EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 6062
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Solving System of Monotone Variational Inclusion Problems with Multiple Output Sets in Banach Spaces

H. A. Abass[1,4,*], M. Aphane[1], O. K. Oyewole[2], O. K. Narain[3], K. I. Mustafoyev[4]

[1] *Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, P.O. Box 94 Medunsa 0204, Pretoria, South Africa*
[2] *Department of Mathematics, Tshwane University of Technology, Arcadia, PMB 0007, Pretoria, South Africa*
[3] *School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa*
[4] *Center of Research and Innovation, Asia International University, Yangiobod MFY, G'ijduvon Street, House 74, Bukhara, Uzbekistan*

Abstract. In this article, we introduce a self-adaptive method for approximating solutions of split common fixed point problem of Bregman demigeneralized mappings and system of monotone variational inclusion problem with multiple output sets in reflexive Banach spaces. By employing our iterative method, we prove a strong convergence theorem for approximating solutions of the aforementioned problems. In summary, we state some consequences of our main result. The result discuss in this paper extends and complements many related results in literature.

2020 Mathematics Subject Classifications: 47H06, 47H09, 47J05, 47J25

Key Words and Phrases: Bregman demigeneralized mapping, monotone operators, self-adapative method, split common fixed point problem

1. Introduction

For modelling inverse problems which arise from phase retrievals and medical image reconstruction, (see [1]), Censor and Elfving [2] introduced the Split Feasibility Problem (SFP) in 1994, which is to find

$$u^* \in C \text{ such that } Fu^* \in Q, \tag{1}$$

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.6062

Email addresses: `hammed.abass@smu.ac.za`, `hammedabass548@gmail.com` (H. A. Abass), `maggie.aphane@smu.ac.za` (M. Aphane), `oyewoleolawalekazeem@gmail.com` (O. K. Oyewole), `Naraino@ukzn.ac.za`, (O. K. Narain), `k.mustafoyev@oxu.uz` (K. I. Mustafoyev)

where C and Q are nonempty, closed and convex subsets of real Banach spaces E_1 and E_2 respectively, and $F : E_1 \to E_2$ is a bounded linear operator. The SFP have been well studied in the framework of real Hilbert spaces, uniformly convex and uniformly smooth Banach spaces, see ([3–5] and other references contained in). Different optimization problems have been formulated in terms of SFP (1), for instance, If $Q = \{b\}$ in SFP (1) is a singleton, then we have the following convexly constrained linear inverse problem (CCLIP) defined as follows:

$$\text{Find a point } u^* \in C \text{ such that } Fu^* = b.$$

Also, if $C = Fix(T) = \{x \in E : x = Tx\}$ and $Q = Fix(S)$, then SFP (1) becomes split common fixed point problem (SCFPP) which is to find a point

$$u^* \in Fix(T) \text{ such that } Fu^* \in Fix(S). \tag{2}$$

Since the introduction of the SCFPP (2), authors have considered several schematic methods for approximating its solution. For instance, Censor and Segal [6] introduced the following iterative algorithm for solving the SCFPP (2) in finite dimensional spaces. They defined the algorithm as follows:

$$x_{n+1} = T(x_n + \tau F^t(S - I)Fx_n),$$

for each $n \geq 1$, where $\tau \in (0, \frac{0}{\gamma})$ with γ being the largest eigenvalue of the matrix F^tF (F^t being the matrix transposition). Also, Moudafi [7] introduced a relaxed algorithm for approximating a solution of SCFPP (2) and proved some weak convergence results in Hilbert spaces with the mappings T and S being quasi-nonexpansive mappings.
The variational inclusion problem consists of finding a point $x^* \in E$ such that

$$0 \in (A + B)x^*, \tag{3}$$

where $A : E \to E^*$ is a single-valued mapping and $B : E \to 2^{E^*}$ is a multi-valued mapping on a real Banach space E with dual space E^*. Combining the notions of SFP and VIP, Moudafi [8] introduced the following Split Variational Inclusion Problem (SVIP): Let H_1 and H_2 be real Hilbert spaces, $A_i : H_i \to H_i,\ i = 1, 2$ be single-valued mappings, $B_i : H_i \to 2^{H_i}$ be multi-valued mappings and $F : H_1 \to H_2$ be a bounded linear operator. The SVIP consists of finding $x^* \in H_1$ such that

$$0 \in (A + B)x^* \tag{4}$$

and such that

$$y^* = Fx^* \ \text{ solves } \ 0 \in (A + B)Fx^*. \tag{5}$$

We note that since its introduction, the SVIP has been considered in other more general frameworks than the Hilbert spaces (see [9–19] and the references therein).

The several variants of the SFP continue to recieve attention of various authors, notably because of the many rich applications, (see [6, 20]). There have been attempts at extending the SFP for more operators to cover the previous studies in the literature. For instance, Reich and Tuyen [21] introduced the Generalized Split Common Monotone Inclusion Problem (GSCMIP): Let $i = 1, 2, \cdots, N$, H_i be real Hilbert spaces, $A_i : H_i \to 2^{H_i}$ be maximal monotone operators. Let $F_i : H_i \to H_{i+1}$ be bounded linear opertors for $i = 1, 2, \cdots, N-1$ such that $T_i \neq 0$. Then the GSCMIP is to find $x^* \in H_1$ such that

$$0 \in A_1(x^*),\ 0 \in A_2(F_1(x^*)), \cdots, 0 \in A_N(T_{N-1}T_{N-2}\cdots T_1(x^*)). \tag{6}$$

Very recently, the same authors in [16] introduced and studied a Split Common Monotone Inclusion Problem with Multiple Output sets (SCMIPOS) in Hilbert spaces. Let $H, H_1, \cdots, H_N$ be real Hilbert spaces, $F_i : H \to H_i, \quad i = 1, 2, \cdots, N$ be bounded linear operators. Let $B : H \to 2^H,\ B_i : H_i \to 2^{H_i}, \quad i = i, 2, \cdots, N$ be maximal monotone operators, then SCMIPOS consists of finding a point $x^* \in H$ such that

$$x^* \in B^{-1}(0) \cap \left(\bigcap_{i=1}^{N} F_i^{-1}(B_i^{-1}(0)) \right). \tag{7}$$

In this paper, our motivation is in two folds. First, we combine the notions of SVIP and the SCMIPOS to introduce a Split Variational Inclusion Problem with Multiple Output sets (SVIPOS) in the framework of real Banach spaces. Let $E = E_0, E_1, E_2, \cdots, E_N$ be real Banach spaces and $F_i : E \to E_i, \quad i = 0, 1, \cdots, N$ with $F_0 = I^E$ be bounded linear operators. For $i = 0, 1, \cdots, N$, let $A_i : H_i \to H_i$ with $A = A_0$ be single-valued mappings and $B_i : H_i \to 2^{H_i}$ with $B = B_0$ be multi-valued mappings. Then the SVIPOS is the problem of finding a point $x^* \in E$ such that

$$x^* \in (A+B)^{-1}(0) \bigcap \left(\bigcap_{i=1}^{N} F_i^{-1}((A_i + B_i)^{-1}(0)) \right). \tag{8}$$

On the other hand, the Fixed Point Problem (FPP) for a multi-valued mapping have been well discussed due to its many applications. For instance, the FPP is used in game theory, control theory, convex optimization differential inclusion and so on [22–26]. The problem of obtaining a common solution of a fixed point problem (in short, FPP) and other optimization problems have been considered in recent articles. We note that these type of problems become more applicable in real life problems whose constraints can be modelled as fixed point and optimization problems. In this direction, Izuchukwu et al. [15] studied the following split monotone variational inclusion and fixed point problem between Hilbert space and a Banach space which is defined as follows:

$$\text{Find } x^* \in Fix(T) \cap (A+B)^{-1}(0) \text{ such that } Fu^* \in G^{-1}(0),$$

where H is a Hilbert space, E is a uniformly convex and uniformly smooth Banach space, T a multivalued quasi-nonexpansive mapping, $B : H \to 2^H$ and $G : E \to 2^E$ are maximal monotone operators, $F : H \to E$ is a bounded linear operator. They proposed a viscosity iterative scheme and under mild conditions and proved a strong convergence theorem.

Inspired by the results discussed above, our second motivation is to propose an iterative algorithm for approximating a common solution of a fixed point problem and split variational inclusion problem with multiple output sets. The proposed method combines the Mann iterative, the Halpern technique and a carefully selected step size to avoid the dependence of the method on prior knowledge of the operator norms. Using this method, we prove a strong convergence method for approximating a common solution of an SVIPOS and a fixed point problem for a Bregman multi-valued mapping in the framework of real reflexive Banach spaces. In particular, the following are some of the highlights of the present study:

(i) The main result in this article generalizes the results in [27] and [14] from p-uniformly Banach spaces which are also uniformly smooth to reflexive Banach spaces.

(ii) The problem considered in [19] is a special case of the one considered in this article and generalizes the results in [6, 7, 11, 19, 28, 29] from real Hilbert spaces to a reflexive Banach spaces.

(iii) It is worth mentioning that the proof of convergence proposed in this paper is different from the ones in [6, 14, 27, 29] in the sense that our approach does not distinguish between whether the sequence generated by our algorithm is Fejer-monotone or not. Our approach is simple and more elegant.

(iv) We dispensed the sets $\{C_n, D_n, Q_n\}_{n\in\mathbb{N}}$ in our algorithm as this gives difficulties in computation. Lastly, our iterative algorithm is designed in such a way that it does not require prior knowledge of operator norm as this also gives difficulties in computation.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\rightarrow$" and "$\rightharpoonup$", respectively. For any $x \in E$, we denote the value of $x^* \in E$ at x by $\langle x, x^*\rangle$.
Let E be a reflexive Banach space with E^* its dual and Q be a nonempty closed and convex subset of E. Let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then the Fenchel conjugate of g is the map $g^* : E^* \to (-\infty, +\infty]$ defined by

$$g^*(x^*) = \sup\{\langle x, x^*\rangle - g(x) : x \in E\},\ \ x^* \in E^*.$$

Let the domain of g be denoted by $domg = \{x \in E : g(x) < +\infty\}$, hence for any $y \in E$, we define the directional derivative of g at x in the direction of y by

$$g^0(x, y) = \lim_{t\to 0^+} \frac{g(x + ty) - g(x)}{t}.$$

The function g is said to be

(i) Gâteaux differentiable at x if $\lim_{t\to 0^+}\frac{g(x+ty)-g(x)}{t}$ exists for any y. At this time, the gradient of g at x is the linear function $\nabla_E^g(x)$ satisfying

$$\langle \nabla_E^g(x), y\rangle := g^0(x,y),\ \forall\ y \in E.$$

(ii) Gâteaux differentiable, if it is Gâteaux differentiable for any $x \in int(domg)$; where int(domg) stands for the interior of domain of g.

(iii) Fréchet differentiable at x, if its limit is attained uniformly in $||y|| = 1$;

(iv) Uniformly Fréchet differentiable on a subset Q of E, if the above limit is attained uniformly for $x \in Q$ and $||y|| = 1$.

Let $g : E \to (-\infty, +\infty]$ be a function, then g is said to be:

(i) essentially smooth, if the subdifferential of g denoted by ∂g is both locally bounded and single-valued on its domain, where $\partial g(x) = \{x^* \in E^* : g(x) + \langle y - x, x^*\rangle \leq g(y),\ y \in E\}$;

(ii) essentially strictly convex, if $(\partial g)^{-1}$ is locally bounded on its domain and g is strictly convex on every convex subset of $dom\ \partial g$;

(iii) Legendre, if it is both essentially smooth and essentially strictly convex. See [30, 31] for more details on Legendre functions.

Alternatively, a function g is said to be Legendre if it satisfies the following conditions:

(i) The $int(domg)$ is nonempty, g is Gâteaux differentiable on $int(dom)g$ and $dom\nabla g = int(domg)$;

(ii) The $int(domg^*)$ is nonempty, g^* is Gâteaux differentiable on $int(domg^*)$ and $dom\nabla_{E^*}^{g^*} = int(domg^*)$.

Definition 1. *[32] Let E be a Banach space. A function $g : E \to (-\infty, \infty]$ is said to be proper if the interior of its domain $dom(g)$ is nonempty. Let $g : E \to (-\infty, \infty]$ be a convex and Gâteaux differentiable function. Then the Bregman distance corresponding to g is the function $D_g : dom(g) \times intdom(g) \to \mathbb{R}$ defined by*

$$D_g(x,y) := g(x) - g(y) - \langle x - y, \nabla_E^g(y)\rangle,\ \forall\ x,y \in E. \tag{9}$$

is called the Bregman distance with respect to g. It is clear that $D_g(x,y) \geq 0$ for all $x,y \in E$.

It is well-known that Bregman distance D_g does not satisfy the properties of a metric because D_g fail to satisfy the symmetric and triangular inequality property. However, the Bregman distance satisfies the following so-called three point identity: for any $x \in domg$ and $y, z \in int(domg)$,

$$D_g(x,z) = D_g(x,y) + D_g(y,z) + \langle x - y, \nabla_E^g(y) - \nabla_E^g(z)\rangle. \tag{10}$$

In particular,

$$D_g(x,y) = -D_g(y,x) + \langle y - x, \nabla_E^g(y) - \nabla_E^g(x)\rangle,\ \forall\ x, y \in E.$$

Let $B : E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of B by $domB = \{x \in E : Bx \neq \emptyset\}$ and $ranB = \bigcup_{x\in E} Bx$, respectively. The graph of B denoted by $G(B) = \{(x, x^*) \in E \times E^* : x^* \in Bx\}$. The mapping $B \subset E \times E^*$ is said to be monotone [33] if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in B$. It is also said to be maximal monotone [34] if its graph is not contained in the graph of any other monotone operator on E. If $B \subset E \times E^*$ is maximal monotone, then we can represent the set $B^{-1}(0) = \{z \in E : 0 \in Bz\}$ is closed and convex.
Let $A : E \to 2^{E^*}$ be a mapping, then the resolvent associated with A and λ for any $\lambda > 0$ is the mapping $Res^g_{\lambda A} : E \to 2^E$ defined by

$$Res^g_{\lambda A} := (\nabla_E^g + \lambda A)^{-1} \circ \nabla_E^g.$$

It is worth mentioning that a mapping $A : E \to 2^{E^*}$ is called Bregman inverse strongly monotone (BISM) on the set C if

$$C \cap (domg) \cap (int\ dom\ g) \neq \emptyset,$$

and for any $x, y \in C \cap (int\ dom\ g),\ \eta \in Ax$ and $\xi \in Ay$, we have

$$\langle \eta - \xi, (\nabla_{E^*}^{g^*}(x) - \eta) - \nabla_{E^*}^{g^*}(\nabla_E^g(y) - \xi)\rangle \geq 0.$$

The anti-resolvent $A^g_\lambda : E \to 2^E$ associated with the mapping $A : E \to 2^{E^*}$ and $\lambda > 0$ is defined by

$$A^g_\lambda := (\nabla_E^g)^{-1} \circ (\nabla_E^g - \lambda A). \tag{11}$$

A point $p \in Q$ is called an asymptotic fixed point of T if Q contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$. We denote by $\hat{Fix}(T)$ the set of asymptotic fixed points of T.
Let Q be a nonempty closed and convex subset of int(dom g), then we define an operator $T : Q \to int(domg)$ to be :

(i) Bregman relatively nonexpansive (BRNE), if $Fix(T) \neq \emptyset$, and

$$D_g(p, Tx) \leq D_g(p, x),\ \forall\ p \in Fix(T),\ x \in Q \text{ and } \hat{Fix}(T) = Fix(T).$$

(ii) Bregman quasi-nonexpansive mapping (BQNE), if $Fix(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x), \forall\ x \in Q \text{ and } p \in Fix(T).$$

(iii) Bregman firmly nonexpansive (BFNE), if

$$\langle \nabla_E^g(Tx) - \nabla_E^g(Ty), Tx - Ty\rangle \leq \langle \nabla_E^g(x) - \nabla_E^g(y), Tx - Ty\rangle,\ \forall\ x, y \in E.$$

Definition 2. *[35] Let C be a nonempty, closed and convex subset of a reflexive Banach space E and $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let β and γ be real numbers with $\beta \in (-\infty, 1)$ and $\gamma \in [0, \infty)$, respectively. Then a mapping $T : C \to E$ with $Fix(T) \neq \emptyset$ is called Bregman (β, γ)-demigeneralized if for any $x \in C$ and $p \in Fix(T)$,*

$$\langle x - p, \nabla_E^g(x) - \nabla_E^g(Tx)\rangle \geq (1 - \beta)D_g(x, Tx) + \gamma D_g(Tx, x),\ \forall\ x \in E \text{ and } p \in F(T).$$

Definition 3. *A function $g : E \to \mathbb{R}$ is said to be strongly coercive if*

$$\lim_{||x_n|| \to \infty} \frac{g(x_n)}{||x_n||} = \infty.$$

Lemma 1. *[19] Let E be a Banach space, $s > 0$ be a constant, ρ_s be the gauge of uniform convexity of g and $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then,*
(i) For any $x, y \in B_s$ and $\alpha \in (0, 1)$, we have

$$D_g\big(x, \nabla_{E^*}^{g^*}[\alpha \nabla_E^g \nabla_E^g(y) + (1 - \alpha)\nabla_E^g(z)]\big) \leq \alpha D_g(x, y) + (1 - \alpha)D_g(x, z) - \alpha(1 - \alpha)\rho_s(||\nabla_E^g(y) - \nabla_E^g(z)||),$$

(ii) For any $x, y \in B_s$,

$$\rho_s(||x - y||) \leq D_g(x, y).$$

Lemma 2. *[36] Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ be a strongly coercive Bregman function and V be a function defined by*

$$V(x, x^*) = g(x) - \langle x, x^*\rangle + g^*(x^*),\ x \in E,\ x^* \in E^*.$$

The following assertions also hold:

$$D_g(x, \nabla_{E^*}^{g^*}(x^*)) = V(x, x^*),\ \text{for all } x \in E \text{ and } x^* \in E^*.$$

$$V(x, x^*) + \langle \nabla_{E^*}^{g^*}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*) \text{ for all } x \in E \text{and } x^*, y^* \in E^*.$$

Lemma 3. *[35] Let E_1 and E_2 be two Banach spaces. Let $F : E_1 \to E_2$ be a bounded linear operator and $T : E_2 \to E_2$ be a Bregman (ϕ, σ)-demigeneralized for some $\phi \in (-\infty, 1)$ and $\sigma \in [0, \infty)$. Suppose that $K = ran(A) \cap Fix(T) \neq \emptyset$ (where $ran(B)$ denotes the range of B). Then for any $(x, q) \in E_1 \times K$,*

$$\begin{aligned}\langle x - q, F^*(\nabla^{g_2}_{E_2}(T(Fx)))\rangle &\geq (1-\phi)D_{g_2}(Fx, T(Fx)) + \sigma D_{g_2}(T(Fx), Fx) \\ &\geq (1-\phi)D_{g_2}(Fx, T(Fx)). \end{aligned} \tag{12}$$

So, given any real numbers ξ_1 and ξ_2, the mapping $L_1 : E_1 \to [0, \infty)$ and $L_2 : E_2 \to [0.\infty)$ formulated for $x \in E_1$ as

$$L_1(x) = \begin{cases} \frac{D_{g_2}(Fx, TFx)}{D^*_{g_1}(F^*(\nabla^{g_2}_{E_2}(Fx)), F^*(\nabla^{g_2}_{E_2}(TFx))}, & \textit{if}\ , \quad (I-T)Fx \neq 0, \\ \xi_1, & \textit{otherwise}, \end{cases} \tag{13}$$

and

$$L_2(x) = \begin{cases} \frac{D^*_{g_1}(\nabla^{g_1}_{E_1}(x) - \gamma F^*(\nabla^{g_2}_{E_2}(Fx) - \nabla^{g_2}_{E_2}(TFx)), \nabla^{g_1}_{E_1}(x))}{D^*_{g_1}(F^*(\nabla^{g_2}_{E_2}(Fx)), F^*(\nabla^{g_2}_{E_2}(TFx))}, & \textit{if}\ , \quad (I-T)Fx \neq 0, \\ \xi_2, & \textit{otherwise}, \end{cases} \tag{14}$$

are well-defined, where γ is any nonnegative real number. Moreover, for any $(x, p) \in E_1 \times K$, we have

$$D_{g_1}(q, y) \leq D_{g_1}(q, x) - (\gamma(1-\phi)L_1(x) - L_2(x))D_{g_1^*}(F^*(\nabla^{g_2}_{E_2}(Fx)), F^*(\nabla^{g_2}_{E_2}(TFx)), \tag{15}$$

where

$$y = (\nabla^{g_1}_{E_1})^{-1}[\nabla^{g_1}_{E_1}(x) - \gamma F^*(\nabla^{g_2}_{E_2}(Fx) - \nabla^{g_2}_{E_2}(TFx))].$$

Lemma 4. *[36] Let E be a Banach space and $g : E \to \mathbb{R}$ a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then,*

$$\lim_{n\to\infty} D_g(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} ||y_n - x_n|| = 0.$$

Lemma 5. *[37] Let $g : E \to (-\infty, +\infty]$ be a Legendre function. Let $\{T_i\}_{i=1}^N : E \to E$ be a BQNE such that $\bigcap\limits_{i=1}^N Fix(T_i) \neq \emptyset$ and $\{\beta_i\}_{i=0}^N \subset (0, 1)$ satisfy $\sum\limits_{i=0}^N \beta_i = 1$. Define a mapping $S : E \to E$ by $Sx := (\nabla^g_E)^{-1}(\beta_0 \nabla^g_E(x) + \sum\limits_{i=1}^N \beta_i \nabla^g_E(T_i x))$ for all $x \in E$. Then S is BQNE such that $Fix(S) = \bigcap\limits_{i=1}^N Fix(T_i)$.*

Lemma 6. *[38] Let $B : E \to 2^{E^*}$ be a maximal monotone operator and $A : E \to E^*$ be a BISM mapping such that $(A + B)^{-1}(0^*) \neq \emptyset$. Let $g : E \to \mathbb{R}$ be a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subset of E. Then,*

$$D_g(u, Res^g_{\lambda B} \circ A^g(x)) + D_g(Res^g_{\lambda B}(x), x) \leq D_g(u, x), \text{ for any } u \in (A + B)^{-1}(0^*), \ x \in E \text{ and } \lambda > 0.$$

Lemma 7. *[38] Let $B : E \to 2^{E^*}$ be a maximal monotone operator and $A : E \to E^*$ be a BISM mapping such that $(A + B)^{-1}(0^*) \neq \emptyset$. Let $g : E \to \mathbb{R}$ be a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subset of E. Then,*
(i) $(A + B)^{-1}(0^*) = Fix(Res^g_{\lambda B} \circ A^g_\lambda)$;
(ii) $Res^g_{\lambda B} \circ A^g_\lambda$ is a BSNE operator with $Fix(Res^g_{\lambda B} \circ A^g_\lambda) = \hat{Fix}(Res^g_{\lambda B} \circ A^g_\lambda)$.

Lemma 8. *[39] Let $g : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_g(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Definition 4. *Let C be a nonempty closed and convex subset of a reflexive Banach space E and $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. A Bregman projection of $x \in int(domg)$ onto $C \subset int(domg)$ is the unique vector $Proj^g_C(x) \in C$ satisfying*

$$D_g(Proj^g_C(x), x) = int\{D_g(y, x) : y \in C\}.$$

Lemma 9. *[40] Let C be a nonempty closed and convex subset of a reflexive Banach space E and $x \in E$. Let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then,*
(i) $z = Proj^g_C(x)$ if and only if $\langle \nabla^g_E(x) - \nabla^g_E(z), y - z \rangle \leq 0, \ \forall \ y \in C$.
(ii) $D_g(y, Proj^g_C(x)) + D_g(Proj^g_C(x), x) \leq D_g(y, x), \ \forall \ y \in C$.

Lemma 10. *[41] Let $\{a_n\}$ and $\{d_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + d_n, \ n \geq 1,$$

where $\{\delta_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a real sequence. Assume that $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\delta_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 11. *[42] Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_j} < \Gamma_{n_{j+1}}$ for all $j \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) := \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq \cdots$ and $\tau(n) \to \infty$,

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \ \forall \ n \geq n_0$.

3. Main Result

Throughout this section, we assume that

Assumption 1.

(i) *Let E_i for $i = 0, 1, 2, \cdots, N$ be reflexive Banach spaces where $E_0 = E$, $g : E \to (-\infty, +\infty]$ and $g_i : E_i \to (-\infty, +\infty]$ be strongly coercive Legendre functions which are bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E and E_i, $i = 1, 2, \cdots, N$, respectively. Let ∇_E^g and $\nabla_{E_i}^{g_i}$ be the gradients of E dependent on g and E_i dependent on g_i respectively.*

(ii) *Let $F_j : E \to E^*, j = 1, 2, \cdots, m$ be BISM mappings and $G_j : E \to E^*, j = 1, 2, \cdots, m$ be maximal monotone mappings respectively. Suppose $A_i : E \to E_i, i = 1, 2, \cdots, N$ be bounded linear operator such that $A_i \neq 0$ and A_i^* be the adjoint of A_i.*

(iii) *$S_i : E_i \to E_i$, $i = 0, 1, 2, \cdots, N$ be Bregman $(\rho_S, \mu_S)-$ demigeneralized mapping such that $\rho_S \in (-\infty, 1)$ and $\mu_S \in [0, \infty)$. Assume that $\Omega := \big\{x^* \in \bigcap_{j=1}^{m} Fix(T_\sigma^j) \cap Fix(S) : A_i x^* \in \bigcap_{i=1}^{N} Fix(S_i)\big\} \neq \emptyset$,*

(iv) *Let $\gamma > 0$ be a real number and $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_j\}_{j=0}^m$ and $\{\lambda_{i,n}\}_{n\in\mathbb{N}}$ be sequences in (0,1) with $\sum_{j=0}^{m} \beta_j = 1$ and $\sum_{i=0}^{N} \lambda_{i,n} = 1$ respectively, satisfying the following control condition:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Let $T_\sigma^j := Res_{\sigma G_j}^g \circ F_j^g$ for $j = 1, 2, \cdots, m$, clearly $Fix(T_\sigma^j) = (F_j + G_j)^{-1}(0)$ for each $j = 1, 2, \cdots, m$ and $\sigma > 0$. Define the sequence $\{x_n\}$ by the following recursive formula:

Algorithm 1. *For fixed $u \in E$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $x_1 \in E$ such that*

$$\begin{cases} z_n = (\nabla_E^g)^{-1}\big[\sum_{i=0}^{N} \lambda_{i,n}\big(\nabla_E^g(x_n) - \gamma A_i^*(\nabla_{E_i}^{g_i}(A_i x_n) - \nabla_{E_i}^{g_i}(S_i A_i x_n))\big)\big] \\ y_n = (\nabla_E^g)^{-1}\big[(\beta_0 \nabla_E^g(z_n) + \sum_{j=1}^{m} \beta_j \nabla_E^g(T_\sigma^j z_n)\big] \\ x_{n+1} = (\nabla_E^g)^{-1}\big[\alpha_n \nabla_E^g(u) + (1-\alpha_n)\nabla_E^g(y_n)\big]. \end{cases} \tag{16}$$

Suppose $\{\xi_{1,n}\}_{n\in\mathbb{N}}$ and $\{\xi_{2,n}\}_{n\in\mathbb{N}}$ are two sequences, where

$$\xi_{1,n} = \begin{cases} \frac{D_{g_i}(A_i x_n, S_i A_i x_n)}{D_g^*(A_i^*(\nabla_{E_i}^{g_i}(A_i x_n)), A_i^*(\nabla_{E_i}^{g_i}(S_i A_i x_n)))}, & \text{if }, \quad (I - S_i)A_i x_n \neq 0, \\ \xi_1, & \text{otherwise}, \end{cases}$$

and

$$\xi_{2,n} = \begin{cases} \frac{D_g^*(\nabla_E^g(x_n) - \gamma A_i^*(\nabla_{E_i}^{g_i}(A_i x_n) - \nabla_{E_i}^{g_i}(S_i A_i x_n)), \nabla_E^g(x_n))}{D_g^*(A_i^*(\nabla_{E_i}^{g_i}(A_i x_n)), A_i^*(\nabla_{E_i}^{g_i}(S_i A_i x_n))}, & if, \quad (I - S_i)A_i x_n \neq 0, \\ \xi_2, & otherwise. \end{cases}$$

Then, the sequence $\{x_n\}$ *defined in* (16) *converges strongly to* $v = Proj_\Omega^g u$, *where* $Proj_\Omega^g$ *is the Bregman projection of* E *onto* Ω.

Proof. Let $V_\sigma = \beta_0 \nabla_E^g + \beta_1 \nabla_E^g(Res_{\sigma G_1}^g \circ F_1^g) + \beta_2 \nabla_E^g(Res_{\sigma G_2}^g \circ F_2^g) + \cdots + \beta_j \nabla_E^g(Res_{\sigma G_m}^g \circ F_m^g)$, then $y_n = V_\sigma z_n$. By applying Lemma 5 and using the fact that T_σ^j is BQNE then we have that $Fix(V_\sigma) = \bigcap_{j=1}^m Fix(T_\sigma^j) = \bigcap_{j=1}^m (F_j + G_j)^{-1}(0)$. Let $v \in \Omega$, then we obtain from Lemma 3 that

$$\begin{aligned} D_g(v, z_n) &= D_g(v, (\nabla_E^g)^{-1}\big[\sum_{i=0}^N \lambda_{i,n}(\nabla_E^g(x_n) - \gamma A_i^*(\nabla_{E_i}^{g_i}(A_i x_n) - \nabla_{E_i}^{g_i}(S_i A_i x_n)))\big] \\ &\leq D_g(v, x_n) - \sum_{i=0}^N \lambda_{i,n}(\gamma(1 - \rho_S)\xi_{1,n} - \xi_{2,n}) D_g^*(A_i^*(\nabla_{E_i}^{g_i}(A_i x_n)), A_i^*(\nabla_{E_i}^{g_i}(S_i A_i x_n)) \qquad (17) \\ &\leq D_g(v, x_n). \qquad (18) \end{aligned}$$

It follows from (16) and (18) that

$$\begin{aligned} D_g(v, y_n) &= D_g(v, (\nabla_E^g)^{-1}\big[\beta_0 \nabla_E^g(z_n) + \sum_{j=1}^m \beta_j \nabla_E^g(T_\sigma^j z_n)\big]) \\ &\leq \beta_0 D_g(v, z_n) + \sum_{j=1}^m \beta_j D_g(v, T_\sigma^j z_n) \\ &\leq \beta_0 D_g(v, z_n) + \sum_{j=1}^m \beta_j D_g(v, z_n) \\ &= D_g(v, z_n) \qquad (19) \\ &\leq D_g(v, x_n). \qquad (20) \end{aligned}$$

Using (16), (18) and (19), we get

$$\begin{aligned} D_g(v, x_{n+1}) &= D_g(v, (\nabla_E^g)^{-1}\big[\alpha_n \nabla_E^g(u) + (1 - \alpha_n)\nabla_E^g(y_n)\big]) \\ &\leq \alpha_n D_g(v, u) + (1 - \alpha_n) D_g(v, y_n) \qquad (21) \\ &\leq \alpha_n D_g(v, u) + (1 - \alpha_n) D_g(v, z_n) \\ &\leq \alpha_n D_g(v, u) + (1 - \alpha_n) D_g(v, x_n) \\ &\leq \max\{D_g(v, u), D_g(v, x_n)\} \end{aligned}$$

$$\vdots$$
$$\leq \max\{D_g(v,u), D_g(v,x_1)\}. \ \forall \ n \geq 1.$$

Thus, we obtain that the sequence $\{D_g(v,x_n)\}_{n\in\mathbb{N}}$ is bounded. Using Lemma 8, then we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Consequently, $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ are bounded. By Lemma 6, (16) and (20), we obtain that

$$\begin{aligned}
D_g(v,y_n) &= D_g(v,(\nabla_E^g)^{-1}[\beta_0\nabla_E^g(z_n) + \sum_{j=1}^m \beta_j\nabla_E^g(T_\sigma^j z_n)]) \\
&\leq \beta_0 D_g(v,z_n) + \sum_{j=1}^m \beta_j D_g(v,T_\sigma^j z_n) \\
&\leq \beta_0 D_g(v,z_n) + \sum_{j=1}^m \beta_j\big(D_g(v,z_n) - D_g(T_\sigma^j z_n,z_n)\big) \\
&= D_g(v,z_n) - \sum_{j=1}^m \beta_j D_g(T_\sigma^j z_n,z_n) \qquad (22) \\
&\leq D_g(v,x_n) - \sum_{j=1}^m \beta_j D_g(T_\sigma^j z_n,z_n) \qquad (23)
\end{aligned}$$

From (17), (21) and (22), we get

$$\begin{aligned}
D_g(v,x_{n+1}) &\leq \alpha_n D_g(v,u) + (1-\alpha_n)D_g(v,y_n) \\
&\leq \alpha_n D_g(v,u) + (1-\alpha_n)\big(D_g(v,z_n) - \sum_{j=1}^m \beta_j D_g(T_\sigma^j z_n,z_n)\big) \\
&= \alpha_n D_g(v,u) + (1-\alpha_n)D_g(v,x_n) - (1-\alpha_n)\sum_{j=1}^m \beta_j D_g(T_\sigma^j z_n,z_n) \\
&- (1-\alpha_n)\sum_{i=0}^N \lambda_{i,n}(\gamma(1-\rho_S)\xi_{1,n} - \xi_{2,n})D_g^*(A_i^*(\nabla_{E_i}^{g_i}(A_i x_n)), A_i^*(\nabla_{E_i}^{g_i}(S_i A_i x_n)).
\end{aligned} \qquad (24)$$

We now divide the remaining proof into two cases.
Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_g(v,x_n)\}$ is non-increasing, then we obtain that $\lim_{n\to\infty} D_g(v,x_n)$ exists. Thus,

$$D_g(v,x_n) - D_g(v,x_{n+1}) \to 0, \ n\to\infty. \qquad (25)$$

From (24), 25 and condition (i) of Assumption (1), we have that

$$(1-\alpha_n)\big[\sum_{i=0}^N \lambda_{i,n}(\gamma(1-\rho_S)\xi_{1,n} - \xi_{2,n})D_g^*(A_i^*(\nabla_{E_i}^{g_i}(A_i x_n)), A_i^*(\nabla_{E_i}^{g_i}(S_i A_i x_n))$$

$$+\sum_{j=1}^{m}\beta_j D_g(T_\sigma^j z_n, z_n))\big] \leq \alpha_n D_g(v,u) + (1-\alpha_n)D_g(v,x_n) - D_g(v,x_{n+1}),$$

which implies from Lemma 4 that

$$\lim_{n\to\infty} D_g(T_\sigma^j z_n, z_n) = 0 = \lim_{n\to\infty} ||T_\sigma^j z_n - z_n||. \tag{26}$$

Also,

$$\lim_{n\to\infty}\sum_{i=0}^{N}\lambda_{i,n}(\gamma(1-\rho_S)\xi_{1,n}-\xi_{2,n})D_g^*(A_i^*(\nabla_{E_i}^{g_i}(A_i x_n)), A_i^*(\nabla_{E_i}^{g_i}(S_i A_i x_n)) = 0. \tag{27}$$

Therefore, we have

$$\lim_{n\to\infty} D_g^*(A_i^*(\nabla_{E_i}^{g_i}(A_i x_n)), A_i^*(\nabla_{E_i}^{g_i}(S_i A_i x_n)) = 0. \tag{28}$$

Hence, by applying Lemma 4, (12) and the properties of D_g^* and A, we get

$$\lim_{n\to\infty} ||A_i x_n - S_i A_i x_n|| = 0,\ i = 0,1,2,\cdots,N. \tag{29}$$

In view of (16), (26), (28) and Lemma 4, we obtain that

$$\lim_{n\to\infty} D_g(z_n, x_n) = 0 = \lim_{n\to\infty} ||z_n - x_n||, \tag{30}$$

and

$$\lim_{n\to\infty} D_g(y_n, z_n) = 0 = \lim_{n\to\infty} ||y_n - z_n|| = 0. \tag{31}$$

By applying (30) and (31) we obtain

$$\lim_{n\to\infty} D_g(y_n, x_n) = 0 = \lim_{n\to\infty} ||y_n - x_n||. \tag{32}$$

More so, employing condition (i) of Assumption 1 and Lemma 4, we arrive at

$$\lim_{n\to\infty} D_g(x_{n+1}, y_n) = 0 = \lim_{n\to\infty} ||x_{n+1} - y_n||. \tag{33}$$

We therefore conclude from (32) and (33) that

$$\lim_{n\to\infty} D_g(x_{n+1}, x_n) = 0 = \lim_{n\to\infty} ||x_{n+1} - x_n||. \tag{34}$$

Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \rightharpoonup x^*$. Also, from (30) and (32), there exist subsequences $\{z_{n_k}\}$ of $\{z_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ which converge weakly to x^* respectively. Thus, for each $i = 0,1,2,\cdots N$, A_i is a bounded linear operator, then it follows that $A_i x_{n_k} \rightharpoonup A_i x^*$. Hence, using the demi-closedness principle and (29), we arrive at $A_i x^* \in Fix(S_i)$ for all $i = 0,1,2,\cdots N$. Also,

from (26), we obtain that $x^* \in \hat{Fix}(T_\sigma^j) = Fix(T_\sigma^j)$ for each $j = 1, 2, \cdots m$. This implies from Lemma 7 that $x^* \in \bigcap_{j=1}^{m} Fix(T_\sigma^j) = \bigcap_{j=1}^{m} (F_j + G_j)^{-1}(0)$. Therefore, we conclude that $x^* \in \Omega$.
Next is to show that

$$\langle \nabla_E^g(u) - \nabla_E^g(z), x_{n+1} - z \rangle \leq 0.$$

Now, from (34), we have

$$\begin{aligned}\limsup_{n\to\infty} \langle \nabla_E^g(u) - \nabla_E^g(z), x_{n+1} - z \rangle &= \lim_{k\to\infty} \langle \nabla_E^g(u) - \nabla_E^g(z), x_{n_k+1} - z \rangle \\ &\leq \langle \nabla_E^g(u) - \nabla_E^g(z), x^* - z \rangle.\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}\limsup_{k\to\infty} \langle \nabla_E^g(u) - \nabla_E^g(z), x_{n+1} - z \rangle &\leq \langle \nabla_E^g(u) - \nabla_E^g(z), x^* - z \rangle \\ &\leq 0. \end{aligned} \tag{35}$$

Next is to prove that $\{x_n\}$ converges strongly to $v \in \Omega$. Using Lemma 2, (18) and (20),

$$\begin{aligned}D_g(v, x_{n+1}) &= D_g(v, (\nabla_E^g)^{-1} \big[\alpha_n \nabla_E^g(u) + (1-\alpha_n)\nabla_E^g(y_n)\big]) \\ &= V_g(v, \alpha_n \nabla_E^g(u) + (1-\alpha_n)\nabla_E^g(y_n)) \\ &\leq V_g(v, \alpha_n \nabla_E^g(u) + (1-\alpha_n)\nabla_E^g(y_n) - \alpha_n(\nabla_E^g(u) - \nabla_E^g(v)) \\ &+ \langle \alpha_n(\nabla_E^g(u) - \nabla_E^g(v)), x_{n+1} - v) \rangle \\ &= V_g(v, \alpha_n \nabla_E^g(v) + (1-\alpha_n)\nabla_E^g(y_n)) + \alpha_n \langle \nabla_E^g(u) - \nabla_E^g(v), x_{n+1} - v \rangle \\ &\leq \alpha_n V_g(v, \nabla_E^g(v)) + (1-\alpha_n) V_g(v, \nabla_E^g(y_n)) + \alpha_n \langle \nabla_E^g(u) - \nabla_E^g(v), x_{n+1} - v \rangle \\ &= \alpha_n D_g(v, v) + (1-\alpha_n) D_g(v, y_n) + \alpha_n \langle \nabla_E^g(u) - \nabla_E^g(v), x_{n+1} - v \rangle \\ &\leq (1-\alpha_n) D_g(v, x_n) + \alpha_n \langle \nabla_E^g(u) - \nabla_E^g(v), x_{n+1} - v \rangle. \end{aligned} \tag{36}$$

In view of Lemma 10 and (35), we conclude that $\lim_{n\to\infty} D_g(v, x_n) = 0$. Therefore $\{x_n\}$ converges strongly to v.
Case 2: Suppose that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $D_g(v, x_{n_k}) < D_g(v, x_{n_{k+1}})$ for all $k \in \mathbb{N}$. We define a positive integer sequence $\{\tau(n)\}$ by

$$\tau(n) := \max\{k \in n : D_g(v, x_k) < D_g(v, x_{k+1})\}$$

for all $n \geq n_0$ (for some n_0 large enough). Applying Lemma 11, we have $\{\tau(n)\}$ to be non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$D_g(v, x_{\tau(n)}) - D_g(v, x_{\tau(n)+1}) \leq 0.$$

Following the same argument to the one used in Case 1 of the proof of (16), we obtain that

$$\begin{cases} \lim\limits_{\tau(n)\to\infty} D_g(T_\sigma^j z_{\tau(n)}, z_{\tau(n)}) = 0, \text{ for } j = 1, 2, \cdots, m, \\ \lim\limits_{\tau(n)\to\infty} ||A_i x_{\tau(n)} - S_i A_i x_{\tau(n)}|| = 0, \text{ for } i = 0, 1, 2, \cdots, N, r \\ \lim\limits_{\tau(n)\to\infty} D_g(z_{\tau(n)}, x_{\tau(n)}) = 0, \\ \lim\limits_{\tau(n)\to\infty} D_g(y_{\tau(n)}, x_{\tau(n)}) = 0, \\ \lim\limits_{\tau(n)\to\infty} \langle \nabla_E^g(u) - \nabla_E^g(v), x_{\tau(n)+1} - v\rangle \leq 0. \end{cases} \tag{37}$$

and

$$D_g(v, x_{\tau(n)+1}) \leq (1 - \alpha_{\tau(n)}) D_g(v, x_{\tau(n)}) + \alpha_{\tau(n)} \langle \nabla_E^g(u) - \nabla_E^g(v), x_{\tau(n)+1} - v\rangle.$$

Using Lemma 11, we arrive at

$$D_g(v, x_{\tau(n)}) \leq D_g(v, x_{\tau(n)+1}).$$

Hence, we conclude that $\lim\limits_{n\to\infty} D_g(v, x_n) = 0$. Therefore, $\{x_n\}$ converges strongly to v. This completes the proof of our theorem.

If we put $m = 1$, then we have the following iterative method which solves $\Omega := \{x^* \in (F+G)^{-1}(0) \cap Fix(S) : A_i x^* \in \bigcap_{i=1}^{N} Fix(S_i)\} \neq \emptyset$.

Corollary 1.

Algorithm 2. *For fixed $u \in E$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $x_1 \in E$ such that*

$$\begin{cases} z_n = (\nabla_E^g)^{-1}\big[\sum_{i=0}^{N} \lambda_{i,n}\big(\nabla_E^g(x_n) - \gamma A_i^*(\nabla_{E_i}^{g_i}(A_i x_n) - \nabla_{E_i}^{g_i}(S_i A_i x_n))\big)\big] \\ y_n = (\nabla_E^g)^{-1}\big[(\beta_n \nabla_E^g(z_n) + (1-\beta_n)\nabla_E^g(Res_{\sigma G}^g \circ F^g)\big] \\ x_{n+1} = (\nabla_E^g)^{-1}\big[\alpha_n \nabla_E^g(u) + (1-\alpha_n)\nabla_E^g(y_n)\big]. \end{cases} \tag{38}$$

where $0 < a \leq \beta_n \leq b < 1$. Suppose $\{\xi_{1,n}\}_{n\in\mathbb{N}}$ and $\{\xi_{2,n}\}_{n\in\mathbb{N}}$ are two sequences, where

$$\xi_{1,n} = \begin{cases} \frac{D_{g_i}(A_i x_n, S_i A_i x_n)}{D_g^*(A_i^*(\nabla_{E_i}^{g_i}(A_i x_n)), A_i^*(\nabla_{E_i}^{g_i}(S_i A_i x_n)))}, & if, \quad (I - S_i)A_i x_n \neq 0, \\ \xi_1, & otherwise, \end{cases}$$

and

$$\xi_{2,n} = \begin{cases} \frac{D_g^*(\nabla_E^g(x_n) - \gamma A_i^*(\nabla_{E_i}^{g_i}(A_i x_n) - \nabla_{E_i}^{g_i}(S_i A_i x_n)), \nabla_E^g(x_n))}{D_g^*(A_i^*(\nabla_{E_i}^{g_i}(A_i x_n)), A_i^*(\nabla_{E_i}^{g_i}(S_i A_i x_n)))}, & if, \quad (I - S_i)A_i x_n \neq 0, \\ \xi_2, & otherwise. \end{cases}$$

Then, the sequence $\{x_n\}$ defined in (38) converges strongly to $v = Proj_{\Omega}^g u$, where $Proj_{\Omega}^g$ is the Bregman projection of E onto Ω.

Here we consider the split common fixed point problem of Bregman demigeneralized mapping which is defined as $\Omega := \{x^* \in Fix(S) : A_i x^* \in \bigcap_{i=1}^{N} Fix(S_i)\} \neq \emptyset$.

Corollary 2.

Algorithm 3. *For fixed $u \in E$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $x_1 \in E$ such that*

$$\begin{cases} z_n = (\nabla_E^g)^{-1}\big[\sum_{i=0}^{N} \lambda_{i,n}\big(\nabla_E^g(x_n) - \gamma A_i^*(\nabla_{E_i}^{g_i}(A_i x_n) - \nabla_{E_i}^{g_i}(S_i A_i x_n))\big)\big] \\ x_{n+1} = (\nabla_E^g)^{-1}\big[\alpha_n \nabla_E^g(u) + (1-\alpha_n)\nabla_E^g(z_n)\big]. \end{cases} \tag{39}$$

Suppose $\{\xi_{1,n}\}_{n\in\mathbb{N}}$ and $\{\xi_{2,n}\}_{n\in\mathbb{N}}$ are two sequences, where

$$\xi_{1,n} = \begin{cases} \frac{D_{g_i}(A_i x_n, S_i A_i x_n)}{D_g^*(A_i^*(\nabla_{E_i}^{g_i}(A_i x_n)), A_i^*(\nabla_{E_i}^{g_i}(S_i A_i x_n)))}, & if, \quad (I - S_i)A_i x_n \neq 0, \\ \xi_1, & otherwise, \end{cases}$$

and

$$\xi_{2,n} = \begin{cases} \frac{D_g^*(\nabla_E^g(x_n) - \gamma A_i^*(\nabla_{E_i}^{g_i}(A_i x_n) - \nabla_{E_i}^{g_i}(S_i A_i x_n)), \nabla_E^g(x_n))}{D_g^*(A_i^*(\nabla_{E_i}^{g_i}(A_i x_n)), A_i^*(\nabla_{E_i}^{g_i}(S_i A_i x_n)))}, & if, \quad (I - S_i)A_i x_n \neq 0, \\ \xi_2, & otherwise. \end{cases}$$

Then, the sequence $\{x_n\}$ defined in (39) converges strongly to $v = Proj_{\Omega}^g u$, where $Proj_{\Omega}^g$ is the Bregman projection of E onto Ω.

4. Numerical Example

In this section, we give a numerical example to illustrate the performance of our method.
Example 1: Let $E = E_i = \mathbb{R}^4$ for $i = 1, 2$. We define $h_m : \mathbb{R} \to (-\infty, +\infty]$ by $h_m(x) = \frac{1}{2}x^2, \quad m = 1, 2, 3, 4$. Also, let $g = g_i$ for $i = 1, 2$ be defined by $g : \mathbb{R}^2 \to (-\infty, +\infty]$, $g(x) = h_1(x) + h_2(x) + h_3(x) + h_4(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + \frac{1}{2}x_4^2$. Then, we have

$$\nabla g(x) = (\nabla h(x_1)), \nabla h(x_2), \nabla h(x_3), \nabla h(x_4) = (x_1, x_2, x_3, x_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

For $i = 0, 1, 2$, let $A_i : \mathbb{R} \to \mathbb{R}$ be defined by $A_i(x) = \frac{x}{(i+1)}$ for $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. We also define the mapping $S_i : \mathbb{R} \to \mathbb{R}$ by $S_i(x) = -(i + 1)x$ for each $i = 0, 1, 2$. Then the mappings S_i are $\left(-\frac{1}{i+1}, 0\right)$-Bregman demigeneralized. Now, define the mappings $F_1, F_2, F_3 : \mathbb{R} \to \mathbb{R}$ respectively by

$$F_1 = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 0 & 3 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 5 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix}$$

and the mappings $G_1, G_2, G_3 : \mathbb{R} \to \mathbb{R}$ respectively by

$$G_1 = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 1 & 2 & -2 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 3 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & -2 & -1 & 2 \\ 0 & 0 & 1 & 3 \\ -1 & 2 & -3 & 4 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 1 & 2 & 0 & 1 \\ 1 & 3 & 0 & 2 \end{pmatrix}.$$

It is easy to see for any $\lambda > 0$, that

$$\begin{aligned} T_1(x) &= (\nabla_E^g + \lambda G_1) \circ \nabla_E^g \circ (\nabla_E^g)^{-1} (\nabla_E^g - \lambda F_1)(x) \\ &= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \lambda \times \begin{pmatrix} 1 & 1 & 0 & -2 \\ 1 & 2 & -2 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 3 \end{pmatrix} \right]^{-1} \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \lambda \times \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \\ &= \left[\begin{pmatrix} 1+\lambda & \lambda & 0 & -2\lambda \\ \lambda & 1+2\lambda & -2\lambda & \lambda \\ -\lambda & 0 & 1 & \lambda \\ 0 & 2\lambda & 0 & 1+3\lambda \end{pmatrix} \right]^{-1} \left[\begin{pmatrix} 1-\lambda & 0 & 0 & 2\lambda \\ -\lambda & 1 & 0 & -\lambda \\ -\lambda & 0 & 1-\lambda & -\lambda \\ -\lambda & 0 & 0 & 1+\lambda \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \end{aligned}$$

Suppose $\lambda = 1$, we obtain

$$T_1(x) = \left[\begin{pmatrix} 2 & 1 & 0 & -2 \\ 1 & 3 & -2 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 4 \end{pmatrix} \right]^{-1} \left[\begin{pmatrix} 0 & 0 & 0 & 2 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Proceeding same way, we obtain

$$T_2(x) = \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

and

$$T_3(x) = \frac{1}{9999} \left[\begin{pmatrix} 15554 & 14443 & 17776 & -2222 \\ -8148 & 11851 & 7407 & -7407 \\ -2963 & 2963 & 11851 & 1852 \\ -370 & 370 & -1481 & 1481 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

For this example, we choose $\alpha_n = \frac{1}{n+1}$, $\beta_0 = \frac{2}{n+15}$, $\beta_1 = \frac{6}{n+15}$, $\beta_2 = \frac{3+n}{n+15}$ and $\beta_3 = \frac{4}{n+15}$. We also choose $\gamma = 0.75$, $\lambda_{0,n} = \frac{5n}{10n+17}$, $\lambda_{1,n} = \frac{3n+10}{10n+17}$ and $\lambda_{2,n} = \frac{2n+7}{10n+17}$. Let $E_n = \|x_{n+1} - x_n\|^2 < 10^{-4}$ be the stopping criterion. We illustrate this example with different initial values of x_1.

(Case 1) $x_1 = (1, 1, 2, 2)'$;

(Case 2) $x_1 = (5, 5, 5, 5)'$;

(Case 3) $x_1 = (0.25, 0.5, 0.25, 0.25)'$;

(Case 4) $x_1 = (10, 5, -5, -20)'$.

The results of this experiment are presented in Figure 1.

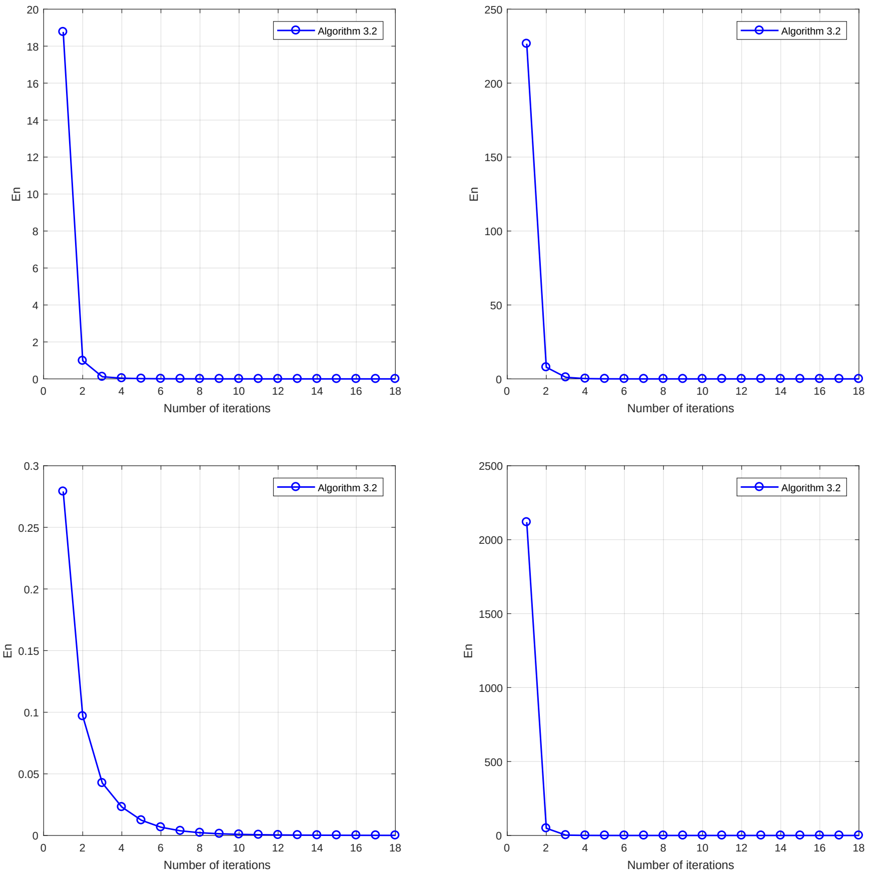

Figure 1: Example 4. Top left: Case 1, Top right: Case 2, Bottom left: Case 3, Bottom right: Case 4.

References

[1] C. Bryne. Iterative oblique projection onto convex subsets and the split feasibility problems. *Inverse Probl.*, 18:441–453, 2002.

[2] Y. Censor and T. Elfving. A multi projection algorithms using bregman projections in a product space. *Numer. Algor.*, 8:221–239, 1994.

[3] P. Sunthrayuth P. Cholamjiak. A halpern-type iteration for solving the split feasibility problem and fixed point problem of bregman relatively nonexpansive semigroup in banach spaces. *Filomat*, 32(9):3211–3227, 2018.

[4] K. R. Kazmi, R. Ali, and S. Yousuf. Generalized equilibrium and fixed point problems for bregman relatively nonexpansive mappings in banach spaces. *J. Fixed Poibt Theory Appl.*, 20(151), 2018.

[5] Y. Shehu, F. U. Ogbuisi, and O. S. Iyiola. Convergence analysis of an iterative

algorithm for fixed point problems and split feasibility problems in certain banach spaces. *Optimization*, 65:299–323, 2016.

[6] Y. Censor and A. Segal. The split common fixed point problem for directed operators. *J. Convex Anal.*, 16(2):587–600, 2009.

[7] A. Moudafi. A note on the split common fixed point problem for quasi-nonexpansive operator. *Nonlinear Anal.*, 74:4083–4087, 2011.

[8] A. Moudafi. Split monotone variational inclusions. *J. Optim. Theory Appl.*, 150:275–283, 2011.

[9] H. A. Abass, C. Izuchukwu, O. T. Mewomo, and Q. L. Dong. Strong convergence of an inertial forward-backward splitting method for accretive operators in real banach space. *Fixed Point Theory*, 20(2):397–412, 2020.

[10] H. A. Abass, K. O. Aremu, L. O. Jolaoso, and O.T. Mewomo. An inertial forward-backward splitting method for approximating solutions of certain optimization problem. *J. Nonlinear Funct. Anal.*, 2020:Article ID 6, 2020.

[11] L. Mokaba, H. A. Abass, and A. Adamu. Two step inertial tseng method for solving monotone variational inclusion problem. *Results in Applied Mathematics*, 25:100545, 2025.

[12] A. Akbar and E. Shahrosvand. Split equality common null point problem for bregman quasi-nonexpansive mappings. *Filomat*, 32(11):3917–3932, 2018.

[13] P. Cholamjiak, D. V. Hieu, and Y. J. Cho. Relaxed forward-backward splitting methods for solving variational inclusions and applications. *J. Sci. Comput.*, 88(3):1–23, 2021.

[14] F. U. Ogbuisi and O. T. Mewomo. Iterative solution of split variational inclusion problem in a real banach spaces. *Afr. Mat.*, 28:295–309, 2017.

[15] C. Izuchukwu, C. C. Okeke, and F. O. Isiogugu. A viscosity iterative technique for split variational inclusion and fixed point problems between a hilbert and a banach space. *J. Fixed Point Theory Appl.*, 20(157), 2018.

[16] S. Reich and T.M. Tuyen. Two new self-adaptive algorithms for solving the split common null point problem with multiple output sets in hilbert spaces. *J. Fixed Point Theory Appl.*, 23(16), 2021.

[17] Y. Shehu and F. U. Ogbuisi. An iterative method for solving split monotone variational inclusion and fixed point problem. *RACSAM*, 110:503–518, 2016.

[18] A. Taiwo, T. O. Alakoya, and O. T. Mewomo. Halpern type iterative process for solving split common fixed point and monotone variational inclusion problem between banach spaces. *Numer. Algor.*, 86:1359–1389, 2021.

[19] S. Timnak, E. Naraghirad, and N. Hussain. Strong convergence of halpern iteration for products of finitely many resolvents of maximal monotone operators in banach spaces. *Filomat*, 31(15):4673–4693, 2017.

[20] B. Tan, H. A. Abass, S. Li, and O. K. Oyewole. Two accelerated double inertial algorithms for variational inequalities on hadamard manifolds. *Commun. Nonlinear Sci. Numer. Simulat.*, 145:108734, 2025.

[21] S. Reich and T.M. Tuyen. Iterative methods for solving the generalized split common null point problem in hilbert spaces. *Optimization*, 69:1013–1038, 2020.

[22] A. A. Mebawondu, H. A. Abass, and O. K. Oyewole. *An accelerated Tseng type method for solving zero point problems and certain optimization problems.*, volume 36. https://doi.org/10.1007/s13370-024-01217-1, 2025.

[23] I. Bartolini, P. Ciaccia, and M. Pattela. *String Matching with Trees Using an Approximate Distance.* SPIR Lecture Notes in Computer Science, vol. 2476. Spring, Berlin, 1999.

[24] S. S. Chang, Y. J. Cho, B. S. Lee, and I. H. Jung. Generalized set-valued variational inclusions in banach spaces. *J. Mathematica; Anal. Appl.*, 246(2):409–422, 2000.

[25] C. Izuchukwu, H. A. Abass, and O. T. Mewomo. Viscosity approximation method for solving minimization problem and fixed point problem for nonexpansive multi-valued mappings in cat(0) spaces. *Ann. Acad. Rom. Sci. Ser. Math. Appl.*, 11(1), 2019.

[26] K.O. Aremu L.O. Jolaoso, O.K. Oyewole and O.T. Mewomo. A new efficient algorithm for finding common fixed points of multi-valued demicontractive mappings and solutions of split generalized equilibrium problems in hilbert spaces. *Intl. J. Comp. Mat.*, 98(9):1892–1919, 2020.

[27] J. Y. Bello and Y. Shehu. An iterative method for split inclusion problem without prior knowledge of operator norm. *J. Fixed Point Theory Appl.*, 19(3):2017–2036, 2017.

[28] O. K. Oyewole, H. A. Abass, and O. J. Ogunsola. An improved subgradient extragradient self-adaptive algorithm based on the golden ratio technique for variational inequality problems in banach spaces. *J. Comput. Appl. Math.*, 460(116420), 2025.

[29] Q. H. Ansari and A. Rehan. Iterative methods for generalized split feasibility problems in banach spaces. *Carapathian J. Math.*, 33(1), 2017.

[30] H. H. Bauschke and J. M. Borwein. Legendre functions and method of random bregman functions. *J. Convex Anal.*, 4:27–67, 1997.

[31] H. H. Bauschke, J. M. Borwein, and P. L. Combettes. Essentially smoothness, essentially strict convexity and legendre functions in banach spaces. *Commun. Contemp. Math.*, 3:615–647, 2001.

[32] L. M. Bregman. The relaxation method for finding the common point of convex sets and its application to solution of problems in convex programming. *U.S.S.R Comput. Math. Phys.*, 7:200–217, 1967.

[33] R.T. Rockafellar. On the maximality of sums of nonlinear monotone operators. *Trans. Amer. Math. Soc.*, 149:75–88, 1970.

[34] R. T. Rockafellar. Characterization of the subdifferentials of convex functions. *Pac. J. Math.*, 17:497–510, 1966.

[35] H. Gazmeh and E. Naraghirad. The split common null point problem for bregman generalized resolvents in two banach spaces. *Optimization*, 70(8):1725–1758, 2020.

[36] D. Butnairu and E. Resmerita. Bregman distances, totally convex functions and a method for solving operator equations in banach spaces. *Abstract and Applied Analysis*, Art. ID 84919:1–39, 2006.

[37] T. M. Tuyen, R. Promkan, and P. Sunthrayuth. Strong convergence of a generalized forward-backward splitting method in reflexive banach spaces. *Optimization*, pages 1–26, 2020.

[38] F. U. Ogbuisi and C. Izuchukwu. Approximating a zero of sum of two monotone operators which solves a fixed point problem in reflexive banach spaces. *Numer. Funct. Anal.*, 41(3):322–343, 2019.

[39] S. Reich and S. Sabach. Two strong convergence theorems for a proximal method in reflexive banach spaces. *Numer. Funct. Anal. Optim.*, 31:24–44, 2010.

[40] S. Reich and S. Sabach. A strong convergence theorem for a proximal-type algorithm in reflexive banach spaces. *J. Nonlinear Convex Anal.*, 10:471–485, 2009.

[41] P. E. Mainge. Approximation methods from common fixed points of nonexpansive mappings in hilbert spaces. *J. Math. Anal. Appl*, 325:469–479, 2007.

[42] P. E. Mainge. Strong convergence of projected subgradient methods for nonsmooth and non strictly convex minimization. *Set-Valued Anal.*, 16:899–912, 2008.