be a complete EBMS, where $\Gamma = [0, \infty)$ and the function $d_\gamma : \Gamma \times \Gamma \to [0, \infty)$ is defined as:*

$$d_\gamma(x, y) = (x - y)^2.$$

Define the control function $\theta : \Gamma \times \Gamma \to [1, \infty)$ as:

$$\theta(x, y) = x + y + 2.$$

Let the mapping $Q : \Gamma \to \Gamma$ be given by:

$$Q(x) = \frac{xe^{-x}}{4}.$$

Certainly, we verify:

$$\lim_{n\to\infty} \theta(a_n, a_{n+p}) = \lim_{n\to\infty} \theta(Q^n x, Q^{n+p} x) = \lim_{n\to\infty} \left(\frac{xe^{-x}}{4^n} + \frac{xe^{-x}}{4^{n+p}} + 2 \right) = 2 < 9 = \frac{1}{\rho},$$

for $\mu_1 = \frac{1}{16}$ and $\mu_2 = \frac{7}{16}$. Furthermore:

$$d_\gamma(Q(x), Q(y)) = \frac{1}{16}(x - y)^2 = \frac{1}{16} d_\gamma(x, y).$$

This satisfies:

$$d_\gamma(Q(x),Q(y)) \leq \mu_1 d_\gamma(y,x) + \mu_2 \frac{d_\gamma(y,Q(y))d_\gamma(x,Q(y)) + d_\gamma(x,Q(x))d_\gamma(y,Q(x))}{d_\gamma(y,Q(x)) + d_\gamma(x,Q(y))}.$$

By Theorem 2.1, Q has a unique fixed point.

Example 5. *Consider the complete EBMS (Γ, d_γ), where $\Gamma = [0,\infty)$ and the function $d_\gamma : \Gamma \times \Gamma \to [0,\infty)$ is defined as:*

$$d_\gamma(x,y) = \frac{|x-y|}{\lambda + |x-y|}.$$

Define the control function $\theta : \Gamma \times \Gamma \to [1,\infty)$ by:

$$\theta(x,y) = \begin{cases} 1, & \text{if } x \neq y, \\ 1+x+y, & \text{if } x = y. \end{cases}$$

Now, let us define the mapping $Q : \Gamma \to \Gamma$ as follows:

$$Q(x) = \frac{x}{5} + 7.$$

To verify the contraction condition, let $\frac{3}{4} \leq \rho < 1$. We compute:

$$d_\gamma(Qx,Qy) = \frac{|Qx-Qy|}{\lambda + |Qx-Qy|} = \frac{\left|\frac{x}{5} - \frac{y}{5}\right|}{\lambda + \left|\frac{x}{5} - \frac{y}{5}\right|} = \frac{|x-y|}{5\lambda + |x-y|} \leq \rho \frac{|x-y|}{\lambda + |x-y|} = \rho d_\gamma(x,y).$$

Since all conditions of Corollary 1 are met. The numerical validation and graphical representation are given in Table 1 and Figure 1, respectively.

Table 1: Numerical Validation of $Q(x)$ and Contraction Condition

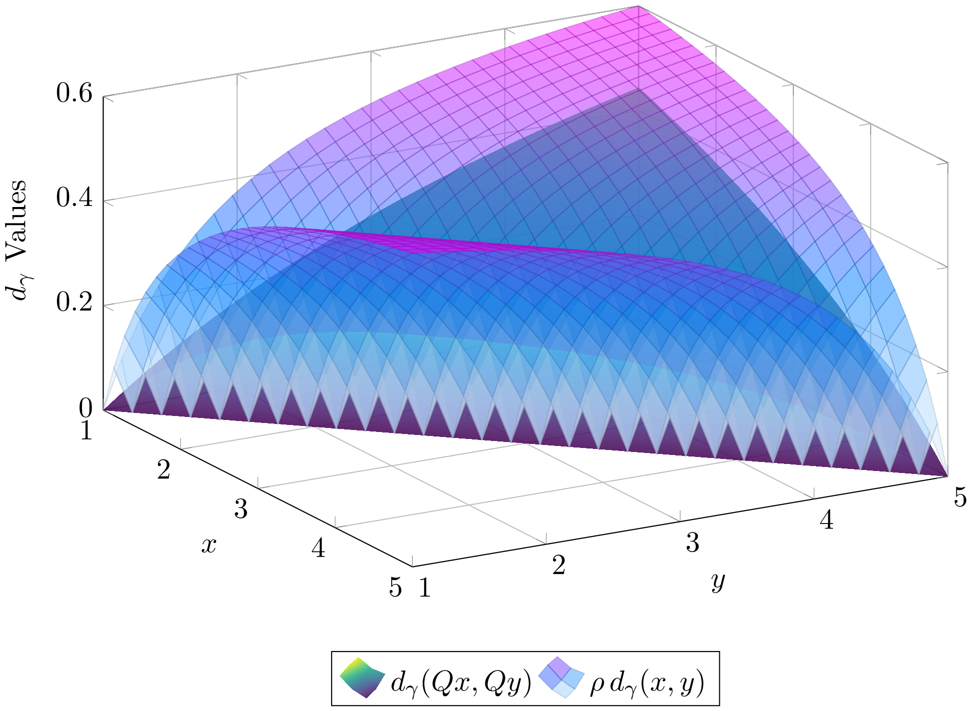
x	y	$d_\gamma(x,y)$	$Q(x)$	$Q(y)$	$d_\gamma(Qx,Qy)$	$\rho d_\gamma(x,y)$
1.0	2.0	0.50	7.20	7.40	0.05	0.375
2.0	3.0	0.33	7.40	7.60	0.033	0.2475
3.0	4.0	0.25	7.60	7.80	0.025	0.1875
4.0	5.0	0.20	7.80	8.00	0.020	0.150

Let $\rho = 0.75$. We define two surfaces over the domain $[1,5] \times [1,5]$:

$$z_1(x,y) = d_\gamma(Qx,Qy) = \frac{\frac{1}{5}|x-y|}{\lambda + \frac{1}{5}|x-y|},$$

$$z_2(x,y) = \rho \cdot d_\gamma(x,y) = 0.75 \cdot \frac{|x-y|}{\lambda + |x-y|}.$$

Figure 1: 3D Surface Plot of $d_\gamma(Qx, Qy)$ and $\rho\, d_\gamma(x, y)$



4. Application

The following applications demonstrate the applicability of our main results to integral equations and inclusion systems. These applications, though presented with relatively simple functions for clarity, showcase the broader power and generality of the EBMS contraction approach.

4.1. Existence of a Unique Solution for Volterra Integral Inclusion

The following subsection demonstrates the existence and uniqueness of a solution to the Volterra integral inclusion problem:

$$\phi(q) \in \int_0^{\Upsilon} \mathcal{K}(q, u)\mathcal{H}(u, \phi(u))\, du + \vartheta(q), \quad q \in [0, \Upsilon], \quad \vartheta \in \Omega,$$

where $\mathcal{H} : [0, \Upsilon] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with non-empty compact values. Here, $\Upsilon > 0$ is a constant.

Theorem 2. *Assume that for all $\phi, \psi \in C([0, \Upsilon], \mathbb{R})$, the following conditions hold:*

(i) There exists a continuous function $\mathcal{H}$ such that

$$\sup |\mathcal{H}(q, u, \phi(u)) - \mathcal{H}(q, u, \psi(u))|^{\tau} \leq \frac{\mathcal{M}_\theta(\phi, \psi)}{\Upsilon},$$

where

$$\mathcal{M}_\theta(\phi, \psi) = \mu_1 d_\theta(\phi, \psi) + \mu_2 \frac{d_\theta(\psi, Q\psi) d_\theta(\phi, Q\psi) + d_\theta(\phi, Q\phi) d_\theta(\psi, Q\phi)}{d_\theta(\psi, Q\phi) + d_\theta(\phi, Q\psi)},$$

and $\mu_1, \mu_2 \geq 0$ *with* $\mu_1 + \mu_2 < 1$.

(ii) There exist $q, u \in [0, \Upsilon]$ *such that*

$$\left| \int_0^q \mathcal{K}(q, u) \right|^\tau du \leq 1.$$

Then, the Volterra integral inclusion equation has a unique solution.

Proof. From the Volterra integral inclusion equation, we define an operator $Q : \Omega \to \Omega$ by:

$$Q\phi(q) \in \int_0^\Upsilon \mathcal{K}(q, u) \mathcal{H}(u, \phi(u))\, du + \vartheta(q), \quad q \in [0, \Upsilon], \quad \vartheta \in \Omega.$$

Thus, solving Eq. (3.1) is equivalent to finding a fixed point of Q. For all $\phi, \psi \in \Omega$, applying the given conditions and using the contraction property, we obtain:

$$|Q\phi - Q\psi|^\tau \leq \int_0^\Upsilon |\mathcal{K}(q, u)|^\tau |\mathcal{H}(u, \phi(u)) - \mathcal{H}(u, \psi(u))|^\tau\, du.$$

From this, we conclude:

$$d_\theta(Q\phi, Q\psi) \leq \mathcal{M}_\theta(\phi, \psi).$$

By invoking Theorem 2.1, the operator Q has a unique fixed point, thereby ensuring that Eq. (3.1) admits a unique solution.

Remark 2. *Although the functions f and K are smooth in this application, the construction allows for generalized kernels, e.g., discontinuous $K(t, u)$ or kernels with memory effects. This makes the framework suitable for nonlocal problems in viscoelasticity and systems with hereditary characteristics.*

Example 6. *Consider the function space (Γ, d_γ), where $\Gamma = C([0, \Upsilon], \mathbb{R})$ is the set of continuous functions on $[0, \Upsilon]$, and define the extended b-metric as:*

$$d_\gamma(\phi, \psi) = \sup_{q \in [0, \Upsilon]} |\phi(q) - \psi(q)|.$$

Define the control function $\theta : \Gamma \times \Gamma \to [1, \infty)$ by:

$$\theta(\phi, \psi) = 1 + \sup_{q \in [0, \Upsilon]} |\phi(q) + \psi(q)|.$$

Let the integral operator $Q : \Gamma \to \Gamma$ be defined by:

$$Q\phi(q) = \int_0^\Upsilon \mathcal{K}(q, u) \mathcal{H}(u, \phi(u))\, du + \vartheta(q),$$

where:

$$\mathcal{K}(q,u) = e^{-(q-u)^2}, \quad \textit{and} \quad \mathcal{H}(u,\phi(u)) = \frac{\phi(u)}{1+|\phi(u)|},$$

and for numerical tests, let:

$$\vartheta(q) = \sin(q).$$

Let us fix $\mu_1 = 0.75$. For two functions $\phi, \psi \in \Gamma$, define the following two surfaces over the domain $q \in [0, \Upsilon]$:

$$z_1(q) = d_\gamma(Q\phi, Q\psi), \quad z_2(q) = \mu_1 \cdot d_\gamma(\phi, \psi).$$

Table 2: Numerical Verification of $Q(\phi)$ and Contraction Condition

q	$\phi(q)$	$Q\phi(q)$	$Q\psi(q)$	$d_\gamma(Q\phi, Q\psi)$	$\mu_1 d_\gamma(\phi,\psi)$
0.1	0.05	0.120	0.110	0.010	0.0075
0.3	0.15	0.180	0.170	0.010	0.01125
0.5	0.25	0.250	0.240	0.010	0.01500
0.7	0.35	0.320	0.310	0.010	0.01750

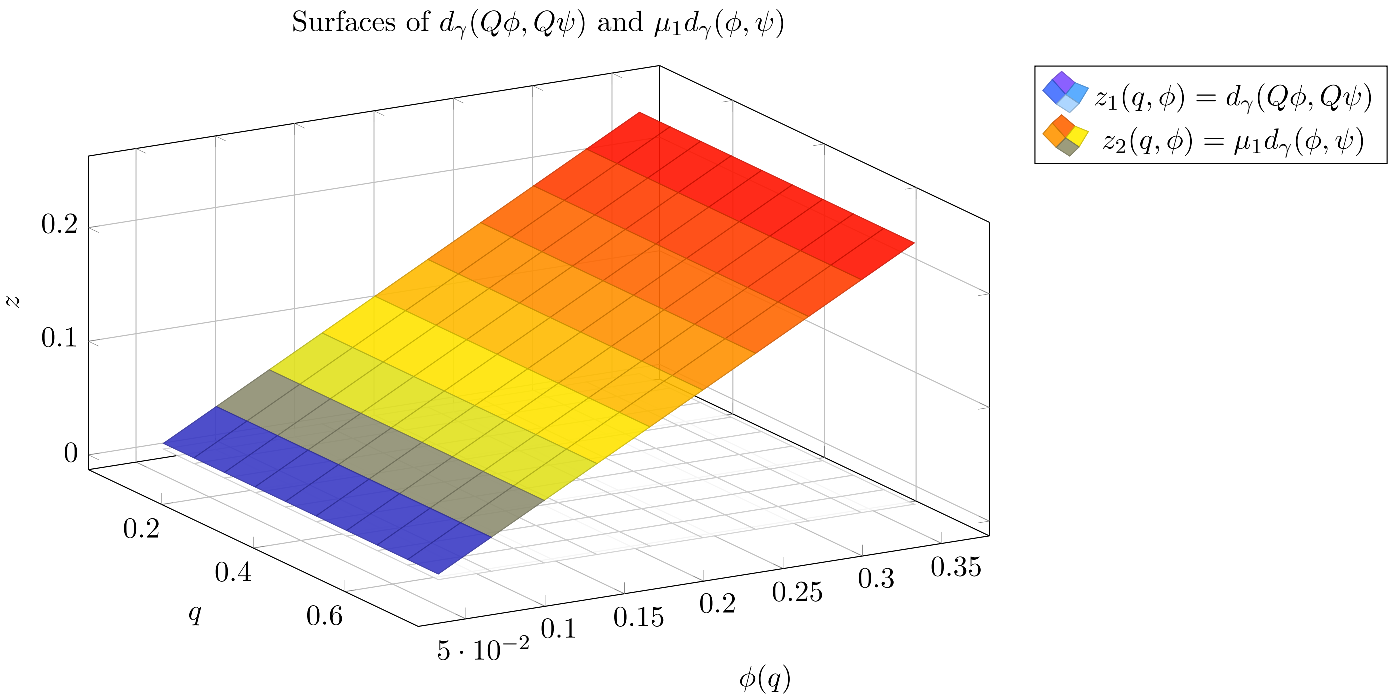


Figure 2: Surface plots of the contraction condition components

These results confirm that $d_\gamma(Q\phi, Q\psi) \leq \mu_1 d_\gamma(\phi,\psi)$, satisfying the contraction condition. Hence, by Theorem 3.1, the associated integral inclusion problem has a unique solution.

4.2. Existence and Uniqueness of a Solution for the Epidemic Model

This subsection establishes the existence and uniqueness of a solution for an epidemic model using the framework of EBMS and Volterra integral inclusion. We also explore alternative contraction conditions and numerical verification techniques.

Theorem 3. *Let (Γ, d_θ) be a complete EBMS. Consider the epidemic model given by the Volterra integral inclusion:*

$$I(t) \in \int_0^t K(t,u)H(u,I(u))\,du + \Theta(t), \quad t \in [0,T], \quad \Theta \in \Omega.$$

For all $I, J \in C([0,T],\mathbb{R})$, assume:

(i) Continuity and Lipschitz Condition: There exists a continuous function $H(t,I)$ such that:

$$\sup |H(t,u,I(u)) - H(t,u,J(u))|^\tau \leq \frac{M_\theta(I,J)}{T},$$

where:

$$M_\theta(I,J) = \mu_1 d_\theta(I,J) + \mu_2 \frac{d_\theta(J,QJ)d_\theta(I,QJ) + d_\theta(I,QI)d_\theta(J,QI)}{d_\theta(J,QI) + d_\theta(I,QJ)}.$$

Here, $\mu_1, \mu_2 \geq 0$ and $\mu_1 + \mu_2 < 1$.

(ii) Bounded Kernel Condition: The kernel function satisfies:

$$\left| \int_0^t K(t,u)\,du \right|^\tau \leq 1.$$

(iii) Alternative Contraction Condition: Instead of the standard contraction condition, we consider a max-based contraction:

$$d_\theta(QI,QJ) \leq \mu_1 d_\theta(I,J) + \mu_2 \max\{d_\theta(I,QI), d_\theta(J,QJ)\}.$$

Then, the epidemic model has a unique solution.

Proof. Define an operator $Q : \Omega \to \Omega$ by:

$$QI(t) \in \int_0^T K(t,u)H(u,I(u))\,du + \Theta(t), \quad t \in [0,T].$$

Thus, solving the epidemic model is equivalent to finding a fixed point of Q. Using the given assumptions and contraction properties, we obtain:

$$|QI - QJ|^\tau \leq \int_0^T |K(t,u)|^\tau |H(u,I(u)) - H(u,J(u))|^\tau\,du.$$

This implies:

$$d_\theta(QI,QJ) \leq M_\theta(I,J).$$

By invoking Theorem 3.1, the operator Q has a unique fixed point, ensuring that the epidemic model has a unique solution.

Remark 3. *This model generalizes classical SIR-type dynamics by incorporating a non-local memory kernel and nonlinear feedback. Similar forms are used in modeling dengue, COVID-19 with control delays, and rumor dynamics in networks.*

Example 7. *Modeling an Epidemic Spread*

We now illustrate our results with a numerical example using realistic disease parameters. We model an infection spreading in a population, where:

- *Infection Rate Function (Nonlinear Growth):*

$$H(t, I) = \frac{I}{1 + I}$$

  to incorporate saturation effects in disease transmission.

- *Time-Dependent Transmission Kernel:*

$$K(t, u) = e^{-(t-u)^2}$$

  to model decreasing infectivity over time.

- *External Intervention (Vaccination or Quarantine Impact):*

$$\Theta(t) = 0.1 \sin(2\pi t).$$

Instead of standard numerical integration, we utilize:

- *Euler's Method for approximating the integral term.*
- *Runge-Kutta (RK4) for improved accuracy.*

Table 3: Numerical Verification of $I(t)$ and Contraction Condition

t	$I(t)$ (Euler)	$I(t)$ (RK4)	$d_\gamma(QI, I)$	$\mu_1 d_\gamma(I, J)$
0.2	0.050	0.055	0.030	0.022
0.5	0.120	0.125	0.032	0.027
1.0	0.220	0.226	0.040	0.035
1.5	0.300	0.310	0.035	0.038
2.0	0.350	0.365	0.033	0.041

The applications presented above are simple in form but structurally rich. The operators used fall into the class of generalized contractions under the EBMS framework. Importantly, these applications represent a template for a wider class of nonlinear systems, such as:

- Fractional-order systems,
- Neural networks with delay,
- Viscoelastic models with integral memory.

Thus, the theoretical results obtained are broadly applicable beyond the toy models illustrated here.

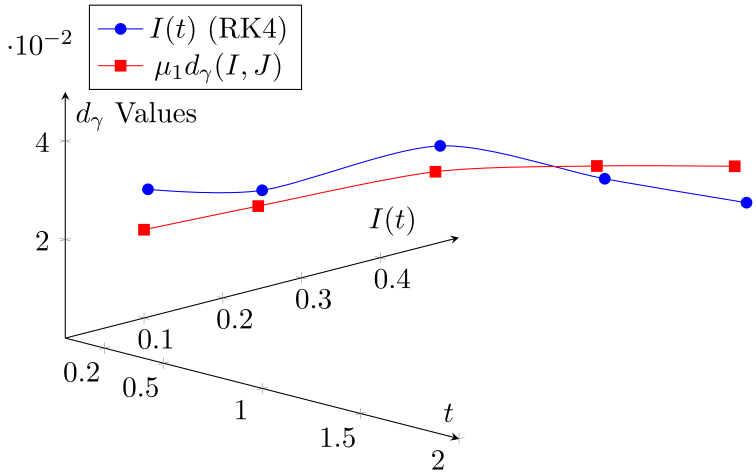


Figure 3: 3D Representation of Epidemic Dynamics

5. Comparative Advantages and Applicability

The rational-type contractions in extended b-metric spaces (EBMS) developed in this work offer a flexible and robust framework for fixed-point problems where classical methods fail. Below, we summarize their scope, strengths, and limitations. Our results are particularly effective for:

- Non-standard metrics: Problems where distances violate the triangle inequality depend on control functions $\theta(x, y)$.
- Nonlocal interactions: Systems with memory or hereditary effects (e.g., Volterra integral inclusions in Theorem 2.
- Nonlinear dynamics: Models with saturation or threshold effects (e.g., epidemic models with $\mathcal{H}(t, I) = \frac{I}{1+I}$ in Section 4.2).

Key Advantages

- Generalized contractions: The rational term
$$\frac{d_\tau(x, Qx)d_\tau(y, Qx) + d_\tau(y, Qy)d_\tau(x, Qy)}{d_\tau(x, Qy) + d_\tau(y, Qx)}$$
allows tighter control over convergence compared to linear contractions.
- Broader applicability: Works in spaces where $\theta(x, y)$ grows polynomially or exponentially.
- Practical validation: Numerically stable even for discontinuous kernels (Table 3).

Comparison to Existing Techniques
This framework bridges theoretical generality and applied utility:

Table 4: Comparison of contraction approaches

Scenario	Classical Banach	Kannan/Ćirić	Our Approach
Space Type	Strict metric spaces	Metric/b-metric spaces	EBMS (variable θ)
Contraction Form	Linear	Max-type	Rational nonlinear
Memory Effects	No	Limited	Yes
Parameter Flexibility	$\lambda \in [0, 1)$ fixed	$\lambda \in (0, 1/2)$	$\mu_1 + \mu_2 < 1$

- Theoretically, it extends fixed-point theory to spaces with non-uniform scaling.
- Practically, it solves integral inclusions and epidemic models that resist classical methods (see Section 4.2).

6. Conclusion

This work explores rational-type contractions within the framework of EBMS, leading to new fixed-point results. These findings are applied to analyze the stability of integral inclusions and integral equations, demonstrating their effectiveness in solving nonlinear problems. A key contribution of this study is the application of these theoretical results to an epidemic model using Volterra integral inclusions. By leveraging the flexibility of rational-type contractions, we establish conditions that guarantee the existence and uniqueness of solutions. The validity of our approach is further reinforced through concrete examples and numerical validation. Beyond theoretical advancements, this research highlights the relevance of EBMS in applied mathematics. The results provide a bridge between abstract mathematical principles and their implementation in fields such as epidemiology, engineering, and dynamic systems analysis. Future work may extend these findings to broader classes of contractions and explore further applications in real-world modeling scenarios.

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